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COMMENTS ON IRSHAD ET AL. (2021) "THE ZOGRAFOS-BALAKRISHNAN LINDLEY DISTRIBUTION: PROPERTIES AND APPLICATIONS"

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1. Introduction

This paper corrects and updates Irshad *et al.* (2021) by some technical comments. The original paper was inadvertently published with some errors, which will be addressed and corrected in the following.

The data analyzed in Irshad *et al.* (2021) are 100 waiting times (in minutes) before service of 100 bank customers reported by Ghitany *et al.* (2008). The waiting times data given in Table 5 of Irshad *et al.* (2021) is incorrect, nevertheless, the right data, given here in Table 1, were used for fitting different models in Irshad *et al.* (2021). Table 2 shows the descriptive summaries of this data.

TABLE 1
Data of waiting times of 100 bank customers.

0.8	2.9	4.3	5.0	6.7	8.2	9.7	11.9	14.1	19.9
0.8	3.1	4.3	5.3	6.9	8.6	9.8	12.4	15.4	20.6
1.3	3.2	4.4	5.5	7.1	8.6	10.7	12.5	15.4	21.3
1.5	3.3	4.4	5.7	7.1	8.6	10.9	12.9	17.3	21.4
1.8	3.5	4.6	5.7	7.1	8.8	11.0	13.0	17.3	21.9
1.9	3.6	4.7	6.1	7.1	8.8	11.0	13.1	18.1	23.0
1.9	4.0	4.7	6.2	7.4	8.9	11.1	13.3	18.2	27.0
2.1	4.1	4.8	6.2	7.6	8.9	11.2	13.6	18.4	31.6
2.6	4.2	4.9	6.2	7.7	9.5	11.2	13.7	18.9	33.1
2.7	4.2	4.9	6.3	8.0	9.6	11.5	13.9	19.0	38.5

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In addition, the correct estimated value for the parameter α of the Extended Inverse Lindley (EIL) distribution in Table 7 of Irshad *et al.* (2021) is approximately 0.00, rather than 0.011. This yields the computed Kolmogorov–Smirnov (K-S) statistic value of this model changes to 0.621 (p-value $= 0$). The same correction for this distribution is needed in Table 8, where $\hat{\alpha} = 0.00$ and K-S $= 0.96$ (p-value $= 0$).

In Table 8 of Irshad *et al.* (2021) the correct estimated values for the parameters of the Zografos-Balakrishnan Lindley (ZBL) distribution are $\hat{a} = 27.66$ and $\hat{\theta} = 12.83$. For this model we have $-\log \mathrm{L} = 35.06$, AIC $= 74.12$, BIC $= 78.47$, AICc $= 74.52$, K-S $= 0.07$ (p-value $= 0.86$). Table 8 of the aforementioned paper contains also a typographical error when reporting the estimated value of the parameter α of the Marshall-Olkin Extended Lindley (MOEL) distribution, where $\hat{\alpha}$ should be changed to 7858.76.

Tables 2 and 3 provide for the fitted models the corresponding standard errors of the estimated parameters. Reporting the standard errors is necessary for measuring the accuracy of the estimates of the parameters and testing their significance, see Pawitan (2013). Especially, as shown in these tables for the EIL distribution fitted to both data sets the α parameter is not significantly different from zero. The same result is inferred for the β parameter of the GL distribution. This means the mentioned parameters do not contributed significantly to the models. It is worth mentioning that the corresponding p-values which lead us to these results are valid under the asymptotic normality of estimators, and the sample sizes are large enough to guarantee this asymptotic approximation to hold.

As mentioned in Mazucheli *et al.* (2016), the EIL distribution in the case $\alpha = 0$ and the GL distribution in the case $\beta = 0$ are reduced to the Inverse Weibull (IW) and the gamma distributions, respectively. The result of fitting these reduced models to data is shown in the bottom rows of Tables 2 and 3. Comparing the goodness-of-fits of the GL and gamma models, the parsimony principle leads us to select the reduced gamma model in favor of the full GL model. This choice is further confirmed by comparing the AIC values of both models since the AIC of the gamma distribution is lower than that of the GL for both data sets. Furthermore, for both data sets, comparing the IW and EIL models using the p-values of the corresponding K-S tests show that the IW model is a significant model for describing data, while the EIL model is not.

After all, it is worth mentioning that the maximum likelihood estimates were implemented using the `R` software, R Core Team (2021), through the package `EstimationTools`, command `maxlogL` with `nlminb` as optimization procedures, see Mosquera and Hernandez (2021).
The final comment, in Figure 4(a), Irshad *et al.* (2021), has not embedded the density plot of the fitted Lindley distribution. Figure 1, depicts the density plot which compares fitted densities of all models with the empirical histogram of the data.

TABLE 2
The estimated parameters, the corresponding standard errors, and the goodness-of-fit statistic for the models fitted to the waiting times of 100 bank customers data.

Model	Estim. (SE)	p-values	AIC	K-S (p-value)
ZBL	$\hat{a} = 1.26$ (0.19)	0	639.75	0.05 (0.95)
	$\hat{\theta} = 0.22$ (0.03)	0		
GL	$\hat{\theta} = 0.20$ (0.06)	0	640.60	0.04 (0.99)
	$\hat{\alpha} = 2.01$ (0.26)	0		
	$\hat{\beta} = 0.00$ (0.11)	1		
EIL	$\hat{\theta} = 6.53$ (1.14)	0	674.76	0.62 (0)
	$\hat{\alpha} = 0.00$ (0.72)	1		
	$\hat{\beta} = 1.16$ (0.08)	0		
GIL	$\hat{\theta} = 7.23$ (0.90)	0	673.56	0.12 (0.11)
	$\hat{\alpha} = 1.15$ (0.08)	0		
PL	$\hat{\theta} = 0.15$ (0.03)	0	640.64	0.05 (0.94)
	$\hat{\alpha} = 1.08$ (0.07)	0		
MOEL	$\hat{\theta} = 0.20$ (0.04)	0	641.83	0.06 (0.89)
	$\hat{\alpha} = 1.24$ (0.53)	0.02		
L	$\hat{\theta} = 0.19$ (0.01)	0	640.07	0.07 (0.72)
Gamma	$\hat{\theta} = 0.20$ (0.03)	0	638.60	0.04 (0.10)
	$\hat{\alpha} = 2.01$ (0.26)	0		
IW	$\hat{\theta} = 6.53$ (0.89)	0	672.76	0.12 (0.12)
	$\hat{\beta} = 1.16$ (0.08)	0		

TABLE 3
The estimated parameters, the corresponding standard errors, and the goodness-of-fit statistic for the models fitted to the the failure stresses of single carbon fibers data.

Model	Estim.(SE)	p-values	AIC	K-S (p-value)
ZBL	$\hat{a} = 27.66$ (4.97)	0	74.12	0.07 (0.86)
	$\hat{\theta} = 12.83$ (2.24)	0		
GL	$\hat{\theta} = 12.83$ (2.30)	0	76.13	0.07 (0.86)
	$\hat{\alpha} = 28.13$ (7.10)	0		
	$\hat{\beta} = 24.20$ (414.78)	0.95		
EIL	$\hat{\theta} = 31.72$ (10.57)	0	93.72	0.96 (0)
	$\hat{\alpha} = 0.00$ (5.34)	1		
	$\hat{\beta} = 4.99$ (0.44)	0		
GIL	$\hat{\theta} = 32.64$ (9.15)	0	91.75	0.13 (0.24)
	$\hat{\alpha} = 4.99$ (0.44)	0		
PL	$\hat{\theta} = 0.05$ (0.02)	0	73.95	0.07 (0.93)
	$\hat{\alpha} = 4.22$ (0.35)	0		
MOEL	$\hat{\theta} = 4.46$ (0.24)	0	75.98	0.08 (0.81)
	$\hat{\alpha} = 7858.76$ (3957.89)	0.05		
L	$\hat{\theta} = 0.71$ (0.06)	0	216.00	0.42 (0)
Gamma	$\hat{\theta} = 12.72$ (2.24)	0	74.14	0.072 (0.86)
	$\hat{\alpha} = 28.55$ (4.98)	0		
IW	$\hat{\theta} = 31.71$ (9.13)	0	91.72	0.13 (0.24)
	$\hat{\beta} = 4.99$ (0.44)	0		

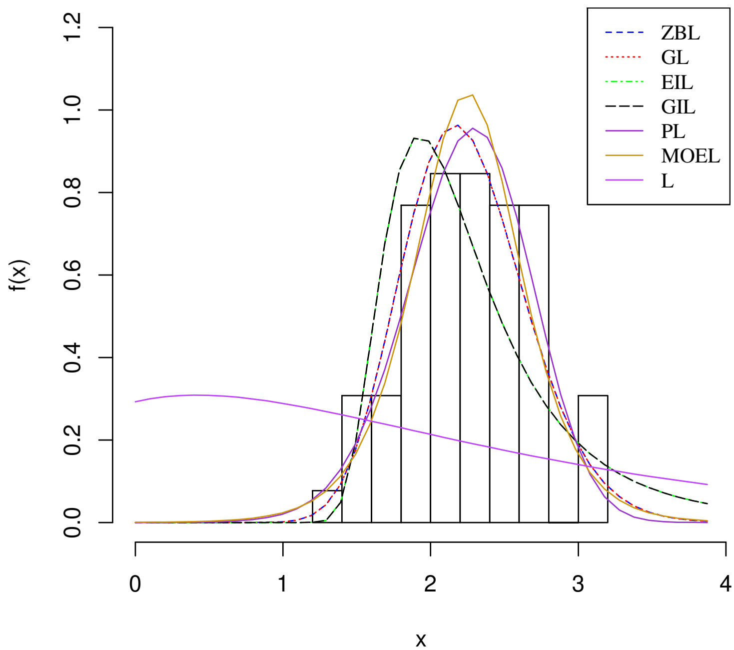

Figure 1 – The fitted densities of data of the failure stresses of single carbon fibers.

2. Conclusion

This paper may be considered as an essential follow-up paper of Irshad *et al.* (2021). Although the theoretical results of the earlier paper were found correct, some errors unknowingly crept into the computational part. This paper corrects all such errors. For the sake of completion in the inference part, we computed standard errors along with the estimates of the parameters of the distributions for both data sets considered here.

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SUMMARY

This paper corrects and updates Irshad *et al.* (2021) by some technical comments. The original paper was inadvertently published with some errors, mainly the computational ones regarding the maximum likelihood (ML) estimate of the parameters of the fitted models which will be addressed and corrected in this paper. Furthermore, the standard errors of the ML estimates of different fitted models, as an important indicator of the accuracy of estimates and particularly are necessary for making the statistical inference for the population parameters, have been added.

Keywords: Goodness-of-fit statistic; Lindley distribution; Maximum likelihood estimation.