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ON SOME CHARACTERIZATIONS OF THE EXTENDED GENERALISED SHIFTED LINDLEY DISTRIBUTION

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SUMMARY

In this article, we unravel an extension of shifted version of Lindley distribution, termed as extended generalized shifted Lindley (EGSL) distribution. Stochastic ordering, moment generating function, reliability characteristics and other relevant properties are studied for this distribution. To estimate the parameters involved, method of maximum likelihood is performed. A detailed simulation study for several choices of parameters is executed as well. Finally, as a comparative exploration, possible fitting of the proposed distribution to a real data along with model fitting by other competent distributions is documented through the aid of a few model checking criteria.

Keywords: Lindley distribution; Stochastic ordering; Parameter estimation; Entropy; Maximum likelihood estimate.

1. INTRODUCTION

In the context of Bayesian statistics, one parameter Lindley distribution was introduced by Lindley in 1958 (Lindley, 1958). Efforts have been made to find statistical properties of this distribution extensively by many authors. This distribution can be considered as a better alternative to the one parameter exponential distribution (Ghitany *et al.*, 2008). One disadvantage of this one parameter Lindley distribution is that it is not flexible enough to analyze different types of lifetime data. It has monotonic increasing failure rates. In many situations, the hazard rate can be upside down bathtub shape, increasing, decreasing and unimodal. In many situations it becomes imperative to increase the flexibility for modelling purposes. Introducing more parameters makes this distribution a

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better fit to data resulting in accurate results and predictions in real life.

A random variable X is said to have Lindley distribution with parameter θ if its probability density function (PDF) is defined as:

$$f(x;\theta)=\frac{\theta^2}{1+\theta}(1+x)e^{-\theta x}, x>0, \theta>0 \tag{1}$$

and corresponding cumulative density function (CDF) is given by

$$F(x;\theta)=1-\frac{\theta+1+\theta x}{1+\theta}e^{-\theta x}, x>0, \theta>0. \tag{2}$$

Last one decade has seen a remarkable development in the area of possible extension of Lindley distribution. A version of generalized three-parameter Lindley distribution was proposed by Zakerzadeh and Dolati (2009). They studied its statistical properties and applications. Another contribution was made by Oluyede and Yang (2015) who have introduced a new class of generalized Lindley (GL) distributions with applications. Another very crucial extension was made by Nadarajah *et al.* (2011) where they have introduced a shape and a scale parameter in the existing generalized Lindley distribution. Ghitany *et al.* (2011) proposed a weighted Lindley (WEL) distribution. An extension using exponentiation of the standard Lindley distribution has been considered by Bakouch *et al.* (2012). Ghitany *et al.* (2013) developed the power Lindley (PL) distribution. However, no attention was given on the shifted version of this widely used Lindley distribution. Of late, an extension to Lindley distribution via location and scale parameters is proposed by Maiti *et al.* (2021). The CDF of a Shifted Lindley distribution with parameters (θ,μ) is given by

$$F(x;\theta,\mu)=1-\frac{1+\theta(1+x)}{1+\theta(1+\mu)}e^{-\theta(x-\mu)}, x>\mu>0. \tag{3}$$

Note that, if we put $\mu=0$ in Eq. (3), these equations become the CDF of a Lindley distribution with a single parameter θ.

This research article delves on an expansion of shifted Lindley distribution through the adoption of introducing parameter technique proposed by Marshall and Olkin (1997) where a new shape parameter (called tilt parameter) is added to the functional exposition of any well-known distribution with CDF $F(x)$. Distributions derived by using Marshall- Olkin (M-O) extension have an interesting failure rate function. Let $h_T(x)$ and $h(x)$ be failure rate functions corresponding to the transformed distribution and the initial distribution respectively. Then, for all $x\geq 0$ one has $h_T(x)\leq h(x)$ if $\delta>1$ and $h_T(x)\geq h(x)$ if $0<\delta\leq 1$. This implies that the failure rate will become lower if the 'tilt' parameter becomes more than 1. By taking the baseline $F(x)$ of any known distribution, new continuous distributions have been generated by several authors. A detailed study on M-O extended inverse Weibull distribution has been considered by Okasha and Kayid (2016). M-O extended generalized Lindley distribution has also received some attentions (Benkhelifa, 2017; Algarni, 2021). Here $F(x)$ is the distribution function of shifted Lindley distribution. This new distribution, hence proposed, is named as Extended Generalised Shifted Lindley (EGSL). Intrinsically, EGSL carries a more general flavour upon the concept of shifted Lindley as well as Lindley distribution.

The contents of paper is sketched as follows. Section 2 discusses the origin of EGSL along with the PDF and CDF. Also the illustrative expositions are presented for different choices of parameters involved. Section 2 also studies stochastic ordering and other distributional properties of EGSL and lights on the reliability characteristic. In Section 3, estimation process of parameters is demonstrated. Monte Carlo simulation study is carried out in Section 4 followed by a real data analysis in Section 5. Finally a short conclusion wraps up the article.

2. Extended generalized shifted Lindley (EGSL) distribution

Following the road map of Marshall and Olkin (1997), we introduce a new shape parameter (called tilt parameter) to the functional exposition of shifted Lindley (vide Eq. (3) having CDF $F(x)$. The distribution function of this newly formed Extended generalized shifted Lindley is, then,

$$G(x,\delta)=\frac{F(x)}{1-(1-\delta)\overline{F}(x)},$$

where $\overline{F}(x)=1-F(x)$. The initial distribution is thus named baseline distribution while the latter one is termed as generalized distribution. Clearly, $\delta=1$ turns the generalized one to the baseline distribution.
The motivation of including tilt parameter δ is to examine the change of hazard rates in baseline distribution and in the generalized one. It is easy to check that the failure rate of the new distribution is shifted below that of original distribution when $\delta>1$. On the contrary the failure rate for the new one shifts above that of the old one when $0<\delta<1$. Keeping shifted Lindley (SL) as the baseline one our quest is to propose an extended generalized shifted Lindley (EGSL) distribution.

Let $\Lambda=(\lambda,\gamma,\delta,\mu)$, then the CDF of EGSL is

$$G(x,\Lambda)=\frac{K}{1-\overline{\delta}(1-K)},\ x>\mu,\ \Lambda>0, \tag{4}$$

where

$$K=\left[1-\frac{\lambda+\lambda x+1}{\lambda+\lambda\mu+1}e^{-\lambda(x-\mu)}\right]^{\gamma}, \tag{5}$$

$\overline{\delta}=1-\delta$, $\delta=$ tilt parameter, $\gamma=$ shape parameter, $\lambda=$ scale parameter.
Correspondingly, the density function can be expressed as

$$f(x,\Lambda)=\frac{\delta\gamma\lambda^2}{\lambda+\lambda\mu+1}\frac{K^{1-\frac{1}{\gamma}}}{(\delta+\overline{\delta}K)^2}(x+1)e^{-\lambda(x-\mu)},\ x>\mu. \tag{6}$$

The proposed EGSL distribution can take different shapes for different combinations of parameter values. As an example, few combinations are given in Figure 1. It may also be noted that this M-O extended shifted Lindley distribution can take care of different failure rates (Figure 3). This will undoubtedly make this distribution more applicable in many situations.

To prove some properties related to EGSL distribution, the following lemma may be very useful.

LEMMA 1. *For $\gamma, \lambda, \mu > 0$ and constant c, let us define*

$$F_{r,s,a,b,c}(\gamma,\lambda,\mu) = \int_{\mu}^{\infty} x^r (1+x)^s e^{-\lambda a x}\left[1 - \frac{\lambda+\lambda x+1}{\lambda+\lambda\mu+1} e^{-\lambda(x-\mu)}\right]^{\gamma(c+b)-b} dx$$

Then, we have

$$F_{r,s,a,b,c}(\gamma,\lambda,\mu) = \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{l=0}^{s}(-1)^i \binom{\gamma(c+b)-b}{i}\binom{i}{j}\binom{s}{l}\frac{(\lambda+1)^{i-j}}{(\lambda+\lambda\mu+1)^i}\frac{\lambda^{-r-l-1}}{(a+i)^{r+l+j+1}} \times e^{\lambda i\mu}\Gamma[r+l+j+1, \lambda\mu(a+i)]$$

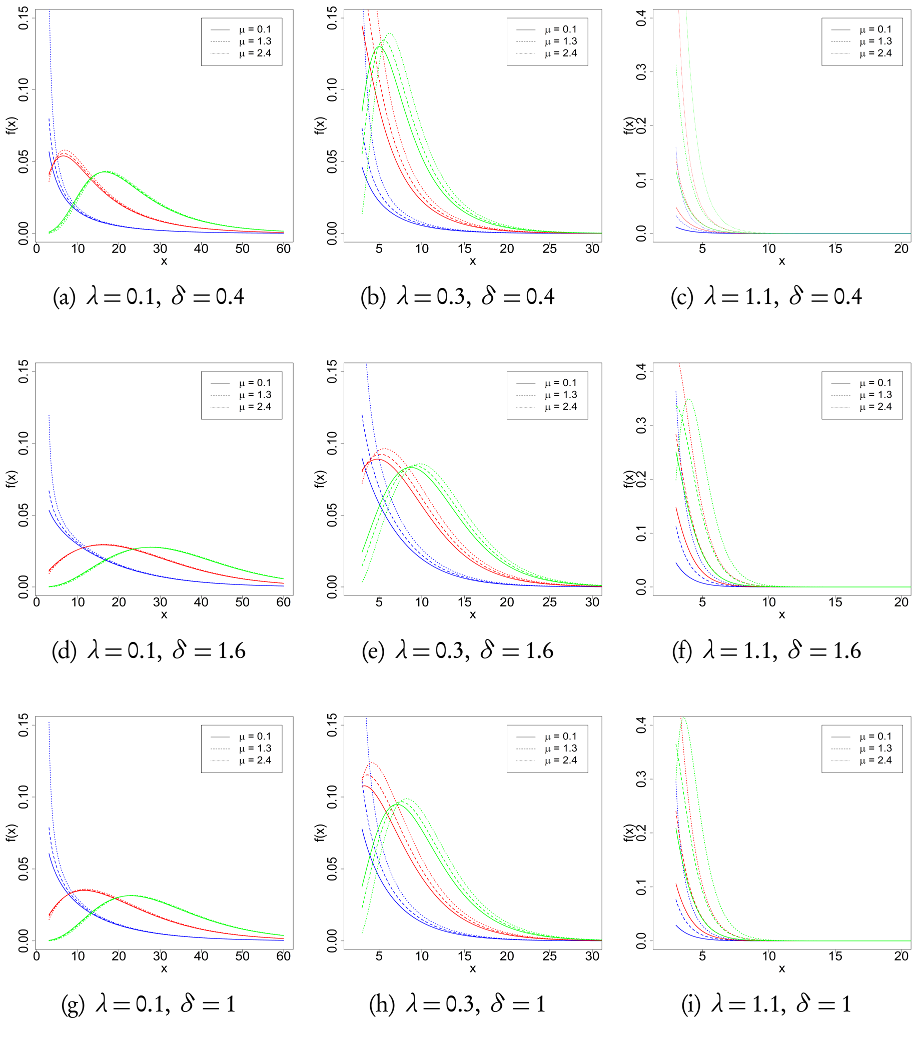
where $\Gamma[\cdot]$ is the upper incomplete Gamma function.

PROOF.

$$F_{r,s,a,b,c}(\gamma,\lambda,\mu) = \int_{\mu}^{\infty} x^r (1+x)^s e^{-\lambda a x}\left[1 - \frac{\lambda+\lambda x+1}{\lambda+\lambda\mu+1} e^{-\lambda(x-\mu)}\right]^{\gamma(c+b)-b} dx$$

Let, $I_1 = \left[1 - \frac{\lambda+\lambda x+1}{\lambda+\lambda\mu+1} e^{-\lambda(x-\mu)}\right]^{\gamma(c+b)-b}$, Then

$$\begin{aligned}
I_1 &= \sum_{i=0}^{\infty}(-1)^i\binom{\gamma(c+b)-b}{i}\left(\frac{1+\lambda+x\lambda}{1+\lambda+\mu\lambda}\right)^i e^{-\lambda i(x-\mu)} \\
&= \sum_{i=0}^{\infty}(-1)^i\binom{\gamma(c+b)-b}{i}\left(\frac{\lambda+1}{\lambda+\lambda\mu+1}\right)^i\left[1+\frac{\lambda}{1+\lambda}x\right]^i e^{-\lambda i(x-\mu)} \\
&= \sum_{i=0}^{\infty}(-1)^i\binom{\gamma(c+b)-b}{i}\left(\frac{\lambda+1}{\lambda+\lambda\mu+1}\right)^i \sum_{j=0}^{i}\binom{i}{j}\left(\frac{\lambda}{1+\lambda}\right)^j x^j e^{-\lambda i(x-\mu)} \\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{i}(-1)^i\binom{\gamma(c+b)-b}{i}\binom{i}{j}\left(\frac{\lambda+1}{\lambda+\lambda\mu+1}\right)^i\left(\frac{\lambda}{1+\lambda}\right)^j x^j e^{-\lambda i(x-\mu)}
\end{aligned}$$



(a) $\lambda = 0.1,\ \delta = 0.4$ (b) $\lambda = 0.3,\ \delta = 0.4$ (c) $\lambda = 1.1,\ \delta = 0.4$

(d) $\lambda = 0.1,\ \delta = 1.6$ (e) $\lambda = 0.3,\ \delta = 1.6$ (f) $\lambda = 1.1,\ \delta = 1.6$

(g) $\lambda = 0.1,\ \delta = 1$ (h) $\lambda = 0.3,\ \delta = 1$ (i) $\lambda = 1.1,\ \delta = 1$

Figure 1 – The PDF's of various EGSL distributions for different values of parameters. Blue: $\gamma = 0.3$, Red: $\gamma = 1.2$, Green: $\gamma = 2.8$.

Therefore,

$$\begin{aligned}
F &= \int_{\mu}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{i}(-1)^i\binom{\gamma(c+b)-b}{i}\binom{i}{j}\left(\frac{\lambda+1}{\lambda+\lambda\mu+1}\right)^i\left(\frac{\lambda}{1+\lambda}\right)^j x^{r+j}e^{\lambda i\mu}e^{-\lambda(a+i)x}(1+x)^s\,dx\\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{l=1}^{s}(-1)^i\binom{\gamma(c+b)-b}{i}\binom{i}{j}\binom{s}{l}\left(\frac{\lambda+1}{\lambda+\lambda\mu+1}\right)^i\left(\frac{\lambda}{1+\lambda}\right)^j e^{\lambda i\mu}\int_{\mu}^{\infty}x^{r+j+l}e^{-\lambda(a+i)x}\,dx\\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{l=1}^{s}(-1)^i\binom{\gamma(c+b)-b}{i}\binom{i}{j}\binom{s}{l}\left(\frac{\lambda+1}{\lambda+\lambda\mu+1}\right)^i\left(\frac{\lambda}{1+\lambda}\right)^j e^{\lambda i\mu}\frac{\Gamma[r+l+j+1,\lambda\mu(a+i)]}{[\lambda(a+i)]^{r+l+j+1}}.
\end{aligned}$$

□

2.1. *Stochastic orders*

The notion of stochastic ordering for random variables is an useful concept which quantifies how much bigger one random variable than the other in terms of probability content. Stochastic orderings covers a wide field of application in probability, statistics, and statistical decision theory. In probability theory, they are useful in deducing probability inequalities, comparing stochastic models, establishing bounds and inequalities in reliability and queueing theory. Let X_1 and X_2 be univariate random variables with distribution function F_1 and F_2 respectively along with the corresponding probability densities f_1 and f_2. X_1 is said to be smaller than a random variable X_2 in the

- Stochastic order ($X_1 \leq_{st} X_2$) if $F_1(x) \geq F_2(x)$ for all x,
- Hazard rate order ($X_1 \leq_{hr} X_2$) if $h_1(x) \geq h_2(x)$ for all x,
- Likelihood ratio order ($X_1 \leq_{Lr} X_2$) if $\frac{f_1(x)}{f_2(x)}$ decreases in x.

In order to establish stochastic ordering of distributions we refer the following result from Shaked and Shanthikumar (1994).

$$X_1 \leq_{LR} X_2 \Longrightarrow X_1 \leq_{hr} X_2 \Longrightarrow X_1 \leq_{st} X_2.$$

Taking a cue from this above-mentioned result, three theorems are proposed regarding the stochastic ordering pattern of EGSL(Λ) for different conditions imposed on $(\lambda, \gamma, \delta, \mu)$. First two theorems are direct-ward from the results on stochastic ordering in shifted Lindley distribution (Maiti *et al.*, 2021).

THEOREM 2. Let $X_1 \sim$ EGSL$(\lambda_1, \gamma_1, \delta_1, \mu_1)$ and $X_2 \sim$ EGSL$(\lambda_2, \gamma_1, \delta_1, \mu_2)$. If $\mu_1 = \mu_2$ and $\lambda_2 < \lambda_1$, then $X_1 \leq_{Lr} X_2$ and hence $X_1 \leq_{hr} X_2$ and $X_1 \leq_{st} X_2$.

THEOREM 3. *Let $X_1 \sim EGSL(\lambda, \gamma, \delta, \mu_1)$ and $X_2 \sim EGSL(\lambda, \gamma, \delta, \mu_2)$. If $\mu_1 > \mu_2$; then $X_1 \geq_{st} X_2$.*

PROOF. Proofs of Theorems 2 and 3 are easy and straightforward. In the Theorem 3 note that, the ratio of two pdf's does not involve x. So the technique adopted in checking of likelihood ratio ordering fails. Therefore, we would head to investigate via ratio of two corresponding distribution functions and hence directly inferring on stochastic ordering of the distribution. □

THEOREM 4. *Let X_1 and X_2 be two univariate random variables such that $X_1 \sim f_1 = EGSL(\lambda,\gamma,\delta_1,\mu)$ and $X_2 \sim f_2 = EGSL(\lambda,\gamma,\delta_2,\mu)$. If $\delta_1 < \delta_2$, then $X_1 \leq_{Lr} X_2$ and hence $X_1 \leq_{hr} X_2$ and $X_1 \leq_{st} X_2$.*

PROOF. Since the entity K is made of λ,γ,μ both the distributions have the same K. Thereafter, the ratio of two pdf's EGSL$_1$ and EGSL$_2$ can be simplified into

$$\frac{f_1}{f_2} = \frac{1+(\frac{1}{\delta_2}-1)K}{1+(\frac{1}{\delta_1}-1)K}.$$

When $\delta_1 > \delta_2$, $1+(\frac{1}{\delta_1}-1)K_1 < 1+(\frac{1}{\delta_2}-1)K$ which follows $\frac{f_1}{f_2} > 1$. Thus $X_1 \leq_{Lr} X_2$ and the other inequalities hold accordingly. □

2.2. Moments

In applications, moments are necessary and very important. Through moments, it is possible to study many of the interesting characteristics and features of a distribution. The r^{th} order raw moment of the EGSL distribution can be obtained as:

$$\begin{aligned} \mu_r' = E(X^r) &= \frac{\delta\gamma\lambda^2}{\lambda+\lambda\mu+1}\int_\mu^\infty x^r \frac{K^{1-\frac{1}{\gamma}}}{(\delta+\bar{\delta}K)^2}(1+x)e^{-\lambda(x-\mu)}dx \\ &= \frac{\delta\gamma\lambda^2 e^{\lambda\mu}}{\lambda+\lambda\mu+1}\int_\mu^\infty \frac{K^{1-\frac{1}{\gamma}}}{(\delta+\bar{\delta}K)^2}x^r(1+x)e^{-\lambda x}dx \\ &= \frac{\delta^{-1}\gamma\lambda^2 e^{\lambda\mu}}{\lambda+\lambda\mu+1}\sum_{j=0}^{\infty}(j+1)\Big(\frac{1-\delta}{\delta}\Big)^j F_{r,1,1,1,j}(\gamma,\lambda,\mu). \end{aligned} \tag{7}$$

Putting $r=1$ we get, $\mu_1' = \dfrac{\delta^{-1}\gamma\lambda^2 e^{\lambda\mu}}{\lambda+\lambda\mu+1}\displaystyle\sum_{j=0}^{\infty}(j+1)\Big(\frac{1-\delta}{\delta}\Big)^j F_{1,1,1,1,j}(\gamma,\lambda,\mu).$

The r^{th} order central moment of the distribution can be obtained as:

$$\begin{aligned}
\mu_r &= E[(X-\mu_1)^r] \qquad (8)\\
&= \frac{\delta\gamma\lambda^2}{\lambda+\lambda\mu+1}\int_{\mu}^{\infty}(x-\mu_1)^r\frac{K^{1-\frac{1}{\gamma}}}{(\delta+\bar{\delta}K)^2}(1+x)e^{-\lambda(x-\mu)}dx\\
&= \frac{\delta\gamma\lambda^2 e^{\lambda\mu}}{\lambda+\lambda\mu+1}\int_{\mu}^{\infty}\frac{K^{1-\frac{1}{\gamma}}}{(\delta+\bar{\delta}K)^2}(x-\mu_1)^r(1+x)e^{-\lambda x}dx\\
&= \frac{\delta^{-1}\gamma\lambda^2 e^{\lambda\mu}}{\lambda+\lambda\mu+1}\sum_{j=0}^{\infty}\sum_{l=0}^{r}(j+1)\Big(\frac{1-\delta}{\delta}\Big)^j\mu_1^{r-l}F_{l,1,1,1,j}(\gamma,\lambda,\mu). \qquad (9)
\end{aligned}$$

Using the Eq. (8) we can find variance (μ_2) and higher order central moments μ_3, μ_4 to find the skewness and kurtosis of the distribution.

In reference to the moments of EGSL distribution, next we present heatmap diagram (Fig. 2) which unravels the intertwining effect of parameters ($\lambda, \gamma, \delta, \mu$) on mean, variance, skewness and kurtosis. Heat plot (or heatmap) is a data visualization technique that shows impact of variables in terms of intensity of color in two dimensions. The variation in color exhibits obvious visual clues about the relationship between two categories. From the matrix layout with reference to first two parameters (λ,γ) color and shading of heat plots in context of mean, variance, skewness and kurtosis project self explanatory rises and drops of these four measures.

2.3. *Moment generating function (MGF)*

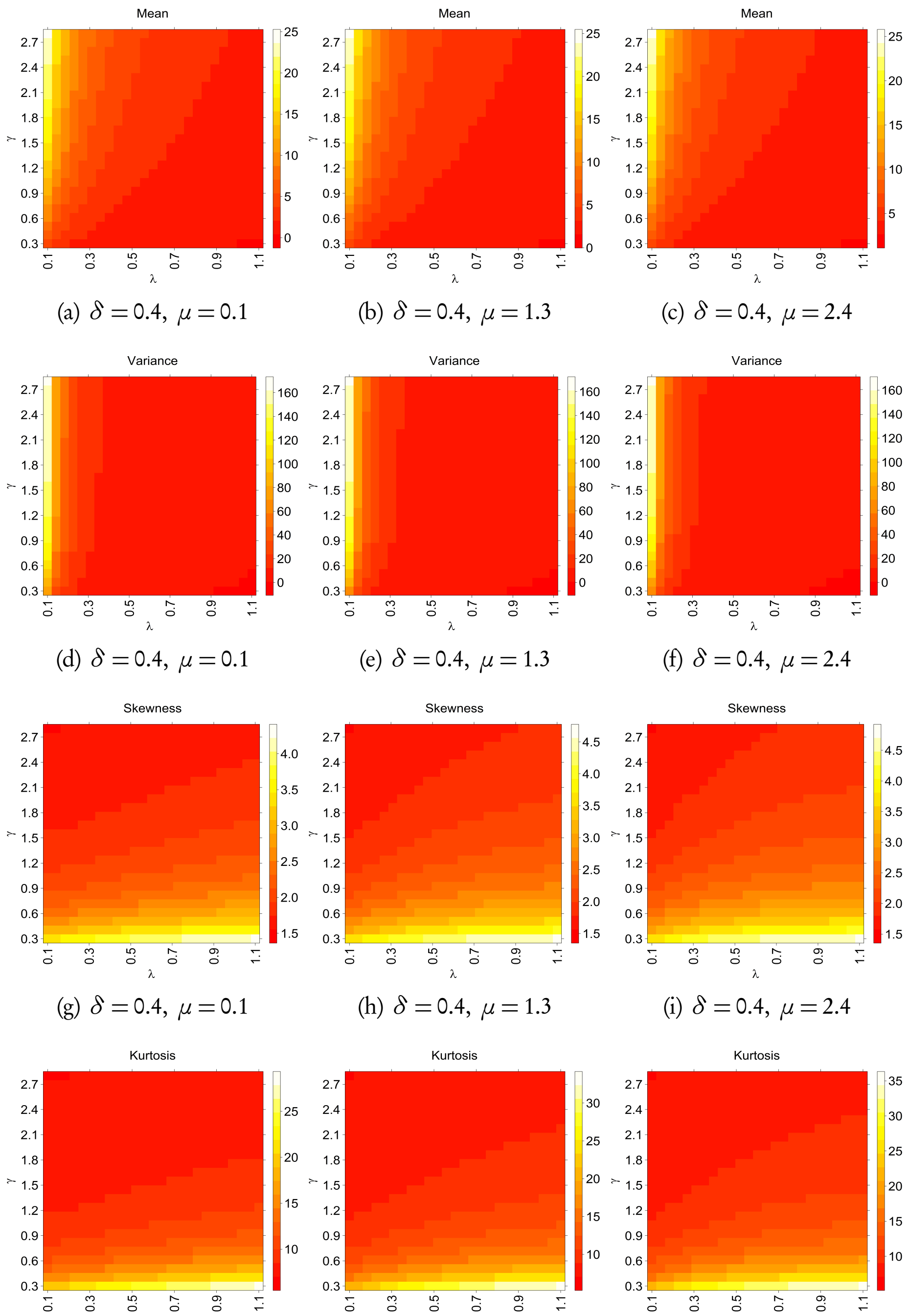
In this Section, we derived the MGF of EGSL($\lambda, \gamma, \delta, \mu$) distribution.

THEOREM 5. *If $X \sim EGSL(\lambda, \gamma, \delta, \mu)$, then the moment generating function $M_X(t)$ has the following form:*

$$M_X(t) = \frac{\delta^{-1}\gamma\lambda^2 e^{\lambda\mu}}{\lambda+\lambda\mu+1}\sum_{j=0}^{\infty}(j+1)\Big(\frac{1-\delta}{\delta}\Big)^j F_{0,1,1,1,j}(\gamma,\lambda-t,\mu). \qquad (10)$$

PROOF.

$$\begin{aligned}
M_X(t) &= E(e^{tX})\\
&= \frac{\delta\gamma\lambda^2}{\lambda+\lambda\mu+1}\int_{\mu}^{\infty}e^{tx}\frac{K^{1-\frac{1}{\gamma}}}{(\delta+\bar{\delta}K)^2}(1+x)e^{-\lambda(x-\mu)}dx\\
&= \frac{\delta\gamma\lambda^2 e^{\lambda\mu}}{\lambda+\lambda\mu+1}\int_{\mu}^{\infty}\frac{K^{1-\frac{1}{\gamma}}}{(\delta+\bar{\delta}K)^2}(1+x)e^{-x(\lambda-t)}dx\\
&= \frac{\delta^{-1}\gamma\lambda^2 e^{\lambda\mu}}{\lambda+\lambda\mu+1}\sum_{j=0}^{\infty}(j+1)\Big(\frac{1-\delta}{\delta}\Big)^j F_{0,1,1,1,j}(\gamma,\lambda-t,\mu).
\end{aligned}$$



(a) $\delta = 0.4,\ \mu = 0.1$ (b) $\delta = 0.4,\ \mu = 1.3$ (c) $\delta = 0.4,\ \mu = 2.4$

(d) $\delta = 0.4,\ \mu = 0.1$ (e) $\delta = 0.4,\ \mu = 1.3$ (f) $\delta = 0.4,\ \mu = 2.4$

(g) $\delta = 0.4,\ \mu = 0.1$ (h) $\delta = 0.4,\ \mu = 1.3$ (i) $\delta = 0.4,\ \mu = 2.4$

(j) $\delta = 0.4,\ \mu = 0.1$ (k) $\delta = 0.4,\ \mu = 1.3$ (l) $\delta = 0.4,\ \mu = 2.4$

Figure 2 – Heat maps for mean, variance, skewness and kurtosis.

□

In the same way the characteristic function of the extended generalized shifted Lindley distribution becomes as follows.

$$\phi_X(t) = M_X(it) = \frac{\delta^{-1}\gamma\lambda^2 e^{\lambda\mu}}{\lambda+\lambda\mu+1}\sum_{j=0}^{\infty}(j+1)\Big(\frac{1-\delta}{\delta}\Big)^j F_{0,1,1,1,j}(\gamma,\lambda-it,\mu), \tag{11}$$

where $i = \sqrt{-1}$ is the unit imaginary number.

2.4. *Quantile function*

Let X denotes a random variable with the CDF mentioned in Eq. (4). The quantile function, say $Q(p)$, defined by $F(Q(p)) = p$ is the root of the equation

$$\frac{K}{1-\overline{\delta}(1-K)} = 1-p \tag{12}$$

for $0 < p < 1$. On further simplification of Eq. (12),

$$\begin{aligned}
K &= \frac{p\delta}{1-p\overline{\delta}} = \alpha \text{ (say)} \\
1-\frac{\lambda+\lambda Q(p)+1}{\lambda+\lambda\mu+1}e^{-\lambda(Q(p)-\mu)} &= \alpha^{1/\gamma} \\
(\lambda+1+\lambda Q(p))e^{-\lambda Q(p)} &= (\lambda+\lambda\mu+1)(1-\alpha^{1/\gamma})e^{-\lambda\mu} \\
-(\lambda+1+\lambda Q(p))e^{-(\lambda+1+\lambda Q(p))} &= -(\lambda+\lambda\mu+1)(1-\alpha^{1/\gamma})e^{-(\lambda+\lambda\mu+1)} \\
\lambda+1+\lambda Q(p) &= -W\Big[-(\lambda+\lambda\mu+1)(1-\alpha^{1/\gamma})e^{-(\lambda+\lambda\mu+1)}\Big] \\
Q(p) &= -\frac{1}{\lambda}W\Big[-(\lambda+\lambda\mu+1)(1-\alpha^{1/\gamma})e^{-(\lambda+\lambda\mu+1)}\Big] \\
&\quad -\frac{1+\lambda}{\lambda}.
\end{aligned}$$

Here, $W[\cdot]$ is the Lambert W function (see Corless *et al.*, 1996) and $0 < p < 1$.

2.5. *Order statistics*

If $X_1, X_2, \cdots, X_n$ be a random sample of size n from EGSL$(\lambda,\gamma,\delta,\mu)$. Then the pdf of the i^{th} order statistic i.e. $f_{X_{(i)}}(x)$ and the joint pdf of the $(j,k)^{\text{th}}$ order statistic i.e. $f_{X_{(j,k)}}(x)$ where $1 \le j \le k \le n$ is given by the following two lemma.

LEMMA 6. *The pdf of the i^{th} order statistic of EGSL$(\lambda,\gamma,\delta,\mu)$ distribution is*

$$f_{X_{(i)}}(x)=\frac{n!}{(i-1)!(n-i)!}\left[\frac{K}{1-\overline{\delta}(1-K)}\right]^{i-1}\left[\frac{\delta(1-K)}{1-\overline{\delta}(1-K)}\right]^{n-i}f(x,\Lambda),\ x>\mu.$$

LEMMA 7. *The pdf of the $(j,k)^{th}$ order statistic of EGSL$(\lambda,\gamma,\delta,\mu)$ distribution is*

$$f_{X_{(j,k)}}(x,y)=\frac{n!}{(j-1)!(k-j-1)!(n-k)!}\left[\frac{K_x}{1-\overline{\delta}(1-K_x)}\right]^{j-1}$$
$$\times\left[\frac{K_y}{1-\overline{\delta}(1-K_y)}-\frac{K_x}{1-\overline{\delta}(1-K_x)}\right]^{k-1-j}\left[\frac{\delta(1-K_y)}{1-\overline{\delta}(1-K_y)}\right]^{n-k}f(x,\Lambda)f(y,\Lambda),$$

where $x\geq y>\mu$. Here $K_x=\left[1-\frac{\lambda+\lambda x+1}{\lambda+\lambda\mu+1}e^{-\lambda(x-\mu)}\right]^{\gamma}$ and $K_y=\left[1-\frac{\lambda+\lambda y+1}{\lambda+\lambda\mu+1}e^{-\lambda(y-\mu)}\right]^{\gamma}$.

2.6. *Reliability characteristics of extended generalized shifted Lindley distribution*

In the present section, we consider Extended Generalized Shifted Lindley distribution as a lifetime model and study different reliability characteristics. The reliability function of the EGSL$(\lambda,\gamma,\delta,\mu)$ distribution is given by:

$$R(t)=P(X>t)=1-F(t). \tag{13}$$

The mean time to system failure (MTSF) is same as we know that the hazard function $h(x)$ can be computed as

$$h(t)=\frac{f(t;\lambda,\gamma,\delta,\mu)}{1-F(t;\lambda,\gamma,\delta,\mu)},$$

which implies

$$h(t)=\frac{\gamma\lambda^2}{\lambda+\lambda\mu+1}\frac{K^{1-\frac{1}{\gamma}}}{(\delta+\overline{\delta}K)(1-K)}(x+1)e^{-\lambda(x-\mu)}. \tag{14}$$

Fig. 3 delineates the hazard rate of EGSL for five choices of the parameters. The choices, undertaken are mentioned in the figure title. The hazard rate diagrams exhibits close proximity of gamma distribution hazard rates.

The cumulative hazard function $H(x)$ is defined as

$$H(x)=-\log[1-F(x;\lambda,\gamma,\delta,\mu)]=-\log\left[\frac{\delta(1-K)}{1-\overline{\delta}(1-K)}\right]=-\log(R(x))$$

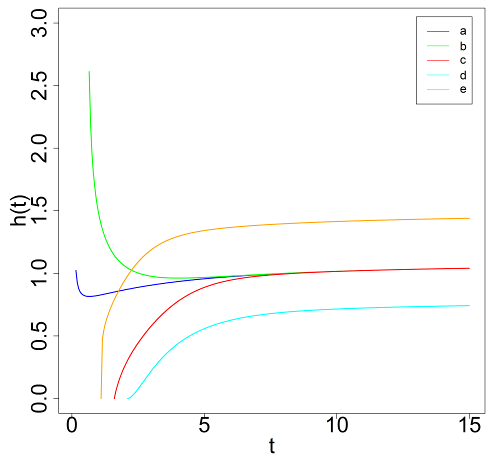


Figure 3 – The Hazard function $h(t)$ with respect to t for the parameter values: a) $(\lambda,\gamma,\delta,\mu)=(1.1,0.8,1,0.1)$, b) $(\lambda,\gamma,\delta,\mu)=(1.1,0.8,0.4,0.6)$, c) $(\lambda,\gamma,\delta,\mu)=(1.1,1.9,1.6,1.6)$, d) $(\lambda,\gamma,\delta,\mu)=(0.8,2.8,1,2.1)$, e) $(\lambda,\gamma,\delta,\mu)=(1.5,1.2,1.6,1.1)$.

and the failure rate average (FRA) is defined by $\text{FRA}(x)=H(x)/x$, where $x>\mu$. The conditional survival of t is

$$R(x|t)=\frac{R(x+t)}{R(t)};\theta,R(.)>0;t,x>\mu,\mu>0.$$

2.6.1. The mean residual life

Let $T\geq 0$ be a continuous random variable with cdf $G(t)$ and pdf $f(t)$, then the mean residual life (MRL) can be obtained by

$$M_T(t)=E(T-t|T>t)=\frac{1}{1-G(t)}\int_t^{\infty}[1-G(x)]\,dx,t>0. \tag{15}$$

LEMMA 8. *Let $T\sim EGSL(\Lambda)$, then the MRL function of a lifetime random variable is given by*

$$M_T(t)=\frac{1-\overline{\delta}(1-K_t)}{K_t}H_t(\Lambda),$$

where

$$H_t(\Lambda)=\sum_{i=0}^{\infty}\sum_{j=0}^{\gamma(i+1)}\frac{(-1)^{i+j+\gamma(i+1)}}{\lambda j^{j+1}(\lambda+\lambda\mu+1)^j}\frac{(1-\delta)^i}{\delta^{i+1}}\binom{\gamma(i+1)}{j}e^{j(\lambda+\lambda\mu+1)}\Gamma[j+1,j(\lambda+\lambda t+1)].$$

2.6.2. *Reversed failure rate*

Let $T\geq 0$ be a continuous random variable with cdf $G(t)$ and pdf $f(t)$, the reversed hazard rate (RHR) function, is defined by

$$r(t)=\frac{f(t)}{G(t)}. \tag{16}$$

LEMMA 9. *Let $T\sim EGSL(\Lambda)$, then the RHR function of a lifetime random variable is given by*

$$r(t)=\frac{\delta\gamma\lambda^2}{\lambda+\lambda\mu+1}\frac{1-\overline{\delta}(1-K)}{K^{\frac{1}{\gamma}}(\delta+\overline{\delta}K)^2}(x+1)e^{-\lambda(x-\mu)}.$$

2.6.3. *Mean inactivity time*

Let $T\geq 0$ be a continuous random variable with cdf $G(t)$ and pdf $f(t)$, the mean inactivity time (MIT) function, is defined by

$$m(t)=E(T-t|T>t)=\frac{1}{G(t)}\int_0^t G(x)\,dx,t>0. \tag{17}$$

LEMMA 10. *Let $T\sim EGSL(\Lambda)$, then the MIT function of a lifetime random variable is given by*

$$m(t)=\frac{\delta+\overline{\delta}K}{K}M_\Lambda(t),$$

where

$$M_\Lambda(t)=\sum_{i=0}^{\infty}\sum_{j=0}^{\gamma(i+1)}\frac{(1-\delta)^i}{\delta^{i+1}}(-1)^{\gamma(i+1)+j}\binom{\gamma(i+1)}{j}$$
$$\times\frac{e^{j(\lambda+\lambda\mu+1)}}{\lambda j^{j+1}(\lambda+\lambda\mu+1)}\Big[\gamma[j+1,(\lambda+\lambda t+1)j]-\gamma[j+1,(\lambda+1)j]\Big].$$

2.7. *Rényi entropy*

Entropy is used to measure the randomness of systems, and it is widely used in areas like physics, molecular imaging of tumors and sparse kernel density estimation. If X has the probability distribution function $f(.)$, Rényi entropy (Rényi, 1961) of order α is defined by

$$I_\alpha(x) = \frac{1}{1-\alpha}\log\Big(\int_{-\infty}^{\infty} f^\alpha(x)\,dx\Big),\ \alpha>0,\ \alpha\neq 1.$$

Using Eq. (6), it is observed that

$$\begin{aligned} f^\alpha(x) &= \frac{\delta^\alpha\gamma^\alpha\lambda^{2\alpha}}{[\lambda+\lambda\mu+1]^\alpha}\frac{K^{\alpha-\frac{\alpha}{\gamma}}}{(\delta+\overline{\delta}K)^{2\alpha}}(x+1)^\alpha e^{-\alpha\lambda(x-\mu)} \\ &= \frac{\delta^{-\alpha}\gamma^\alpha\lambda^{2\alpha}}{[\lambda+\lambda\mu+1]^\alpha}K^{\alpha-\frac{\alpha}{\gamma}}\Big(1+\frac{\overline{\delta}}{\delta}K\Big)^{-2\alpha}(x+1)^\alpha e^{-\alpha\lambda(x-\mu)} \\ &= \frac{\delta^{-\alpha}\gamma^\alpha\lambda^{2\alpha}e^{\alpha\lambda\mu}}{[\lambda+\lambda\mu+1]^\alpha}\sum_{i=0}^{\infty}\binom{-2\alpha}{i}\Big(\frac{1-\delta}{\delta}\Big)^i K^{i+\alpha-\frac{\alpha}{\gamma}}(1+x)^\alpha e^{-\alpha\lambda x}. \end{aligned}$$

After some algebra, the Rényi entropy of X is reduces to

$$I_\alpha(x) = \tfrac{\alpha\lambda\mu}{1-\alpha}+\tfrac{\alpha}{1-\alpha}\log\Big(\tfrac{\delta^{-1}\gamma\lambda^2}{\lambda+\lambda\mu+1}\Big)+\tfrac{1}{1-\alpha}\log\Big[\textstyle\sum_{i=0}^{\infty}\binom{-2\alpha}{i}\Big(\tfrac{1-\delta}{\delta}\Big)^i F_{0,\alpha,\alpha,\alpha,i}(\gamma,\lambda,\mu)\Big]$$

where the function, F is defined on Lemma 1.

3. Estimation

3.1. *Maximum likelihood (ML) estimation of parameters*

Let $X_1, X_2, \cdots, X_n$ be a random sample from EGSL with parameter vector Λ. The log likelihood function would take the form

$$\begin{aligned} \Phi(\mathbf{x}|\Lambda) &= n(\log\delta+\log\gamma+2\log\lambda)-n\log(\lambda+\lambda\mu+1)+(1-\frac{1}{\gamma})\sum_{i=1}^{n}\log K_i \\ &- 2\sum_{i=1}^{n}\log[\delta+(1-\delta)K_i]+\sum_{i=1}^{n}\log(x_i+1)-\lambda\sum_{i=1}^{n}(x_i-\mu). \end{aligned} \tag{18}$$

It is to be noted that, the mle of μ is

$$\hat{\mu} = \min_i X_i = X_{(1)}. \tag{19}$$

The MLE of three unknown parameters can be obtained from equating the partial derivatives of $\Phi(\mathbf{x}|\lambda,\gamma,\delta,\mu=\hat{\mu})$ with respect to three parameters λ, γ and δ to zero, i.e. forming likelihood equations

$$\begin{aligned}\Phi_\lambda = & \frac{n(\lambda+\lambda\mu+2)}{\lambda(\lambda+\lambda\mu+1)} + \sum_{i=1}^{n} e^{-\lambda(x_i-\mu)} K_i^{-1/\gamma} \\ & \times \frac{\lambda[(x_i^2+x_i-\mu x_i-\mu)(\lambda+1+\lambda\mu)+(1+\mu)(x_i-\mu)]}{(\lambda+\lambda\mu+1)^2} \\ & \times\left[\gamma-1-\frac{2\gamma(1-\delta)K_i}{\delta+(1-\delta)K_i}\right] - \sum_{i=1}^{n}(x_i-\mu) \end{aligned} \tag{20}$$

$$\Phi_\gamma = \frac{n}{\gamma} + \frac{1}{\gamma}\sum_{i=1}^{n}\log K_i - \frac{2(1-\delta)}{\gamma}\sum_{i=1}^{n}\frac{K_i \log K_i}{\delta+(1-\delta)K_i} \tag{21}$$

$$\Phi_\delta = \frac{n}{\delta} - 2\sum_{i=1}^{n}\frac{1-K_i}{\delta+(1-\delta)K_i} \tag{22}$$

Clearly, these four equations are not in the explicit form. These equations can be solved through method of iteration by employing statistical softwares.

4. Simulation study

Using uniform distribution and Lambert's W function it is possible to generate samples using the inversion of the CDF. Upon the generation of random samples from uniform distribution, we can simply invert the CDF of EGSL$(\lambda,\gamma,\delta,\mu)$ at those points. Thereafter, using the aforesaid function a convenient sampling scheme for data generation of EGSL can be framed as follows:

1. Select values of λ,γ,δ and μ.

2. Generate u from $U(0,1)$.

3. Generate x such that $x=F^{-1}(u)$, i.e.

$$x = -\frac{1}{\lambda}W\left[-(\lambda+\lambda\mu+1)(1-\alpha^{1/\gamma})e^{-(\lambda+\lambda\mu+1)}\right] - \frac{1+\lambda}{\lambda},$$

where $\alpha=\frac{u\delta}{1-(1-\delta)u}$ and $W[\cdot]$ is the Lambert W function.

4. Repeat N times to generate samples from EGSL$(\lambda,\gamma,\delta,\mu)$.

To estimate the parameters λ, γ, δ and μ, we generate 50,000 samples from the EGSL distribution. We consider sixteen different combinations of the parameter to study their influence. Then using Eq. (19) together with the likelihoods in Eq. (20), Eq. (21) and Eq. (22), we obtain the ML estimates of μ, λ, γ and δ. We replicate these processes 50, 100 and 500 times and compute the estimate and the standard error of corresponding estimates.

The simulation study is carried out with sample size $N = 50,000$ for choices of two values of each parameter $(\lambda, \gamma, \delta, \mu) = (0.5, 0.3)$, $(1.5, 1.1)$, $(0.5, 1.1), (1.5, 0.3)$ and replications $n = (50, 100, 500)$. The following measures are calculated to assess the simulation results:

$$\mathrm{Bias}_\lambda = \sum_{i=1}^{n} \frac{\widehat{\lambda}_j - \lambda}{n},$$

$$\text{Magnitude of Relative Error } \mathrm{MRE}_\lambda = \sum_{i=1}^{n} \frac{\widehat{\lambda}_j / \lambda}{n},$$

$$\text{Mean Square Error } \mathrm{MSE}_\lambda = \sum_{i=1}^{n} \frac{(\widehat{\lambda}_j - \lambda)^2}{n},$$

$$\mathrm{Bias}_\gamma = \sum_{i=1}^{n} \frac{\widehat{\gamma}_j - \gamma}{n},$$

$$\mathrm{MRE}_\gamma = \sum_{i=1}^{n} \frac{\widehat{\gamma}_j / \gamma}{n},$$

$$\mathrm{MSE}_\gamma = \sum_{i=1}^{n} \frac{(\widehat{\gamma}_j - \gamma)^2}{n},$$

$$\mathrm{Bias}_\delta = \sum_{i=1}^{n} \frac{\widehat{\delta}_j - \delta}{n},$$

$$\mathrm{MRE}_\delta = \sum_{i=1}^{n} \frac{\widehat{\delta}_j / \delta}{n},$$

$$\mathrm{MSE}_\delta = \sum_{i=1}^{n} \frac{(\widehat{\delta}_j - \delta)^2}{n},$$

$$\mathrm{Bias}_\mu = \sum_{i=1}^{n} \frac{\widehat{\mu}_j - \mu}{n},$$

$$\mathrm{MRE}_\mu = \sum_{i=1}^{n} \frac{\widehat{\mu}_j / \mu}{n},$$

$$\text{MSE}_{\mu} = \sum_{i=1}^{n} \frac{(\widehat{\mu}_j - \mu)^2}{n}.$$

Result against parameter λ, γ, δ and μ are reported in Table 1, Table 2, Table 3 and Table 4 respectively. MRE for each parameters emerge as very low, around 10^{-2} level, indicating better precision of the estimates.

For further comparison between two distributions fitted to the data, we also report some model selection criteria, such as Komogorov-Smirnoff (KS) statistic, Akaike information criterion (AIC), Bayesian information Criterion (BIC), Corrected AIC (CAIC) and Hannan and Quinn information criterion (HQIC). The definitions used for these selection tools are as: AIC$= -2\ln L(\theta) + 2k$, CAIC$= -2\ln L(\theta) + 2k\frac{n}{n-k-1}$; BIC$= -2\ln L(\theta) + k\ \ln(n)$; and HQIC$= -2\ln L(\theta) + 2k\ \ln\{\ln(n)\}$, where $\ln L(\theta)$ denotes log likelihood, n being the number of observations and k being the number of parameters of the distribution. These are reported in Table 6. Considering all the model selection criteria, reported in Table 6, we found that EGSL fits the data well compared to Lindley and Shifted Lindley distribution.

TABLE 1
ML estimate of λ along with SE, Bias, MSE & MRE.

Replication	λ	γ	δ	μ	**ML Estimator** (λ)				
					$\widehat{\lambda}$	SE	Bias	MSE	MRE
	0.3	0.8	0.4	0.5	0.301	0.0004	1.327×10^{-5}	8.80×10^{-9}	0.020
	0.3	0.8	0.4	1.5	0.300	0.0005	3.753×10^{-5}	7.042×10^{-10}	0.020
	0.3	0.8	1.6	0.5	0.300	0.0003	1.90×10^{-5}	1.817×10^{-8}	0.021
	0.3	0.8	1.6	1.5	0.299	0.0004	-1.1290×10^{-5}	6.382×10^{-9}	0.019
	0.3	1.9	0.4	0.5	0.301	0.0005	1.95×10^{-5}	1.901×10^{-5}	0.020
	0.3	1.9	0.4	1.5	0.300	0.0006	1.090×10^{-5}	5.941×10^{-9}	0.020
50	0.3	1.9	1.6	0.5	0.301	0.0004	9.618×10^{-6}	9.249×10^{-9}	0.010
	0.3	1.9	1.6	1.5	0.301	0.0003	2.078×10^{-5}	2.159×10^{-8}	0.020
	1.1	0.8	0.4	0.5	1.098	0.0015	-2.683×10^{-5}	3.599×10^{-8}	0.019
	1.1	0.8	0.4	1.5	1.101	0.0022	2.293×10^{-5}	2.628×10^{-8}	0.020
	1.1	0.8	1.6	0.5	1.099	0.0010	-1.649×10^{-6}	1.360×10^{-8}	0.019
	1.1	0.8	1.6	1.5	1.098	0.0012	-3.762×10^{-5}	7.078×10^{-8}	0.019
	1.1	1.9	0.4	0.5	1.103	0.0002	5.26×10^{-5}	1.38×10^{-8}	0.020
	1.1	1.9	0.4	1.5	1.103	0.0017	5.828×10^{-5}	1.698×10^{-7}	0.020
	1.1	1.9	1.6	0.5	1.101	0.0011	1.463×10^{-5}	1.070×10^{-8}	0.020
	1.1	1.9	1.6	1.5	1.102	0.0012	3.578×10^{-5}	6.189×10^{-8}	0.020
	0.3	0.8	0.4	0.5	0.300	0.0003	3.405×10^{-6}	1.159×10^{-9}	0.010
	0.3	0.8	0.4	1.5	0.299	0.0003	-5.788×10^{-6}	3.351×10^{-9}	0.0099
	0.3	0.8	1.6	0.5	0.299	0.0002	-5.502×10^{-7}	$3{,}02\times10^{-11}$	0.0099
	0.3	0.8	1.6	1.5	0.300	0.0002	3.656×10^{-6}	1.336×10^{-9}	0.010
	0.3	1.9	0.4	0.5	0.300	0.0003	4.393×10^{-6}	1.930×10^{-9}	0.010
	0.3	1.9	0.4	1.5	0.301	0.0001	1.817×10^{-6}	1.652×10^{-9}	0.002
	0.3	1.9	1.6	0.5	0.300	0.0002	5.474×10^{-6}	2.996×10^{-9}	0.010
	0.3	1.9	1.6	1.5	0.301	0.0002	8.343×10^{-6}	6.96×10^{-9}	0.010
100	1.1	0.8	0.4	0.5	1.101	0.0011	6.090×10^{-6}	3.07×10^{-9}	0.010
	1.1	0.8	0.4	1.5	1.101	0.0013	1.499×10^{-5}	2.248×10^{-8}	0.010
	1.1	0.8	1.6	0.5	1.100	0.0001	3.337×10^{-6}	1.113×10^{-9}	0.010
	1.1	0.8	1.6	1.5	1.101	0.0008	1.487×10^{-5}	2.212×10^{-8}	0.010
	1.1	1.9	0.4	0.5	1.102	0.0014	2.37×10^{-5}	5.62×10^{-8}	0.010
	1.1	1.9	0.4	1.5	1.104	0.0012	4.58×10^{-5}	2.099×10^{-7}	0.010
	1.1	1.9	1.6	0.5	1.104	0.0007	3.55×10^{-5}	1.501×10^{-7}	0.010
	1.1	1.9	1.6	1.5	1.103	0.0007	3.755×10^{-5}	1.411×10^{-7}	0.010
	0.3	0.8	0.4	0.5	0.300	0.0001	8.669×10^{-7}	3.758×10^{-10}	0.002
	0.3	0.8	0.4	1.5	0.299	0.0002	-1.711×10^{-7}	1.464×10^{-11}	0.0019
	0.3	0.8	1.6	0.5	0.300	0.0004	3.514×10^{-7}	6.173×10^{-11}	0.020
	0.3	0.8	1.6	1.5	0.300	0.0001	2.554×10^{-7}	3.160×10^{-9}	0.020
	0.3	1.9	0.4	0.5	0.301	0.0001	2.077×10^{-6}	2.157×10^{-9}	0.020
	0.3	1.9	0.4	1.5	0.301	0.0001	1.817×10^{-6}	1.652×10^{-9}	0.020
	0.3	1.9	1.6	0.5	0.301	0.0000	1.585×10^{-6}	1.257×10^{-9}	0.020
	0.3	1.9	1.6	1.5	0.301	0.0001	1.417×10^{-6}	1.004×10^{-9}	0.020
500	1.1	0.8	0.4	0.5	1.10	0.0006	4.395×10^{-7}	9.65×10^{-11}	0.020
	1.1	0.8	0.4	1.5	1.099	0.0006	-1.753×10^{-6}	1.533×10^{-9}	0.0019
	1.1	0.8	1.6	0.5	1.099	0.0003	-8.528×10^{-8}	3.638×10^{-12}	0.0019
	1.1	0.8	1.6	1.5	1.100	0.0003	3.109×10^{-7}	4.834×10^{-11}	0.0020
	1.1	1.9	0.4	0.5	1.104	0.0005	7.796×10^{-6}	3.039×10^{-8}	0.0020
	1.1	1.9	0.4	1.5	1.105	0.0006	9.247×10^{-6}	4.275×10^{-8}	0.0020
	1.1	1.9	1.6	0.5	1.103	0.0003	-5.906×10^{-6}	1.744×10^{-8}	0.0020
	1.1	1.9	1.6	1.5	1.103	0.0004	5.06×10^{-6}	1.628×10^{-8}	0.0020

TABLE 2
ML estimate of γ along with SE, Bias, MSE & MRE.

Replication	λ	γ	δ	μ	**ML Estimator** (γ)				
					$\widehat{\gamma}$	SE	Bias	MSE	MRE
	0.3	0.8	0.4	0.5	0.799	0.0009	-1.798×10^{-5}	1.617×10^{-8}	0.0199
	0.3	0.8	0.4	1.5	0.801	0.0009	2.366×10^{-5}	2.799×10^{-8}	0.0200
	0.3	0.8	1.6	0.5	0.797	0.0013	-5.2347×10^{-5}	1.3701×10^{-7}	0.0199
	0.3	0.8	1.6	1.5	0.802	0.00154	3.3471×10^{-5}	5.6017×10^{-8}	0.0200
	0.3	1.9	0.4	0.5	1.883	0.0021	-0.00031	5.2459×10^{-6}	0.0198
	0.3	1.9	0.4	1.5	1.879	0.0024	-0.0004	8.260×10^{-6}	0.0197
50	0.3	1.9	1.6	0.5	1.862	0.00383	-0.0007	2.8452×10^{-5}	0.0196
	0.3	1.9	1.6	1.5	1.863	0.0041	-0.00073	2.6571×10^{-5}	0.0196
	1.1	0.8	0.4	0.5	0.799	0.0008	-1.4098×10^{-5}	9.9387×10^{-9}	0.0199
	1.1	0.8	0.4	1.5	0.8002	.0008	4.1552×10^{-6}	8.6330×10^{-10}	0.0200
	1.1	0.8	1.6	0.5	0.799	0.0013	-1.7086×10^{-5}	1.459×10^{-8}	0.0199
	1.1	0.8	1.6	1.5	0.801	0.0012	1.5909×10^{-5}	1.2655×10^{-8}	0.0200
	1.1	1.9	0.4	0.5	1.8813	0.0020	-0.0004	6.9511×10^{-6}	0.0198
	1.1	1.9	0.4	1.5	1.8701	0.0021	-.00035	6.1353×10^{-6}	0.0198
	1.1	1.9	1.6	0.5	1.8701	0.0043	-.00059	1.7889×10^{-5}	0.0198
	1.1	1.9	1.6	1.5	1.8661	0.0038	-.0007	2.2989×10^{-5}	0.0196
	0.3	0.8	0.4	0.5	0.8000	0.0006	1.768×10^{-7}	3.127×10^{-12}	0.0100
	0.3	0.8	0.4	1.5	0.8118	0.0005	6.153×10^{-6}	3.786×10^{-9}	0.01000
	0.3	0.8	1.6	0.5	0.7980	0.0009	-1.1461×10^{-5}	1.313×10^{-8}	0.00990
	0.3	0.8	1.6	1.5	0.7991	0.0010	-8.300×10^{-6}	7.6213×10^{-9}	0.00998
	0.3	1.9	0.4	0.5	1.8780	0.0015	-.00021	4.5521×10^{-6}	0.00988
	0.3	1.9	0.4	1.5	1.8770	0.0015	-.00022	5.1049×10^{-6}	0.00989
	0.3	1.9	1.6	0.5	1.8630	0.0026	-0.0004	1.3798×10^{5}	0.00981
	0.3	1.9	1.6	1.5	1.8610	.0030	-0.0004	1.512×10^{-5}	0.00979
100	1.1	0.8	0.4	0.5	0.8003	0.00059	1.7919×10^{-6}	3.211×10^{-10}	0.0101
	1.1	0.8	0.4	1.5	0.8002	0.0006	2.3775×10^{-6}	5.653×10^{-10}	0.0100
	1.1	0.8	1.6	0.5	0.7993	0.0009	-6.0124×10^{-6}	3.6149×10^{-9}	0.0099
	1.1	0.8	1.6	1.5	0.8003	0.0008	3.5385×10^{-6}	1.2521×10^{-9}	0.0100
	1.1	1.9	0.4	0.5	1.8804	0.00172	-0.00019	3.8264×10^{-6}	0.0098
	1.1	1.9	0.4	1.5	1.8774	0.00169	-0.00022	5.0728×10^{-6}	0.00988
	1.1	1.9	1.6	0.5	1.8586	0.00276	-0.00041	1.7165×10^{-5}	0.00978
	1.1	1.9	1.6	1.5	1.8635	0.00288	-0.00036	1.3329×10^{-5}	0.00981
	0.3	0.8	0.4	0.5	0.799	0.0002	-9.453×10^{-7}	4.467×10^{-10}	.00199
	0.3	0.8	0.4	1.5	0.799	0.00025	-3.0509×10^{-7}	4.654×10^{-11}	0.00199
	0.3	0.8	1.6	0.5	0.799	0.0003	-5.7409×10^{-7}	1.6479×10^{-10}	0.00199
	0.3	0.8	1.6	1.5	0.7998	0.0004	-1.1956×10^{-7}	7.147×10^{-12}	0.00199
	0.3	1.9	0.4	0.5	1.8790	0.0007	-4.227×10^{-5}	8.9347×10^{-7}	0.00197
	0.3	1.9	0.4	1.5	1.8789	0.0007	-4.0237×10^{-5}	8.0953×10^{-7}	0.00198
	0.3	1.9	1.6	0.5	1.8635	0.00012	-7.3043×10^{-5}	2.6676×10^{-6}	0.00196
	0.3	1.9	1.6	1.5	1.8649	0.00015	-7.013×10^{-5}	2.4590×10^{-6}	0.00196
500	1.1	0.8	0.4	0.5	0.8003	0.00027	5.8536×10^{-7}	1.7133×10^{-10}	0.002
	1.1	0.8	0.4	1.5	0.8000	0.0004	1.2767×10^{-8}	8.1504×10^{-14}	0.002
	1.1	0.8	1.6	0.5	0.7998	0.0004	-3.272×10^{-7}	5.3537×10^{-11}	0.0019
	1.1	0.8	1.6	1.5	0.8000	0.0004	1.2767×10^{-8}	8.1504×10^{-14}	0.0020
	1.1	1.9	0.4	0.5	1.8790	0.00078	-4.1896×10^{-5}	8.7765×10^{-7}	0.0019
	1.1	1.9	0.4	1.5	1.8646	0.00117	-7.0716×10^{-5}	2.5004×10^{-6}	0.0020
	1.1	1.9	1.6	0.5	1.8785	0.0008	-4.2826×10^{-5}	9.104×10^{-7}	0.0020
	1.1	1.9	1.6	1.5	1.8653	.0012	-6.942×10^{5}	2.4093×10^{-6}	0.0019

TABLE 3
ML estimate of δ along with SE, Bias, MSE & MRE.

Replication	λ	γ	δ	μ	**ML Estimator** (δ)				
					$\hat{\delta}$	SE	Bias	MSE	MRE
	0.3	0.8	0.4	0.5	0.402	0.0016	3.435×10^{-5}	5.899×10^{-8}	0.0201
	0.3	0.8	0.4	1.5	0.399	0.0018	-1.883×10^{-5}	1.773×10^{-8}	0.01995
	0.3	0.8	1.6	0.5	1.619	0.006	0.004	7.21×10^{-6}	0.0202
	0.3	0.8	1.6	1.5	1.5905	0.0074	-0.0002	1.787×10^{-6}	0.0199
	0.3	1.9	0.4	0.5	0.4087	0.0019	0.0002	1.503×10^{-6}	0.0204
	0.3	1.9	0.4	1.5	0.4072	.0026	0.00014	1.036×10^{-6}	0.0204
50	0.3	1.9	1.6	0.5	1.6626	0.0079	0.0013	7.833×10^{-5}	0.0208
	0.3	1.9	1.6	1.5	1.6483	0.0079	0.001	4.664×10^{-5}	0.0206
	1.1	0.8	0.4	0.5	0.3998	0.0014	-3.988×10^{-6}	7.953×10^{-10}	0.02
	1.1	0.8	0.4	1.5	0.4088	0.0017	1.652×10^{-5}	1.365×10^{-8}	0.020
	1.1	0.8	1.6	0.5	1.6029	0.0058	5.837×10^{-5}	1.704×10^{-7}	0.02
	1.1	0.8	1.6	1.5	1.5928	0.0062	-0.00014	1.0305×10^{-6}	0.0199
	1.1	1.9	0.4	0.5	0.4078	0.002	0.0002	1.219×10^{-6}	0.0204
	1.1	1.9	0.4	1.5	0.4078	0.0018	0.0002	1.203×10^{-6}	0.0204
	1.1	1.9	1.6	0.5	1.6281	0.0074	0.0006	1.577×10^{-5}	0.0204
	1.1	1.9	1.6	1.5	1.6364	0.0084	0.0007	2.648×10^{-5}	0.0204
	0.3	0.8	0.4	0.5	0.401	0.0013	1.099×10^{-5}	1.159×10^{-9}	0.01003
	0.3	0.8	0.4	1.5	0.3988	0.0012	-1.238×10^{-5}	1.533×10^{-8}	0.00997
	0.3	0.8	1.6	0.5	1.605	0.0043	4.548×10^{-8}	2.0688×10^{-7}	0.01002
	0.3	0.8	1.6	1.5	1.6089	0.0047	8.8507×10^{-7}	7.8334×10^{-7}	0.01001
	0.3	1.9	0.4	0.5	0.4083	0.0015	8.269×10^{-5}	6.838×10^{-7}	0.0102
	0.3	1.9	0.4	1.5	0.4106	0.0016	0.00011	1.12×10^{-6}	0.0103
	0.3	1.9	1.6	0.5	1.6446	0.006	0.0004	1.992×10^{-5}	0.0103
	0.3	1.9	1.6	1.5	1.6476	0.0056	0.0005	2.262×10^{-5}	0.0103
100	1.1	0.8	0.4	0.5	0.4001	0.0009	1.245×10^{-6}	1.549×10^{-10}	0.01
	1.1	0.8	0.4	1.5	0.4011	0.0011	1.143×10^{-5}	1.306×10^{-8}	0.010
	1.1	0.8	1.6	0.5	1.6013	0.0042	1.279×10^{-5}	1.636×10^{-8}	0.010
	1.1	0.8	1.6	1.5	1.6054	0.0041	5.381×10^{-5}	2.896×10^{-7}	0.010
	1.1	1.9	0.4	0.5	0.4082	0.0016	8.157×10^{-5}	6.654×10^{-7}	0.010
	1.1	1.9	0.4	1.5	0.4114	0.0015	0.0001	1.303×10^{-6}	0.0103
	1.1	1.9	1.6	0.5	1.6552	0.0052	0.0006	3.052×10^{-5}	0.0103
	1.1	1.9	1.6	1.5	1.6492	0.005	0.0005	2.422×10^{-5}	0.0103
	0.3	0.8	0.4	0.5	0.4016	0.0005	3.263×10^{-6}	5.324×10^{-9}	0.002
	0.3	0.8	0.4	1.5	0.4001	0.0005	2.137×10^{-7}	2.282×10^{-11}	0.002
	0.3	0.8	1.6	0.5	1.6032	0.00195	6.326×10^{-6}	2.0009×10^{-8}	0.002
	0.3	0.8	1.6	1.5	1.6023	0.0019	4.5604×10^{-6}	1.0399×10^{-8}	0.002
	0.3	1.9	0.4	0.5	0.4102	0.0007	2.043×10^{-5}	2.0862×10^{-7}	0.0021
	0.3	1.9	0.4	1.5	0.409	0.0007	1.79×10^{-5}	1.602×10^{-7}	0.002
	0.3	1.9	1.6	0.5	1.6458	0.0024	9.153×10^{-5}	4.189×10^{-6}	0.002
	0.3	1.9	1.6	1.5	1.6442	0.0025	8.836×10^{-5}	3.903×10^{-6}	0.0021
500	1.1	0.8	0.4	0.5	0.3999	0.0005	-2.453×10^{-7}	3.01×10^{-11}	0.002
	1.1	0.8	0.4	1.5	0.3992	0.0005	-1.632×10^{-6}	1.332×10^{-9}	0.002
	1.1	0.8	1.6	0.5	1.6006	0.0019	1.222×10^{-6}	7.465×10^{-10}	0.002
	1.1	0.8	1.6	1.5	1.6022	0.0018	4.319×10^{-6}	9.327×10^{-9}	0.0020
	1.1	1.9	0.4	0.5	0.4097	0.0007	1.949×10^{-5}	1.9×10^{-7}	0.0020
	1.1	1.9	0.4	1.5	0.4104	0.0007	2.079×10^{-5}	2.161×10^{-7}	0.0020
	1.1	1.9	1.6	0.5	1.6447	0.0024	8.935×10^{-5}	3.992×10^{-6}	0.0020
	1.1	1.9	1.6	1.5	1.6434	0.0024	8.687×10^{-5}	3.773×10^{-6}	0.0020

TABLE 4
ML estimate of μ along with SE, Bias, MSE & MRE.

Replication	λ	γ	δ	μ	**ML Estimator** (μ)				
					$\widehat{\mu}$	SE	Bias	MSE	MRE
	0.3	0.8	0.4	0.5	0.5000	5.961×10^{-7}	5.155×10^{-8}	1.329×10^{-13}	0.0200
	0.3	0.8	0.4	1.5	1.500	5.989×10^{-7}	5.155×10^{-8}	1.329×10^{-13}	0.0200
	0.3	0.8	1.6	0.5	0.5000	6.807×10^{-6}	6.5267×10^{-7}	2.1299×10^{-11}	0.0201
	0.3	0.8	1.6	1.5	01.5001	3.716×10^{-6}	4.1912×10^{-7}	8.7832×10^{-12}	0.0200
	0.3	1.9	0.4	0.5	0.5185	.000135	.00037	6.8555×10^{-6}	0.0207
	0.3	1.9	0.4	1.5	1.5157	.00112	.00031	4.9894×10^{-6}	0.0202
50	0.3	1.9	1.6	0.5	0.5377	0.00275	0.00075	2.8569×10^{-5}	0.0215
	0.3	1.9	1.6	1.5	1.5297	0.00249	0.00059	1.768×10^{-5}	0.0204
	1.1	0.8	0.4	0.5	0.4999	8.7658×10^{-8}	-6.6055×10^{-9}	2.1816×10^{-15}	0.0199
	1.1	0.8	0.4	1.5	1.50	8.586×10^{-8}	-7.7311×10^{-9}	2.9885×10^{-15}	0.0199
	1.1	0.8	1.6	0.5	0.5001	5.7424×10^{-7}	5.5939×10^{-8}	1.5645×10^{-13}	0.0200
	1.1	0.8	1.6	1.5	1.5000	3.9185×10^{-7}	4.2001×10^{-8}	8.8202×10^{-14}	0.0200
	1.1	1.9	0.4	0.5	0.5022	.00019	4.5374×10^{-5}	1.0294×10^{-7}	0.0203
	1.1	1.9	0.4	1.5	1.5021	0.00016	4.3612×10^{-5}	9.5101×10^{-8}	0.0200
	1.1	1.9	1.6	0.5	0.5055	.00015	.00011	6.10299×10^{-7}	0.0200
	1.1	1.9	1.6	1.5	1.5056	.00034	0.00010	5.0406×10^{-7}	0.0201
	0.3	0.8	0.4	0.5	0.5000	5.795×10^{-7}	3.939×10^{-8}	1.5521×10^{-13}	0.0100
	0.3	0.8	0.4	1.5	1.500	4.6272×10^{-7}	3.122×10^{-8}	9.7488×10^{-14}	0.0100
	0.3	0.8	1.6	0.5	0.5000	3.7224×10^{-6}	2.8596×10^{-7}	8.1776×10^{-12}	0.0101
	0.3	0.8	1.6	1.5	1.5001	2.3845×10^{-6}	1.8831×10^{-7}	3.546×10^{-12}	0.0100
	0.3	1.9	0.4	0.5	0.5201	.00096	.000202	4.0728×10^{-12}	0.0104
	0.3	1.9	0.4	1.5	1.5150	.00073	.00015	2.2503×10^{6}	0.0101
	0.3	1.9	1.6	0.5	0.5396	.00200	.00039	1.5682×10^{-5}	0.0108
	0.3	1.9	1.6	1.5	1.5300	.00157	.000030	9.0177×10^{-6}	0.0102
100	1.1	0.8	0.4	0.5	0.4999	8.4348×10^{-8}	-2.9687×10^{-9}	8.8133×10^{-16}	0.0099
	1.1	0.8	0.4	1.5	1.5303	.00158	.00030	9.0177×10^{-6}	0.0102
	1.1	0.8	1.6	0.5	0.5000	$6.5208\times10^{-}$	4.2867×10^{-8}	1.8376×10^{-13}	0.0100
	1.1	0.8	1.6	1.5	1.500	3.1981×10^{-7}	1.7781×10^{-8}	3.1617×10^{-14}	0.0100
	1.1	1.9	0.4	0.5	0.5022	.000197	4.537×10^{-5}	1.0294×10^{-8}	0.0201
	1.1	1.9	0.4	1.5	1.5024	.00012	2.4196×10^{-5}	5.8545×10^{-8}	0.0100
	1.1	1.9	1.6	0.5	0.5061	.0003	6.1624×10^{-5}	3.975×10^{-7}	0.0100
	1.1	1.9	1.6	1.5	1.5048	.00028	4.7879×10^{-5}	2.2923×10^{-7}	0.0101
	0.3	0.8	0.4	0.5	.5000	2.4890×10^{-7}	6.8498×10^{-9}	2.3460×10^{-14}	0.0020
	0.3	0.8	0.4	1.5	1.5020	1.9122×10^{-7}	5.1621×10^{-9}	1.3323×10^{-14}	0.0020
	0.3	0.8	1.6	0.5	0.5000	1.7485×10^{-6}	5.4772×10^{-8}	1.5003×10^{-12}	0.0020
	0.3	0.8	1.6	1.5	1.5002	1.2248×10^{-6}	4.0669×10^{-8}	8.2702×10^{-13}	0.0020
	0.3	1.9	0.4	0.5	0.5193	.00049	3.8705×10^{-5}	7.4905×10^{-7}	0.0021
	0.3	1.9	0.4	1.5	1.5144	0.00033	2.831×10^{-5}	4.1273×10^{-7}	0.0020
	0.3	1.9	1.6	0.5	0.5406	0.00098	8.1305×10^{-5}	3.3052×10^{-6}	0.0022
	0.3	1.9	1.6	1.5	1.5291	0.00072	5.8297×10^{-5}	1.6992×10^{-6}	0.0020
500	1.1	0.8	0.4	0.5	0.4999	3.9413×10^{-8}	-5.5195×10^{-10}	1.5232×10^{-16}	0.0019
	1.1	0.8	0.4	1.5	1.5000	3.1691×10^{-8}	-8.2119×10^{-10}	3.3718×10^{-16}	0.0019
	1.1	0.8	1.6	0.5	0.5004	2.1463×10^{-7}	5.5986×10^{-9}	1.5670×10^{-14}	0.0020
	1.1	0.8	1.6	1.5	1.5001	1.9168×10^{-7}	5.1069×10^{-9}	1.3040×10^{-14}	0.0020
	1.1	1.9	0.4	0.5	0.5027	6.4291×10^{-5}	5.5339×10^{-6}	1.5312×10^{-8}	0.0021
	1.1	1.9	0.4	1.5	1.5022	5.8285×10^{-5}	4.5120×10^{-6}	1.0179×10^{-8}	0.0020
	1.1	1.9	1.6	0.5	0.5054	0.00014	1.0783×10^{-5}	5.8133×10^{-8}	0.0020
	1.1	1.9	1.6	1.5	1.5047	0.00011	9.3496×10^{-6}	4.3708×10^{-8}	0.0020

5. Data analysis

The proposed distribution is fitted for a data set available in Duffy *et al.* (1993). The data consists of measurements on strength of the sintered silicon nitride after four-point bend system is applied. On four point bend specimen, the support span of test fixture was 40.373 mm and the inner load span of 19.622 mm. All specimens are subjected to pure four-point bending. Number of complete specimens in the data set is found to be 27. We apply Lindley, Shifted Lindley and Extended Generalized Shifted Lindley in order to fit this data. Subject to the fitting of extended generalized shifted Lindley distribution on the data we figure out estimates of the parameters λ, γ, δ and μ through maximum likelihood method. Estimates alongwith standard errors (SE) are given in Table 5.

TABLE 5
Parameter estimates for the four point bend data.

Distribution	$\widehat{\lambda}$ (SE)	$\widehat{\gamma}$ (SE)	$\widehat{\delta}$ (SE)	$\widehat{\mu}$ (SE)
Lindley	0.0027	1	1	0
	(0.0001)	(-)	(-)	(-)
Shifted Lindley	0.0096	1	1	613.9
	(0.0001)	(-)	(-)	(0.0006)
EGSL	0.0135	0.4188	5.1418	613.9
	(3.694×10^{-6})	(3.456×10^{-4})	(6.434×10^{-3})	(7.115×10^{-9})

TABLE 6
Model selection criteria for the four point bend data.

Distribution	KS	AIC	BIC	$CAIC$	$HQIC$
Lindley	0.4619	391.7628	393.0586	391.9109	392.1481
Shifted Lindley	0.1216	315.8982	318.49	316.36	316.669
EGSL	0.107	304.4224	309.6057	306.0891	305.9637

6. Conclusion

In this study we have proposed a new family of distributions based on Marshall-Olkin extension of shifted Lindley distribution. Some mathematical properties along with estimation issues are addressed. The hazard rate function of shifted Lindley distribution shows that the subject distribution can be used to model reliability data as well. We derived the moment and maximum likelihood estimates of the parameters along with the biases, mean square error and mean relative errors. A real data application of the EGSL distribution projects that it could provide a meaningful fit than a set of usual statistical distributions, while being considered specially in the discourse of life time data analysis.

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