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ESTIMATION OF THE SCALE PARAMETER OF THE CAUCHY DISTRIBUTION USING ABSOLVED ORDER STATISTICS

Poruthiyudian Yageen Thomas [1]
Department of Statistics, University of Kerala, Thiruvananthapuram, Kerala, India
Vipinachandran Pillai Anjana
Department of Statistics,University of Kerala, Thiruvananthapuram, Kerala, India.
Manoj Chacko
Department of Statistics,University of Kerala, Thiruvananthapuram, Kerala, India.

SUMMARY

A new set of ordered random variables generated from a sample from a scale dependent Cauchy distribution known as Absolved Order Statistics (AOS) of the sample forms the problem of investigation in this paper. The distribution theory of these AOS is developed. The vector of AOS is found to be the minimal sufficient statistic for the Cauchy distribution, which is contrary to the existing perception that the vector of order statistics of the sample is minimal sufficient. The best linear unbiased estimate $\hat{\sigma}$ of σ based on AOS is derived and its variance is also explicitly expressed. Though only $n-4$ intermediate order statistics are usable to determine the BLUE of σ based on order statistics, it is found that $n-2$ AOS are usable to determine $\hat{\sigma}$. This makes $\hat{\sigma}$ a more efficient estimate of σ than all of its competitors especially when the sample size is small. Illustration for the above result is made through a real life example. It is found that censoring based on AOS is more realistic and the estimate obtained from it for σ is more efficient than the case of censoring with order statistics. A new ranked set sampling called Adjusted Ranked Set Sampling which is suitable for the Cauchy distribution and results in observations distributed as AOS is developed in this paper. Its role in producing a better estimate for σ is analyzed.

Keywords: Cauchy distribution; Logistic distribution; Order statistics; Absolved order statistics; Minimal sufficient statistics; Best linear unbiased estimate; Estimation from censored samples; Ranked set sampling; Adjusted ranked set sampling.

1. INTRODUCTION

The Cauchy distribution is considered an important heavy tailed statistical model, which has the potential for applications in several areas of studies. The scale dependent Cauchy

[1] Corresponding Author. E-mail: yageenthomas@gmail.com

distribution is defined by the probability density function (pdf) given by

$$f(x,\sigma)=\frac{1}{\sigma\pi}\frac{1}{(1+(\frac{x}{\sigma})^2)}, -\infty < x < \infty, \tag{1}$$

where $\sigma > 0$ is the scale parameter of the distribution. Onen *et al.* (2001) have called the above distribution a heavy tailed error distribution with scale parameter σ. A discussion on various applications of this error distribution in physics, econometrics and engineering is also made available in Onen *et al.* (2001). The third Hagen's hypothesis on the theory of errors (see, Rao and Gupta, 1989,P.14) specifies that each component of error has an equal chance of being positive or negative. Clearly, as the Cauchy distribution defined by the pdf of Eq. (1) is symmetric about zero, it is also an error model coming under the governance of Hagen's third hypothesis. For details on error models either generated from the Cauchy distribution or appearing similar to the Cauchy distribution and satisfying the third Hagen's hypothesis one may refer to Thomas and Priya (2017, 2015). The heavy tailed Cauchy distribution is a better model to describe financial returns than the Gaussian model which fails to capture large fluctuations observed in real assets (for details see, Nolan, 2014). Stock market return distributions involve a sharp peak around zero but with heavy tails. This behaviour means that stock market researchers (for example see, Mahdizadeh and Zamanzade, 2019) depend largely on the Cauchy distribution as defined in Eq. (1) in their investigations. Some more real life applications of the Cauchy distribution are (i) the recommendation of the Cauchy distribution by Roe (1992) to describe the distribution of the energy width of a state that decays exponentially with time; (ii) the description by Winterton *et al.* (1992) to use the Cauchy distribution for studying physical systems involving contact resistivity; (iii) the narration of Kagan (1992), attributing the Cauchy distribution to explain well the distribution of hypocentres on focal spheres of earth quakes; and (iv) the illustration by Min *et al.* (1996) in using the Cauchy distribution as a good model to study the distribution of velocity differences induced by different vortex elements.

It is textbook knowledge that the Cauchy distribution fails to admit its first two moments. Consequently the problem of the estimation of σ by the method of moments fails, and this makes the problem of the estimation of σ a more challenging task. For a discussion on the maximum likelihood estimator (MLE) of σ see Copas (1975), Haas *et al.* (1970) and Howlader and Weiss (1988). For details on the optimal linear rank estimator of σ see Durbin and Knott (1972) and Kravchuk (2005). For the Hodges-Lehmann estimator of σ see Hodges and Lehmann (1963) and Kravchuk and Pollett (2012). Barnett (1966) has evaluated and tabulated the means, variances and covariances of middle order statistics obtained after excluding two smallest and two largest order statistics of a sample arising from the standard Cauchy distribution for $n = 6(1)16(2)20$. Making use of these values, Thomas (1990) has derived the best linear unbiased estimate (BLUE) of the scale parameter σ of the Cauchy distribution based on the middle order statistics of a sample.

Suppose $U = (X_{1:n}, X_{2:n}, ..., X_{n:n})$ is the order statistics of a random sample of observations $X_1, X_2, ..., X_n$ drawn from a distribution belonging to the family $\mathscr{F}$ of all absolutely continuous distributions. Then from Lehmann and Scheffe (1950) we observe that the statistic U is complete sufficient for $\mathscr{F}$. Some sub-families of distributions contained in $\mathscr{F}$ may have a sufficient statistic of reduced dimension than n. But usually a distribution of a sub-family to $\mathscr{F}$ that does not exhibit a reduced dimension of the sufficient statistic than n, is considered as one with $U = (X_{1:n}, X_{2:n}, ..., X_{n:n})$ as the minimal sufficient statistic. In this sense the usual perception is that for the Cauchy distribution as given by Eq. (1), $U = (X_{1:n}, X_{2:n}, ..., X_{n:n})$ is a minimal sufficient statistic. But contrary to this perception, U is not a minimal sufficient statistic.

Thomas and Anjana (2022) have considered the family $\mathscr{F}_1$ of all absolutely continuous distributions that are symmetrically distributed about zero. They defined a new variety of ordered random variables, which is given below.

DEFINITION 1. *(Thomas and Anjana, 2022). Suppose $X_1, X_2, ..., X_n$ is a random sample of size n drawn from a distribution with pdf $f(x)$ such that $f(x) \in \mathscr{F}_1$. If we take the absolute values of the observations and order them in the increasing order of magnitude as $X_{(1:n)} \leq X_{(2:n)} \leq ... \leq X_{(n:n)}$, then we say that $X_{(1:n)}, X_{(2:n)}, ..., X_{(n:n)}$ are the absolved order statistics (AOS) of the given sample.*

Thomas and Anjana (2022) further proved the following theorem on the minimal sufficiency of the statistics introduced in the above definition.

THEOREM 2. *See, Theorem 2.1 of (Thomas and Anjana, 2022). Suppose $\underset{\sim}{X} = (X_1, X_2, ..., X_n)$ is a vector of observations of a random sample of size n drawn from a distribution with pdf $f(x) \in \mathscr{F}_1$. Let $T = (X_{(1:n)}, X_{(2:n)}, ..., X_{(n:n)})$ be a statistic based on the AOS $X_{(1:n)}, X_{(2:n)}, ..., X_{(n:n)}$ constructed from the observations $X_1, X_2, ..., X_n$. Then T is minimal sufficient for the family $\mathscr{F}_1$.*

The remaining part of the paper is organized as given below. In Section 2, first we describe the distribution theory of AOS arising from the Cauchy distribution, and we also deduce from Theorem 2 the minimal sufficient statistic for the Cauchy family of distributions defined by the pdf in Eq. (1). The method of estimating the scale parameter σ involved in Eq. (1) by the best linear unbiased estimate $\hat{\sigma}$ based on the first $n-2$ AOS of the sample is described in Section 3. The explicit expression for the variance of the estimate $\hat{\sigma}$ as well is given in this section. The coefficients of the AOS involved in $\hat{\sigma}$ for each of $n = 4(1)20$ have been computed, and those values are provided in Table 1. We have computed further Var($\hat{\sigma}$), the variance of the BLUE σ^* based on order statistics, the variance of the maximum likelihood estimate $\tilde{\sigma}$ of σ and the relative efficiencies $e(\hat{\sigma}/\sigma^*)$, $e(\hat{\sigma}/\tilde{\sigma})$ of $\hat{\sigma}$ relative to σ^* and $\tilde{\sigma}$, respectively, for $n = 4(1)20$. These are presented in Table 2. We have made a discussion of estimating σ by censored AOS in Section 4. Section 5 deals with the estimation of σ based on a newly defined ranked set sampling (RSS). In Section 6, we define an RSS called adjusted ranked set sampling (ARSS) so as to make it suitable to deal with a Cauchy distribution. The estimation of the scale parameter of the Cauchy distribution based on ARSS observations as well is described in this section. Section 7 shows a real life application for modelling with the Cauchy distribution by using the AOS of a sample. Lastly the conclusions of the study are given in Section 8.

2. Minimal sufficient statistic for the Cauchy distribution

Let $X_1, X_2, ..., X_n$ be a random sample of size n drawn from the Cauchy distribution with pdf as given in Eq. (1). Suppose $X_{(1:n)}, X_{(2:n)}, ..., X_{(n:n)}$ are the AOS of the sample. Then the distribution theory of AOS is given in the following theorem.

THEOREM 3. *Suppose $X_{(1:n)}, X_{(2:n)}, ..., X_{(n:n)}$ are the AOS of a random sample of size n drawn from the distribution with pdf $f(x,\sigma)$ as given in Eq. (1). Let $Z_{1:n}, Z_{2:n}, ..., Z_{n:n}$ be the order statistics of a random sample of size n arising from the folded distribution with density $g(z,\sigma) = 2f(z,\sigma), z \geq 0$. Then*

$$(X_{(1:n)}, X_{(2:n)}, ..., X_{(n:n)}) \stackrel{d}{=} (Z_{1:n}, Z_{2:n}, ..., Z_{n:n}),$$

where $X \stackrel{d}{=} Z$ is the usual notation representing the identically distributed property between two random variables X and Z.

The proof of this theorem is provided in Appendix A.1.

Now we discuss a minimal sufficient statistic for the Cauchy family of distributions as defined by the pdf in Eq. (1). From Theorem 2 due to Thomas and Anjana (2022), we observe that the statistic $T = (X_{(1:n)}, X_{(2:n)}, ..., X_{(n:n)})$ formed from the AOS of a random sample of size n is the minimal sufficient statistic for the family $\mathscr{F}_1$ of distributions which are all symmetric about zero. If we consider a specific member of $\mathscr{F}_1$ with pdf $f(x)$, then sometimes it is possible to get a dimension reduction in the minimal sufficient statistic for $f(x)$. For example some distributions belonging to the exponential family (such as the normal distribution), which are distributed symmetrically about zero, allow for minimal sufficient statistic which is different from $T = (X_{(1:n)}, X_{(2:n)}, ..., X_{(n:n)})$ and whose dimension is less than n. The essential point is that by merely observing a Cauchy distribution as defined in (1) belonging to $\mathscr{F}_1$, we cannot claim that $T = (X_{(1:n)}, X_{(2:n)}, ..., X_{(n:n)})$ is also minimal sufficient for the restricted Cauchy distribution. Clearly, the Cauchy family of distributions is not an exponential family, and similarly we can verify that the Cauchy distribution is not a member to any sub-family of the class of all continuous distributions for which an improved version of minimal sufficient statistic exists with a reduced dimension compared to $T = (X_{(1:n)}, X_{(2:n)}, ..., X_{(n:n)})$. Observing the above arguments about the irreducible nature of the sufficient statistic than T, we can prove from basic principles the following theorem.

THEOREM 4. *Suppose $\underline{X} = (X_1, X_2, ..., X_n)$ is a random sample of size n drawn from the Cauchy distribution with pdf $f(x,\sigma)$ as given in Eq. (1). Let $T = (X_{(1:n)}, X_{(2:n)}, ..., X_{(n:n)})$*

be the vector of AOS constructed from $X_1, X_2, \ldots, X_n$. Then T is minimal sufficient for $f(x, \sigma)$.

The proof of this theorem is similar to the proof given by Thomas and Anjana (2022) for proving the minimal sufficiency of $T = (X_{(1:n)}, X_{(2:n)}, \ldots, X_{(n:n)})$ for the family $\mathscr{F}_1$ and is given in Appendix A.2.

It is known in statistical inference that any inference procedure developed based on the minimal sufficient statistic excels in performance more than those based on other statistics. Thus, at this stage we are destined to show that the BLUE of σ based on the AOS is better than that based on the classical order statistics. In the next section we discuss this aspect.

3. Best linear unbiased estimation of the scale parameter of the Cauchy distribution based on AOS

The standard form of the Cauchy distribution as given in Eq. (1) has the pdf

$$f(x, 1) = \frac{1}{\pi} \frac{1}{1 + x^2}, -\infty < x < \infty. \tag{2}$$

One can easily verify that (see Barnett, 1966) the variances of the first two and last two order statistics of a random sample of size n arising from Eq. (2) are not finite. However, the means, variances and covariances of the $(n-4)$ intermediate order statistics exist. Barnett (1966) took this into consideration and thereby evaluated the means, variances and covariances of the intermediate $n-4$ order statistics. Those values were tabulated in his paper for $n = 6(1)16(2)20$. From Thomas (1990) we observe that the BLUE based on order statistics of the scale parameter σ of a distribution that is symmetric about zero reduces to that based on quasi-ranges of the sample. Consequently he tabulated the coefficients of the quasi-ranges of the middle most $n-4$ observations arising from the Cauchy distribution in Eq. (1) as an estimate of its scale parameter σ together with their variances for $n = 6(1)16(2)20$. From Eq. (1) the pdf of the half-Cauchy distribution can be written as

$$g(x, \sigma) = \frac{2}{\sigma\pi} \frac{1}{1 + (\frac{x}{\sigma})^2}, x \geq 0, \sigma > 0. \tag{3}$$

The pdf of the corresponding standard form of the half-Cauchy distribution is

$$g(x, 1) = \frac{2}{\pi} \frac{1}{1 + x^2}, x \geq 0. \tag{4}$$

Clearly, Equations (3) and (4) are left truncated distributions of two varieties of the Cauchy distribution at the truncation point $x = 0$. Hence both Equations (3) and (4) are long tailed only at the right tail and consequently one can easily verify that among all order statistics of a random sample of size n arising from Eq. (4), only the largest two order statistics fail to admit finite variances. Then, from Theorem 3 we conclude that among all AOS of a sample of size n arising from the Cauchy distribution in Eq. (1), only the largest two AOS fail to admit finite variances. This leads us to conclude that for inference purposes out of a random sample of size n arising from Eq. (1), only $n-4$ order statistics are usable, whereas for the same purpose $n-2$ AOS of the sample are usable. This is one major advantage of using the minimal sufficiency established for the statistic based on AOS when compared with the sufficient statistic based on order statistics. In the following theorem we obtain the expression for the BLUE and its variance for the scale parameter σ of the Cauchy distribution based on the first $n-2$ AOS.

THEOREM 5. *Let $\underset{\sim}{X} = (X_{(1:n)}, X_{(2:n)}, ..., X_{(n-2:n)})'$ be the first $n-2$ AOS of a random sample of size n drawn from the Cauchy distribution defined in Eq. (1). Suppose $\underset{\sim}{Y} = (Y_{1:n}, Y_{2:n}, ..., Y_{n-2:n})'$ is the vector of first $n-2$ order statistics arising from the standard half-Cauchy distribution defined by the pdf of Eq. (4). Let $E(\underset{\sim}{Y}) = \underset{\sim n-2}{\alpha} = (\alpha_{1:n}, \alpha_{2:n}, ..., \alpha_{n-2:n})'$ and let the dispersion matrix of $\underset{\sim}{Y}$ be given by $D(\underset{\sim}{Y}) = A_{n-2} = ((\alpha_{i,j:n}))$, where $\alpha_{i,j:n} = Cov(Y_{i:n}, Y_{j:n})$, $i, j = 1, 2, ..., n-2$, for $i \neq j$ and $\alpha_{i,i:n} = Var(Y_{i:n})$ for $i = 1, 2, ..., n-2$. Then, the BLUE $\hat{\sigma}$ of σ based on the first $n-2$ AOS is given by*

$$\hat{\sigma} = (\underset{\sim n-2}{\alpha}' \mathbf{A}_{n-2}^{-1} \underset{\sim n-2}{\alpha})^{-1} \underset{\sim n-2}{\alpha}' \mathbf{A}_{n-2}^{-1} \underset{\sim}{X}, \tag{5}$$

and its variance is given by

$$Var(\hat{\sigma}) = (\underset{\sim n-2}{\alpha}' \mathbf{A}_{n-2}^{-1} \underset{\sim n-2}{\alpha})^{-1} \sigma^2. \tag{6}$$

PROOF. Let $X_{(1:n)}, X_{(2:n)}, ..., X_{(n:n)}$ be the AOS of a random sample of size n drawn from the Cauchy distribution with pdf in Eq. (1). Then, $(\frac{X_{(1:n)}}{\sigma}, \frac{X_{(2:n)}}{\sigma}, ..., \frac{X_{(n-2:n)}}{\sigma})' \stackrel{d}{=} (Y_{1:n}, Y_{2:n}, ..., Y_{n-2:n})'$, where $Y_{1:n}, Y_{2:n}, ..., Y_{n-2:n}$ are the first $n-2$ order statistics of a random sample of size n drawn from the standard half-Cauchy distribution with pdf of Eq. (4). Clearly the means, variances and covariances of all order statistics involved in the right side vector of above distributional identity exists finitely and they are independent of σ. Thus for $\underset{\sim}{X} = (X_{(1:n)}, X_{(2:n)}, ..., X_{(n-2:n)})'$ and $\underset{\sim}{Y} = (Y_{1:n}, Y_{2:n}, ..., Y_{n-2:n})'$ we can write

$$E(\underset{\sim}{X}) = \sigma E(\underset{\sim}{Y}) = \sigma \underset{\sim n-2}{\alpha}, \tag{7}$$

and

$$D(\underset{\sim}{X}) = \sigma^2 D(\underset{\sim}{Y}) = \sigma^2 A_{n-2}. \tag{8}$$

Then, Equations (7) and (8) together form a generalized Gauss-Markov setup, and by least-squares theory we get

$$\hat{\sigma} = (\underset{\sim n-2}{\alpha}' \mathbf{A}_{n-2}^{-1} \underset{\sim n-2}{\alpha})^{-1} \underset{\sim n-2}{\alpha}' \mathbf{A}_{n-2}^{-1} \underset{\sim}{X}, \tag{9}$$

and

$$\mathrm{Var}(\hat{\sigma}) = (\underset{\sim n-2}{\alpha}' \mathbf{A}_{n-2}^{-1} \underset{\sim n-2}{\alpha})^{-1} \sigma^2. \tag{10}$$

This proves the theorem. □

The linear estimate $\hat{\sigma}$ of σ as given in Eq. (9) may also be written as

$$\hat{\sigma} = \sum_{i=1}^{n-2} c_{i,n} X_{(i:n)}, \tag{11}$$

where $c_{i,n}$, $i = 1, 2, ..., n-2$ are appropriate constants.

REMARK 6. *It is trivial to observe that any deletion of component random variables from a minimal sufficient set of random variables makes the resulting set lose its minimal sufficient property. However, the proposed BLUE based on a censored vector of AOS makes the estimator unbiased as well as possessing least variance among all possible linear estimates obtained from the AOS in the censored vector. One merit of the estimator is that it utilizes the subset of variables in the minimal sufficient set of random variables that are usable for deriving the BLUE, so the desirable condition of having smaller variance of this estimate than other linear estimators of σ holds for this estimator.*

In Eq. (11), it is strange to note that $X_{(i:n)}$, $i = 1, 2, ..., n-2$ are the first $n-2$ AOS arising from the Cauchy distribution defined in Eq. (1), while $c_{i,n}$, $i = 1, 2, ..., n-2$ are the coefficients of the BLUE of σ based on the first $n-2$ order statistics of a random sample of size n arising from the half-Cauchy distribution defined by the pdf in Eq. (3) (a consequence of Theorem 3). This leads us to conclude that provided the BLUE of σ based on the first $n-2$ order statistics of a sample of size n arising from the half-Cauchy distribution as defined by the pdf of Eq. (3) with its variance is available, then the BLUE of σ based on the first $n-2$ AOS of a sample of size n arising from the Cauchy distribution as defined by the pdf in Eq. (1) with its variance can be obtained without any direct evaluation of means, variances and covariances of those AOS.

Though moment relations for order statistics arising from general distributions which are symmetrically distributed about zero such as those described in Arnold *et al.* (1992), David and Nagaraja (2003) and Thomas and Samuel (1996) are useful for easy evaluation of moments of order statistics from symmetric distributions, there is a limitation for applying those relations as such to Cauchy order statistics. However application of Theorem 3 helps us to deal only with the evaluation of moments of order statistics arising from the half-Cauchy distribution for using those values for the development of inference procedures on the scale parameter σ of the Cauchy distribution. Moments of order statistics arising from a standard half-Cauchy distribution has not been discussed in the available literature as far as we know. Hence we have used the Mathematica software to evaluate the means, variances and covariances of all order statistics $Y_{1:n}, Y_{2:n}, ..., Y_{n-2:n}$ for $n = 4(1)20$ arising from the standard half-Cauchy distribution defined by the pdf in Eq. (4). Using those values we have determined the coefficients $c_{i,n}$ of the AOS $X_{(i:n)}$ in the estimate $\hat{\sigma}$ of σ as given in Eq. (11) for $i = 1,2,...,n-2$; $n = 4(1)20$, and those values are tabulated in Table 1. We have obtained further $\sigma^{-2}\text{Var}(\hat{\sigma})$ for $n = 4(1)20$, and those values are given in Table 2.

If we depend on order statistics of a random sample of size n arising from the Cauchy distribution with pdf of Eq. (1), then the estimate σ^* of σ based on the intermediate order statistics $X_{3:n}, X_{4:n}, ..., X_{n-2:n}$ as given by Thomas (1990) can be written as

$$\sigma^* = \sum_{i=3}^{n-2} d_{i,n} X_{i:n}, \tag{12}$$

and the variance of σ^* is given by

$$\text{Var}(\sigma^*) = (\underset{\sim n-4}{\beta'} V_{n-4}^{-1} \underset{\sim n-4}{\beta})^{-1}\sigma^2, \tag{13}$$

where to describe the constants in Equations (12) and (13) we write $\underset{\approx n-4}{Y}$ to denote the vector $(Y_{3:n}, Y_{4:n}, ..., Y_{n-2:n})'$ of $n-4$ intermediate order statistics of a random sample of size n drawn from the standard Cauchy distribution defined by the pdf in Eq. (2) with $E(\underset{\approx n-4}{Y}) = \underset{\sim n-4}{\beta}$, $D(\underset{\approx n-4}{Y}) = V_{n-4}$, and $d_{i,n}$, $i = 3,4,...,n-2$, are appropriate constants. Thomas (1990) tabulated $\text{Var}(\sigma^*)$ for $n = 6(1)16(2)20$. As variances of σ^* for $n = 17, 19$ are not available in the literature, we have computed those variances as well and tabulated $\text{Var}(\sigma^*)$ for $n = 6(1)20$ in Table 2 with an objective of comparing the performance of the new estimator $\hat{\sigma}$ proposed in this paper with the already available estimator σ^*.

If we write $S(X_1, X_2, ..., X_n)$ to denote the likelihood of n observations $X_1, X_2, ..., X_n$ drawn from the Cauchy distribution with pdf in Eq. (1), then the MLE $\tilde{\sigma}$ of σ is obtained as a solution of

$$\sum_{i=1}^{n} \frac{\sigma^2}{\sigma^2 + X_i^{\,2}} = \frac{n}{2}. \tag{14}$$

Although for some initial values of n, we can solve Eq. (14) for $\tilde{\sigma}$, generally $\tilde{\sigma}$ is computed by numerical methods. To discuss the large sample property of $\tilde{\sigma}$, we obtain the asymptotic variance of $\tilde{\sigma}$ from the second derivative of $S(X_1, X_2, ..., X_n)$, and one can easily obtain it as $AV(\tilde{\sigma}) = \frac{2\sigma^2}{n}$. Kravchuk and Pollett (2012) have carried out a computational study on the MLE of σ. They commented that the MLE of σ is not consistent with its asymptotic distribution. To illustrate the inappropriateness of $\text{Var}(\tilde{\sigma})$ for the small sample cases, they pointed out a situation for $n = 4$ involving observations w, x, y, z such that $w < x < y < z$ with $\tilde{\sigma} = \frac{\sqrt{(z-y)(y-x)(x-w)(z-w)}}{|(z-y+x-w)|}$ and $\text{Var}(\tilde{\sigma}) = 1.90\sigma^2$. Using Table 1, we can estimate σ by $\hat{\sigma}$ for n=4, by leaving the largest two AOS yielding $\hat{\sigma} = 1.2955X_{(1:4)} + 0.5904X_{(2:4)}$ with $\text{Var}(\hat{\sigma}) = 0.8007\sigma^2$. Note that for the Cauchy distribution, the asymptotic variance of the MLE given by $\frac{2\sigma^2}{n}$ is unattainable by any estimator of σ in the finite sample case as it does not admit a minimum variance bound

TABLE 1

Coefficient $c_{i,n}$ of $X_{(i:n)}$ involved in $\hat{\sigma} = \sum_{i=1}^{n-2} c_{i,n} X_{(i:n)}$ as an estimate of σ of Cauchy distribution for $n = 4(1)20$.

n	Coefficient $c_{i,n}$ of $X_{(i:n)}$ involved in $\hat{\sigma} = \sum_{i=1}^{n-2} c_{i,n} X_{(i:n)}$																	
	$X_{(1:n)}$	$X_{(2:n)}$	$X_{(3:n)}$	$X_{(4:n)}$	$X_{(5:n)}$	$X_{(6:n)}$	$X_{(7:n)}$	$X_{(8:n)}$	$X_{(9:n)}$	$X_{(10:n)}$	$X_{(11:n)}$	$X_{(12:n)}$	$X_{(13:n)}$	$X_{(14:n)}$	$X_{(15:n)}$	$X_{(16:n)}$	$X_{(17:n)}$	$X_{(18:n)}$
4	1.300	0.560	-	-														
5	0.870	0.682	0.218	-	-													
6	0.612	0.618	0.367	0.101	-	-												
7	0.454	0.521	0.410	0.209	0.053	-	-											
8	0.350	0.432	0.397	0.271	0.126	0.031	-	-										
9	0.278	0.360	0.364	0.292	0.182	0.080	0.019	-	-									
10	0.226	0.302	0.325	0.290	0.214	0.125	0.053	0.012			-							
11	0.188	0.257	0.288	0.276	0.227	0.157	0.088	0.036	0.008	-								
12	0.158	0.220	0.255	0.257	0.228	0.176	0.117	0.063	0.026	0.006	-							
13	0.136	0.190	0.226	0.236	0.221	0.185	0.138	0.088	0.047	0.019	0.004	-	-					
14	0.117	0.166	0.201	0.216	0.211	0.187	0.150	0.108	0.068	0.035	0.014	0.003	-	-				
15	0.102	0.147	0.179	0.197	0.198	0.184	0.157	0.122	0.085	0.052	0.027	0.011	0.002	-	-			
16	0.090	0.130	0.161	0.180	0.186	0.178	0.158	0.131	0.099	0.068	0.041	0.021	0.008	0.002	-	-		
17	0.080	0.116	0.145	0.164	0.173	0.170	0.157	0.135	0.109	0.081	0.055	0.033	0.017	0.006	0.001	-	-	
18	0.072	0.104	0.131	0.150	0.161	0.162	0.153	0.137	0.116	0.091	0.067	0.045	0.026	0.013	0.005	0.001	-	
19	0.065	0.094	0.119	0.138	0.149	0.153	0.148	0.137	0.119	0.099	0.077	0.055	0.037	0.022	0.011	0.004	0.001	
20	0.058	0.085	0.109	0.127	0.139	0.144	0.142	0.134	0.121	0.104	0.084	0.065	0.046	0.030	0.018	0.009	0.003	0.001

estimator for σ. However, to make a comparison of the estimator $\hat{\sigma}$ with $\tilde{\sigma}$, we have included $AV(\tilde{\sigma})$ also in Table 2. Clearly, as $\hat{\sigma}$ involves a larger number of observations, which are the components of minimal sufficient statistic, than those involved in σ^*, the estimate $\hat{\sigma}$ is likely to be more efficient than the estimate σ^*. Thus our proposed new estimator $\hat{\sigma}$ of σ based on AOS is preferable, especially when the sample size is small. To compare the performance of our estimator $\hat{\sigma}$, we have defined the relative efficiencies $e(\hat{\sigma}/\sigma^*)$ and $e(\hat{\sigma}/\tilde{\sigma})$ as $e(\hat{\sigma}/\sigma^*) = \frac{\text{Var}(\sigma^*)}{\text{Var}(\hat{\sigma})}$, and $e(\hat{\sigma}/\tilde{\sigma}) = \frac{AV(\tilde{\sigma})}{\text{Var}(\hat{\sigma})}$, respectively. We have calculated the above relative efficiencies for $n = 6(1)20$, and they are presented in Table 2.

TABLE 2
Variances of (i) BLUE $\hat{\sigma}$ based on AOS (ii) BLUE σ^* based on order statistics (iii) Asymptotic variance of MLE $\tilde{\sigma}$ of the scale parameter σ of the Cauchy Distribution, the relative efficiencies : $e(\hat{\sigma}/\sigma^*)$ and $e(\hat{\sigma}/\tilde{\sigma})$ for $n = 4(1)20$.

n	$\sigma^{-2}\text{Var}(\hat{\sigma})$	$\sigma^{-2}\text{Var}(\sigma^*)$	$\sigma^{-2}AV(\tilde{\sigma})$	$e(\hat{\sigma}/\sigma^*)$	$e(\hat{\sigma}/\tilde{\sigma})$
4	0.88	-	0.50	-	0.57
5	0.61	-	0.40	-	0.66
6	0.46	1.76	0.33	3.78	0.72
7	0.38	1.00	0.29	2.67	0.76
8	0.32	0.68	0.25	2.16	0.79
9	0.27	0.52	0.22	1.90	0.81
10	0.24	0.42	0.20	1.73	0.83
11	0.21	0.35	0.18	1.62	0.85
12	0.19	0.30	0.17	1.53	0.86
13	0.18	0.26	0.15	1.47	0.87
14	0.16	0.23	0.14	1.42	0.88
15	0.15	0.21	0.13	1.38	0.89
16	0.14	0.19	0.13	1.35	0.90
17	0.13	0.17	0.12	1.28	0.90
18	0.12	0.16	0.11	1.30	0.91
19	0.12	0.14	0.11	1.23	0.91
20	0.11	0.14	0.10	1.26	0.92

REMARK 7. *It is clear that the asymptotic variance of the MLE is also equal to the variance of the minimum variance bound (unbiased) estimator of σ. In the case of the Cauchy distribution such an estimator is unattainable. However, for efficiency comparison, this variance is usually taken and used as a basic scale to observe the efficiency of other unbiased estimators. As the estimator $\hat{\sigma}$ proposed by us is unbiased for σ, it is natural to obtain the relative efficiency of this estimator when compared with the minimum variance bound (which is the asymptotic variance of the MLE). It may be noted that as the MLE is only evaluated numerically, the exact variance of the MLE is not derived in the existing literature. From*

Table 2, it is clear that our estimator $\hat{\sigma}$ is more efficient than σ^*, the BLUE based on usable order statistics.

From Table 2, we conclude that the estimate $\hat{\sigma}$ based on AOS is remarkably better than σ^* based on order statistics. The gain in efficiency observed in $\hat{\sigma}$ when compared with σ^* ranges from 25% to 277%. The relative efficiency $e(\hat{\sigma}/\tilde{\sigma})$ steadily increases as n increases, and this relative efficiency approaches 0.9166 for $n = 20$.

4. Estimation of the scale parameter from censored samples

When an outlier occurs in a sample drawn from a distribution which is symmetric about zero, it must be far away from zero either on the positive side or on the negative side. It is quite curious to know that unlike order statistics of the data, AOS capture that far off observation uniquely as the largest absolved order statistic $X_{(n:n)}$. But while working with order statistics this uniqueness is not materialized, as the farthest observation from zero may be either the smallest order statistic $X_{1:n}$ or the largest order statistic $X_{n:n}$. So to eliminate the effect of a suspected outlier from the sample with ordered data, theoretically a double censoring with one observation in the left ($X_{1:n}$) and one observation in the right ($X_{n:n}$) is required. Although for the Cauchy distribution the above descriptions applies as well, the BLUE $\hat{\sigma}$ of σ based on AOS as given in Eq. (9) has not included the largest two AOS, $X_{(n-1:n)}$ and $X_{(n:n)}$, and hence the estimate $\hat{\sigma}$ has the inbuilt capacity to eliminate the effect of two possible outliers in the data.

If we suspect more than two outliers in the data, then we have to generalize the estimate of σ by modifying the censoring scheme appropriately. In particular if we have the reason to believe that there are k outlying observations in the data, then we go for right censoring of k of the AOS (those corresponding to the k observations in the data which lie most far away from zero) and then estimate σ by using the available AOS $X_{(1:n)}, X_{(2:n)}, \ldots, X_{(n-k:n)}$ where k is any positive integer such that $3 \le k \le n-2$. The estimation procedure for σ for the censoring scheme then follows from the theorem given below.

THEOREM 8. *Suppose $X_{(1:n)}, X_{(2:n)}, \ldots, X_{(n:n)}$ are the AOS of a random sample of size n drawn from the Cauchy distribution with pdf defined in Eq. (1). Let $Y_{1:n}, Y_{2:n}, \ldots, Y_{n:n}$ be the order statistics of a random sample of size n drawn from the standard half-Cauchy distribution given in Eq. (4). Suppose the k observations with largest k absolute values are censored so that the vector of the remaining AOS is $\underset{\sim}{X}_{n-k} = (X_{(1:n)}, X_{(2:n)}, \ldots, X_{(n-k:n)})'$. Define $\underset{\sim}{Y}_{n-k} = (Y_{1:n}, Y_{2:n}, \ldots, Y_{n-k:n})'$, $E(\underset{\sim}{Y}_{n-k}) = \underset{\sim}{\alpha}_{n-k} = (\alpha_{1:n}, \alpha_{2:n}, \ldots, \alpha_{n-k:n})'$, and let the dispersion matrix of $\underset{\sim}{Y}_{n-k}$ be denoted by A_{n-k}. In this case we write $E(\underset{\sim}{X}_{n-k}) = \underset{\sim}{\alpha}_{n-k}\sigma$, $D(\underset{\sim}{X}_{n-k}) = A_{n-k}\sigma^2$. Then the BLUE $\hat{\sigma}_{k,n}$ of σ based on the censored AOS is given by*

$$\hat{\sigma}_{k,n} = (\underset{\sim}{\alpha}'_{n-k}\mathbf{A}^{-1}_{n-k}\underset{\sim}{\alpha}_{n-k})^{-1}\underset{\sim}{\alpha}'_{n-k}\mathbf{A}^{-1}_{n-k}\underset{\sim}{X}_{n-k}. \tag{15}$$

The variance of $\hat{\sigma}_{k,n}$ is given by

$$Var(\hat{\sigma}_{k,n}) = (\underset{\sim n-k}{\alpha'} \mathbf{A}^{-1}_{n-k} \underset{\sim n-k}{\alpha})^{-1} \sigma^2. \quad (16)$$

PROOF. The proof of the above theorem follows easily by application of the Gauss-Markov theorem. □

REMARK 9. *One can write Eq. (15) as a linear function of $X_{(1:n)}, X_{(2:n)}, ..., X_{(n-k:n)}$ as*

$$\hat{\sigma}_{k,n} = \sum_{i=1}^{n-k} c^{(k)}_{i,n} X_{(i:n)}, \quad (17)$$

where $c^{(k)}_{i,n}$, $i = 1, 2, ..., n-k$ are appropriate constants.

The method of estimation of σ by $\hat{\sigma}_{k,n}$ as described in Theorem 8 using censored AOS arising from the Cauchy distribution is attempted for a sample of size 10 and for each of $k = 2, 3, ..., 8$. We have for each $k = 2, 3, ..., 8$ computed the numerical value of the coefficient $c^{(k)}_{i,10}$ of $X_{(i:10)}$ involved in $\hat{\sigma}_{k,10}$ for $i = 1, 2, ..., 10-k$, and $\sigma^{-2}\text{Var}(\hat{\sigma}_{k,10})$. These computed values are given in Table 3. The relative efficiency $e(\hat{\sigma}_{k,10}/\hat{\sigma}) = \frac{\text{Var}(\hat{\sigma})}{\text{Var}(\hat{\sigma}_{k,10})}$ of $\hat{\sigma}_{k,10}$ when compared with the BLUE $\hat{\sigma}$ as derived in Eq. (11) is again computed for each $k = 2, 3, ..., 8$, and the computed values are also presented in Table 3. Note that when $k = 2$, the estimate $\hat{\sigma}_{2,10}$ is the same as $\hat{\sigma}$ as given in Eq. (11) for $n = 10$.

From Table 3 we observe that initially for $k = 3, 4, 5$, there is not much reduction noticed on the relative efficiencies, as in all those cases the relative efficiency exceeds 93%. However the reduction noticed in the relative efficiencies becomes a little more but changes at a sluggish rate when a larger number of extreme absolved order statistics are censored. For example, if $n = 10$ and $k = 8$, then in the estimator $\hat{\sigma}_{8,10}$ altogether eight AOS are censored, and hence it utilizes only two AOS, $X_{(1:10)}$ and $X_{(2:10)}$ for estimating σ. From these two AOS, the efficiency observed on $\hat{\sigma_{6,10}}$ relative to the estimator $\hat{\sigma}$ (in which 8 out of 10 AOS are involved) is more than 49%. This prompts us to comment that the AOS based estimator for the scale parameter σ of Cauchy distribution appears to be robust.

5. A NEW RANKED SET SAMPLING AND ITS APPLICATION IN ESTIMATING THE SCALE PARAMETER OF DISTRIBUTIONS OF $\mathscr{F}_1$.

McIntyre (1952) has introduced Ranked Set Sampling (RSS) to find a more efficient estimate of the yield of pastures. This method consists in first selecting at random n^2 units and arranging them randomly in n sets of n units each. Next the units in the i^{th} set are ranked using a judgement method or by a method not involving any cost, and the i^{th} ranked unit is selected and measured for the characteristic of interest for each

TABLE 3

Coefficient $c_{i,10}^{(k)}$ *of* $X_{(i:10)}$ *involved in* $\hat{\sigma}_{k,10} = \sum_{i=1}^{10-k} c_{i,10}^{(k)} X_{(i:10)}$, *Var*$(\hat{\sigma}_{k,10})$ *and efficiency* $e(\hat{\sigma}_{k,10}/\hat{\sigma})$ *of* $\hat{\sigma}_{k,10}$ *relative to* $\hat{\sigma}$ *for the scale parameter* σ *of the Cauchy distribution.*

Statistic used	**Coefficient** $c_{i,10}^{(k)}$ **of** $X_{(i:10)}$ **involved in** $\hat{\sigma}_{k,10} = \sum_{i=1}^{10-k} c_{i,10}^{(k)} X_{(i:10)}$ **given along columns.**						
$X_{(i:10)}$	$\hat{\sigma}(k=2)$	$\hat{\sigma}_{1,10}\,(k=3)$	$\hat{\sigma}_{2,10}\,(k=4)$	$\hat{\sigma}_{3,10}\,(k=5)$	$\hat{\sigma}_{4,10}\,(k=6)$	$\hat{\sigma}_{5,10}\,(k=7)$	$\hat{\sigma}_{6,10}\,(k=8)$
$X_{(1:10)}$	0.226	0.227	0.230	0.240	0.264	0.315	0.434
$X_{(2:10)}$	0.302	0.303	0.307	0.320	0.351	0.419	3.049
$X_{(3:10)}$	0.325	0.326	0.330	0.344	0.376	1.697	
$X_{(4:10)}$	0.290	0.291	0.295	0.307	0.959		
$X_{(5:10)}$	0.214	0.214	0.217	0.500			
$X_{(6:10)}$	0.125	0.125	0.220				
$X_{(7:10)}$	0.053	0.071					
$X_{(8:10)}$	0.012						
$\sigma^{-2}\mathrm{Var}(\hat{\sigma}_{k,n})$	0.240	0.241	0.245	0.256	0.283	0.342	0.482
$e(\hat{\sigma}_{k,n}/\hat{\sigma})$	-	0.997	0.982	0.939	0.849	0.702	0.498

$i = 1,2,...,n$. This method of collecting n units from the population is known as RSS, and the sample of observations measured on the selected units is known as the ranked set sample.

For a discussion on the initial developments of RSS, one may refer to Chen and Wang (2004). When the population of interest is infinite and the variable of interest measured follows a continuous distribution, then again RSS is applied successfully to draw inferences on the parameters of the distribution of the variable of interest. For some recent discussion on the above application of RSS see Lam *et al.* (1994, 1996), Lesitha and Thomas (2013); Lesitha *et al.* (2010), Sinha *et al.* (1996) and Wolfe (2004). For some new variants of RSS one may refer to Al-Saleh and Al-Omari (2002), Muttlak (1998), Paul and Thomas (2017), Priya and Thomas (2016), Salehi and Ahmadi (2014), Thomas and Priya (2016) and Thomas and Philip (2018).

If $\mathscr{F}_1$ is the family of distributions that are all symmetrical about zero, then from Thomas and Anjana (2022) we observe that the AOS of the sample is minimal sufficient for $\mathscr{F}_1$. Hence, if we rely on the classical method of RSS as defined by McIntyre, then it does not have a ranking system which utilizes the advantages of the minimal sufficient statistic. Hence, for distributions belonging to $\mathscr{F}_1$, in order to explore higher efficiency on the methods for inference problems using RSS, we require a modification on it by using the judgement by an expert in advance about the relative largeness of the possible absolute values that will be measured on the units with respect to the variable of interest. In particular, we have to modify the RSS, which involves a ranking of units, which when performed without any ranking error results in observations each of which is an absolved order statistic. If such an RSS is developed, then from it we can modify an appropriate RSS that can be performed for a population random variable which follows a Cauchy distribution with pdf in Eq. (1). With this objective in this mind, we define the following.

DEFINITION 10. *Suppose the variable of interest on which we make measurements on the units of an infinite population follows a distribution belonging to* $\mathscr{F}_1$. *Let* n^2 *units be drawn randomly and those units arranged randomly in n sets each with n units. Suppose without direct measurement that an expert's judgement gives a perfect ranking on the relative largeness of the absolute values of the possible measurement values that one may obtain on the units. Now from the* i^{th} *set choose the unit ranked i and make the measurement on the variable of interest from this unit, take its absolute value and denote this observation as* $X_{(i:n)i}, i = 1,2,...,n$. *The above procedure of drawing units from the population is called RSS-A, as under perfect ranking each observation* $X_{(i:n)i}$ *is distributed as the* i^{th} *"Absolved" order statistic of a sample of size* n, *for* $i = 1,2,...,n$.

REMARK 11. *In the above definition by the judgement method, only the rank of the AOS on a unit of a sample is judged, so that to obtain the AOS we depend on the judged values of the absolute values of the units to determine the* i^{th} *observation of RSS-A. Also* $X_{(1:n)1}, X_{(2:n)2}, ..., X_{(n:n)n}$ *are independently distributed, as they are selected from different independent samples.*

Whenever a new methodology is developed, the immediate concern of a statistician is about its applicability. One example where RSS-A can be profitably applied is described below. It is of interest to observe that by the last two decades of 20th century, under social forestry scheme 'Acacia Auriculiformis, A.cunn.ex Benth' (for convenience we call this species of trees as the locally called name acacia trees) was planted over a large extent of barren lands of the Kerala state in India, especially in Government lands given to universities, other institutes and public sector units. But recently wood processing and furniture manufacturing companies as well as retail users have shown much interest in the acacia woods, and this makes the acacia woods wealth so valuable to those who planted it extensively. But presently, while expansion projects of universities are undertaken, the administrators are compelled to dispose of the acacia trees. But they face the problem of getting an estimate of the timber volume of the trees without felling them. In this case we can apply RSS-A very effectively to model the timber volume data and associated inference procedures.

For applying RSS-A, we choose randomly n independent lines of trees planted, and from each line we select randomly n trees. We know that the timber volume of a tree is somewhat directly proportional to its height, and that both tree height and timber volume are symmetrically distributed around their medians. We can evaluate the height of the trees very easily from the ground using hypsometer. Thus, we measure the height of all n^2 trees using a hypsometer. Let the median of the n^2 observations be denoted by m_0. We know that if $n \geq 6$, n^2 can be regarded as large (as $n^2 \geq 36$) and hence m_0 is a consistent and asymptotically normal (CAN) estimator of the population median. Now we write down the absolute values of deviations of height from m_0 of each tree in the i^{th} set and identify from these values the tree in the set (selected trees in the i^{th} line) with i^{th} smallest value. This tree is selected for making a measurement of its timber volume, which is somewhat harder to measure. To do so, we can employ a labourer (to climb or use an elevator) to measure the perimeter of the usable pieces of timbers (those pieces whose top part has at least 20 inches perimeter) at each multiple of 5 feet height from the bottom of the tree. Note that for each piece of timber the average of the top and bottom perimeters, say p, may be taken as the perimeter of that cylindrically approximated piece so that its radius is equal to $\frac{p}{2\pi}$. Hence we can evaluate piece by piece the volume and thereby the total usable timber volume of the tree. Similarly, we evaluate the timber volume of all selected trees and further the volume v_0 of the tree whose height is equal to the median m_0. Let $V_{1,n,1}, V_{2,n,2}, ..., V_{n,n,n}$ be the volumes of the selected trees. Define $X_{(i:n)i} = |V_{i,n,i} - v_0|$, $i = 1, 2, ...,$n. Then $X_{(i:n)i}$, $i = 1, 2, ..., n$ may be considered as an RSS-A sample drawn from a distribution belonging to $\mathscr{F}_1$. Similarly more examples of applications of RSS-A as well can be described.

Now we consider the problem of estimating the scale parameter of a distribution belonging to $\mathscr{F}_1$ with density $f(x, \sigma) = \frac{1}{\sigma} f_0(\frac{x}{\sigma})$, $-\infty < x < \infty, \sigma > 0$ using RSS-A. Suppose $X_{(1:n)1}, X_{(2:n)2}, ..., X_{(n:n)n}$ are the observations of RSS-A. We assume that the ordering of the units in the sample is perfect. Let $g(z, \sigma) = \frac{2}{\sigma} f_0(\frac{z}{\sigma})$, $0 \leq z < \infty, \sigma > 0$

be the folded form of the distribution with the pdf $f(x,\sigma)$. Then we can write $g_0(y) = 2f_0(y), 0 \le y < \infty$, as the standard form of the folded density $g(z,\sigma)$. If $Z_{1:n}, Z_{2:n}, ..., Z_{n:n}$ are the order statistics of a random sample of size n drawn from the folded distribution with pdf $g(z,\sigma)$, then clearly $X_{(i:n)i} \stackrel{d}{=} Z_{i:n}, i = 1,2,...,n$. Thus, if $Y_{1:n}, Y_{2:n}, ..., Y_{n:n}$ are the order statistics of a random sample of size n arising from the distribution with density $g_0(y)$, which admits the first two moments, then we write $E(Y_{i:n}) = \alpha_{i:n}$ and $\text{Var}(Y_{i:n}) = \alpha_{i,i:n}$, $i = 1,2,...,n$. Hence we write $E(X_{(i:n)i}) = \sigma\alpha_{i:n}$ and $\text{Var}(X_{(i:n)i}) = \sigma^2\alpha_{i,i:n}$ for $i = 1,2,...,n$.

Now if we write $\underset{\sim}{X}_A = (X_{(1:n)1}, X_{(2:n)2}, ..., X_{(n:n)n})'$, then,

$$E(\underset{\sim}{X}_A) = \underset{\sim}{\alpha}\sigma, \tag{18}$$

and since $X_{(1:n)1}, X_{(2:n)2}, ..., X_{(n:n)n}$ are independently distributed, we have

$$D(\underset{\sim}{X}_A) = B\sigma^2, \tag{19}$$

where $\underset{\sim}{\alpha} = (\alpha_{1:n}, \alpha_{2:n}, ..., \alpha_{n:n})'$ and B is a diagonal matrix defined by $B = \text{diag}(\alpha_{1,1:n}, \alpha_{2,2:n}, \cdots, \alpha_{n,n:n})$. Clearly Equations (18) and (19) together constitute a generalized Gauss-Markov model, and hence the BLUE of σ based on the observations of RSS-A is given by

$$\hat{\sigma}_A = (\underset{\sim}{\alpha}' B^{-1} \underset{\sim}{\alpha})^{-1} \underset{\sim}{\alpha}' B^{-1} \underset{\sim}{X}_A,$$

and the variance is given by

$$\text{Var}(\hat{\sigma}_A) = (\underset{\sim}{\alpha}' B^{-1} \underset{\sim}{\alpha})^{-1} \sigma^2.$$

Thus we have proved the following theorem.

THEOREM 12. *Let* $\underset{\sim}{X}_A = (X_{(1:n)1}, X_{(2:n)2}, ..., X_{(n:n)n})'$ *be the vector of observations of RSS-A drawn from a distribution belonging to* $\mathscr{F}_1$ *with density* $f(x,\sigma) = \frac{1}{\sigma} f_0(\frac{x}{\sigma})$, $-\infty < x < \infty, \sigma > 0$. *Let* $Y_{1:n}, Y_{2:n}, ..., Y_{n:n}$ *be the order statistics of a random sample of size n that arises from the distribution with pdf* $g_0(y) = 2f_0(y), 0 \le y < \infty$, *which is the standard form of the distribution of* $f(x,\sigma)$ *folded about* $x = 0$. *If we define* $E(Y_{i:n}) = \alpha_{i:n}$, $Var(Y_{i:n}) = \alpha_{i,i:n}$, *for i=1, 2,..., n,* $\underset{\sim}{\alpha} = (\alpha_{1:n}, \alpha_{2:n}, ..., \alpha_{n:n})'$ *and* $B = diag(\alpha_{1,1:n}, \alpha_{2,2:n}, ..., \alpha_{n,n:n})$, *then the BLUE of* σ *is given by*

$$\hat{\sigma}_A = (\underset{\sim}{\alpha}' B^{-1} \underset{\sim}{\alpha})^{-1} \underset{\sim}{\alpha}' B^{-1} \underset{\sim}{X}_A, \tag{20}$$

and

$$Var(\hat{\sigma}_A) = (\underset{\sim}{\alpha}' B^{-1} \underset{\sim}{\alpha})^{-1} \sigma^2. \tag{21}$$

The linear estimate of Eq. (20) can also be written as

$$\hat{\sigma}_A = \sum_{i=1}^{n} d_{i,n} X_{(i:n)i}, \tag{22}$$

where $d_{i,n}$, $i = 1, 2, ..., n$ *are appropriate constants.*

Now we illustrate the advantage of RSS-A when compared with the McIntyre (1952) RSS for the case of sampling from the logistic distribution with scale parameter σ, which we consider as a member of the family $\mathscr{F}_1$ of symmetric distributions. The pdf of the logistic distribution that we consider for discussion is given by

$$f(x,\sigma) = \frac{1}{\sigma}\frac{e^{-\frac{x}{\sigma}}}{(1+e^{-\frac{x}{\sigma}})^2}, -\infty < x < \infty, \sigma > 0. \tag{23}$$

Then the pdf of the standard form of the half-logistic distribution is given by

$$g_0(y) = 2\frac{e^{-y}}{(1+e^{-y})^2}, 0 \leq y < \infty. \tag{24}$$

Let $X_{(1:n)1}, X_{(2:n)2}, ..., X_{(n:n)n}$ be the RSS-A observations available from Eq. (23). We have computed the means and variances of the order statistics of a random sample of size n drawn from Eq. (24) and used them in Equations (20) and (21) to determine the constants $d_{i,n}$ of $X_{(i:n)i}$ for $i = 1, 2, ..., n$ in the estimate $\hat{\sigma}_A$ and $\text{Var}(\hat{\sigma}_A)$ for $n = 2(1)10$. These values are given in Table 4. Although from Abu-Dayyeh *et al.* (2004) one can obtain the estimate $\hat{\sigma}_M$ of the scale parameter of the logistic distribution by McIntyre's RSS, we have directly obtained the variance $\text{Var}(\hat{\sigma}_M)$ for $n = 2(1)10$. Those values are also presented in Table 4. The efficiency of the RSS-A based estimate $\hat{\sigma}_A$ relative to the estimate $\hat{\sigma}_M$ defined by $e(\hat{\sigma}_A/\hat{\sigma}_M) = \frac{\text{Var}(\hat{\sigma}_M)}{\text{Var}(\hat{\sigma}_A)}$ has been evaluated for $n = 2(1)10$. This is also presented in Table 4.

REMARK 13. *From the definition of RSS-M under perfect ranking, the resulting observations are order statistics. But from the definition of RSS-A under perfect ranking, the resulting observations are absolved order statistics. This is the basic difference between RSS-M and RSS-A. Though it is known that order statistics form a sufficient statistic, it is noticeable that AOS is minimal sufficient. So naturally an RSS-A dependent estimate also should possess better results.*

From Table 4, we observe that the efficiency of the estimate of σ based on RSS-A is uniformly larger than for the estimate based on McIntyre's RSS. The gain in efficiency ranges from 178% to 352%. For perfect ranking it is always better to choose the set size of RSS-A small. If we intend to gain more efficiency in the estimate of σ, then we may repeat the RSS-A sampling in k cycles with the RSS-A estimate for $i = 1$, as given in Eq. (20) from the i^{th} cycle denoted by $\hat{\sigma}_{Ai}$ for $i = 2, ..., k$. Then the estimate of σ based on all RSS-A samples may be taken as $\hat{\hat{\sigma}}_A = \frac{\sum_{i=1}^{k} \hat{\sigma}_{Ai}}{k}$ so that $\text{Var}(\hat{\hat{\sigma}}_A) = \frac{\text{Var}(\hat{\sigma}_A)}{k}$.

TABLE 4

Coefficients $d_{i,n}$ of $X_{(i:n)i}$ involved in $\hat{\sigma}_A = \sum_{i=1}^{n} d_{i,n} X_{(i:n)i}$, $Var(\hat{\sigma}_A)$, $Var(\hat{\sigma}_M)$ and $e(\hat{\sigma}_A/\hat{\sigma}_M)$ using observations of RSS-A samples from the logistic distribution for $n = 2(1)10$.

n	**Coefficient $d_{i,n}$ of $X_{(i:n)i}$ in $\hat{\sigma}_A = \sum_{i=1}^{n} d_{i,n} X_{(i:n)i}$**										σ^{-2}**Var**$(\hat{\sigma}_A)$	σ^{-2}**Var**$(\hat{\sigma}_M)$	$e(\hat{\sigma}_A/\hat{\sigma}_M)$
	$X_{(1:n)1}$	$X_{(2:n)2}$	$X_{(3:n)3}$	$X_{(4:n)4}$	$X_{(5:n)5}$	$X_{(6:n)6}$	$X_{(7:n)7}$	$X_{(8:n)8}$	$X_{(9:n)9}$	$X_{(10:n)10}$			
2	0.447	0.328									0.253	1.145	4.525
3	0.316	0.293	0.196								0.131	0.453	3.455
4	0.241	0.240	0.211	0.133							0.081	0.252	3.132
5	0.194	0.198	0.189	0.160	0.097						0.055	0.163	2.984
6	0.162	0.167	0.166	0.153	0.126	0.074					0.039	0.114	2.903
7	0.138	0.144	0.145	0.140	0.127	0.102	0.059				0.030	0.085	2.855
8	0.121	0.126	0.128	0.127	0.120	0.107	0.084	0.048			0.023	0.066	2.823
9	0.107	0.112	0.114	0.114	0.111	0.104	0.092	0.071	0.040		0.019	0.052	2.799
10	0.096	0.100	0.103	0.104	0.103	0.099	0.091	0.079	0.061	0.034	0.015	0.043	2.785

6. ADJUSTED RANKED SET SAMPLING FOR THE CAUCHY DISTRIBUTION

The RSS as defined by McIntyre (1952) is not applicable as such for the Cauchy distribution defined in Eq. (1), since the variances of the observations obtained from the 1^{st},2^{nd}, $(n-1)^{th}$ and n^{th} sets are not finite due to the fact that the variances of the smallest two and largest two observations of a Cauchy sample are not finite (for details see Barnett, 1966). Similarly we see that though the Cauchy distribution defined by Eq. (1) belongs to $\mathscr{F}_1$, the RSS-A defined in section 5 for distributions of $\mathscr{F}_1$ also cannot be applied as such to the Cauchy distribution, for the reason that the units ranked $n-1$ and n do not possess finite variances. However we modify RSS-A developed in Section 5 in such a way that it is suitable to apply to the Cauchy distribution, and this is developed below.

DEFINITION 14. *Draw $n(n-2)$ independent units from an infinite population where the characteristic measured on the units follows a Cauchy distribution with pdf as given in Eq. (1). Now arrange the units randomly in $n-2$ sets each with n units. Suppose without direct measurement that an expert's judgement gives a perfect ranking on the relative largeness of the absolute values of the possible measurement values that one may obtain on the units. Now from the i^{th} set choose the unit ranked i, make the measurement on the variable of interest from this unit, take its absolute value, and denote the observation as $X_{(i:n)i}$, $i=1,2,...,n-2$. The above procedure of drawing units from the population is called Adjusted Ranked Set Sampling (ARSS-A).*

REMARK 15. *ARSS-A is related to RSS-A in the sense that the sample observations generated by ARSS-A are just obtained by censoring two of the most largely ranked observations from the sample to be generated by RSS-A.*

Now to estimate the scale parameter σ of the Cauchy distribution by the observations of ARSS-A, we state the following theorem.

THEOREM 16. *Let $\underset{\sim}{X}_{AA} = (X_{(1:n)1}, X_{(2:n)2}, ..., X_{(n-2:n)n-2})'$ be the vector of observations generated by an ARSS-A carried out on an infinite population and the characteristic measured on the units follow a Cauchy distribution defined in Eq. (1). Let $Y_{1:n}, Y_{2:n}, ...,$ $Y_{n-2:n}$ be the first $n-2$ order statistics of a random sample of size n drawn from the standard half-Cauchy distribution as defined by the pdf given in Eq. (4). Define $\alpha_{i:n} = E(Y_{i:n})$ and $\alpha_{i,i:n} = V(Y_{i:n})$ for $i=1,2,...,n-2$. Let $\underset{\sim}{\alpha}_{n-2} = (\alpha_{1:n}, \alpha_{2:n}, ..., \alpha_{n-2:n})'$ and let D_{n-2} be a diagonal matrix of order $n-2$ given by $D_{n-2} = diag(\alpha_{1,1:n}, \alpha_{2,2:n}, ..., \alpha_{n-2,n-2:n})$. Then the BLUE of σ based on the observations of ARSS-A is given by*

$$\hat{\sigma}_{AA} = (\underset{\sim}{\alpha}'_{n-2} D_{n-2}^{-1} \underset{\sim}{\alpha}_{n-2})^{-1} \underset{\sim}{\alpha}'_{n-2} D_{n-2}^{-1} \underset{\sim}{X}_{AA}, \tag{25}$$

and the variance is given by

$$Var(\hat{\sigma}_{AA}) = (\underset{\sim}{\alpha}'_{n-2} D_{n-2}^{-1} \underset{\sim}{\alpha}_{n-2})^{-1} \sigma^2. \tag{26}$$

The proof of the theorem is similar to that of Theorem 12 and hence omitted. The BLUE $\hat{\sigma}_{AA}$ as given in Eq. (25) may be also written as

$$\hat{\sigma}_{AA} = \sum_{i=1}^{n-2} h_{i,n} X_{(i:n)i}, \tag{27}$$

where $h_{i,n}$ for $i = 1, 2, ..., n-2$ are appropriate constants.

We have used the `Mathematica` software to evaluate $h_{i,n}, i = 1, 2, ..., n-2$, $\text{Var}(\hat{\sigma}_{AA})$ for $n = 4(1)12$. Results are presented in Table 5. There is an advantage with ARSS-A when compared to the corresponding modified form of McIntyre's RSS, which we denote as ARSS-M (Adjusted McIntyre's RSS) since in ARSS-A we get $n-2$ observations where as in ARSS-M we could utilize only $n-4$ observations for a fixed set size of units equal to n by leaving out the two lowest ranked and the two largest ranked units of the sets. However, to compare the advantage of ARSS-A, we consider the estimate of σ based on ARSS-M observations. In this case for a set size equal to n, we choose at random $n(n-4)$ units from the infinite population wherein the variable of interest follows the Cauchy distribution as defined in Eq. (1). Now we arrange the primary units selected randomly in $n-4$ sets each with n units. Now using a judgement method rank the units of the 1^{st} set, and select the unit ranked 3 from this set, similarly rank the units in the 2^{nd} set and select the unit ranked 4, proceed similarly and from the $(n-4)^{\text{th}}$ set select the unit which is ranked $n-2$. Now measure the characteristic of interest from the selected units. Let the observed observations be denoted by $X_{3:n(1)}, X_{4:n(2)}, ..., X_{n-2:n(n-4)}$. Clearly the above observations are independently distributed as they arise from different independent samples. Also $X_{i:n(i)}$ is distributed as the i^{th} order statistic of a random sample of size n drawn from Eq. (1). Suppose $Y_{3:n}, Y_{4:n}, ..., Y_{n-2:n}$ are the $n-4$ intermediate order statistics of a random sample of size n drawn from the standard Cauchy distribution with pdf of Eq. (2). Let $\text{E}(Y_{i:n}) = \beta_{i:n}$, $\text{V}(Y_{i:n}) = \beta_{i,i:n}$, $i = 3, 4, ..., n-2$, $\underset{\sim}{\beta}_{n-4} = (\beta_{3:n}, \beta_{4:n}, ..., \beta_{n-2:n})'$, and let the matrix G_{n-4} be defined as the diagonal matrix $G_{n-4} = diag(\beta_{3,3:n}, \beta_{4,4:n}, ..., \beta_{n-2,n-2:n})$. Then the estimate $\hat{\sigma}_M$ of σ of the Cauchy distribution based on the vector $\underset{\sim}{X}_M = (X_{3:n(1)}, X_{4:n(2)}, ..., X_{n-2:n(n-4)})'$ of ARSS-M is given by

$$\hat{\sigma}_M = (\underset{\sim}{\beta}'_{n-4} G_{n-4}{}^{-1} \underset{\sim}{\beta}_{n-4})^{-1} \underset{\sim}{\beta}'_{n-4} G_{n-4}{}^{-1} \underset{\sim}{X}_M, \tag{28}$$

and the variance is given by

$$\text{Var}(\hat{\sigma}_M) = (\underset{\sim}{\beta}'_{n-4} G_{n-4}{}^{-1} \underset{\sim}{\beta}_{n-4})^{-1} \sigma^2. \tag{29}$$

The numerical values of $\beta_{3:n}, \beta_{4:n}, ..., \beta_{n-2:n}$ and $\beta_{3,3:n}, \beta_{4,4:n}, ..., \beta_{n-2,n-2:n}$ have been computed and tabulated by Barnett (1966) for $n = 4(1)16(2)20$. Hence we have computed $\sigma^{-2}\,\text{Var}(\hat{\sigma}_M)$ and those values are presented in Table 5. To compare the efficiency of $\hat{\sigma}_{AA}$ relative to $\hat{\sigma}_M$ we have computed $e(\hat{\sigma}_{AA}/\hat{\sigma}_M) = \frac{\text{Var}(\hat{\sigma}_M)}{\text{Var}(\hat{\sigma}_{AA})}$ for $n = 6(1)12$ and those values as well are provided in Table 5.

TABLE 5

Coefficients $h_{i,n}$ of $X_{(i:n)i}$ involved in $\hat{\sigma}_{AA} = \sum_{i=1}^{n-2} h_{i,n} X_{(i:n)i}$, $Var(\hat{\sigma}_{AA})$, $Var(\hat{\sigma}_M)$ and $e(\hat{\sigma}_{AA}/\hat{\sigma}_M)$ using the observations of ARSS-A samples from the Cauchy distribution for $n = 4(1)12$.

n	**Coefficient** $h_{i,n}$ **of** $X_{(i:n)i}$ **in** $\hat{\sigma}_{AA} = \sum_{i=1}^{n-2} h_{i,n} X_{(i:n)i}$										σ^{-2}**Var**$(\hat{\sigma}_{AA})$	σ^{-2}**Var**$(\hat{\sigma}_M)$	$e(\hat{\sigma}_{AA}/\hat{\sigma}_M)$
	$X_{(1:n)1}$	$X_{(2:n)2}$	$X_{(3:n)3}$	$X_{(4:n)4}$	$X_{(5:n)5}$	$X_{(6:n)6}$	$X_{(7:n)7}$	$X_{(8:n)8}$	$X_{(9:n)9}$	$X_{(10:n)10}$			
4	1.326	0.548									0.579	-	
5	0.972	0.615	0.228								0.299	-	
6	0.756	0.571	0.330	0.116							0.184	4.166	22.704
7	0.613	0.509	0.358	0.196	0.067						0.124	1.594	12.831
8	0.512	0.451	0.353	0.236	0.125	0.042					0.090	0.852	9.497
9	0.439	0.400	0.335	0.251	0.163	0.084	0.028				0.068	0.532	7.853
10	0.382	0.358	0.313	0.252	0.184	0.116	0.059	0.019			0.053	0.365	6.872
11	0.338	0.322	0.290	0.246	0.193	0.137	0.085	0.043	0.014		0.043	0.266	6.227
12	0.303	0.292	0.269	0.236	0.195	0.150	0.105	0.065	0.032	0.010	0.035	0.203	5.769

From Table 5, we observe that the estimate of σ based on ARSS-A possess remarkably high efficiency when compared with that based on ARSS-M. The gain in efficiency of the estimate $\hat{\sigma}_{AA}$ ranges from 476% to 2170%.

7. A REAL LIFE APPLICATION

Mahdizadeh and Zamanzade (2019) have used data on annual returns for the year 1991 scored from the prices and other characteristics of 30 major companies traded on the German based Frankfurt Stock Exchange for the illustration of some inference problems of the Cauchy distribution. We take the data as such and present it below.

TABLE 6
Scores on annual returns of 30 major companies traded on Frankfurt Stock Exchange.

0.0011848	-0.0057591	-0.0051393	-0.0051781	0.0020043	0.0017787
0.0026787	-0.0066238	-0.0047866	-0.0052497	0.0004985	0.0068006
0.0016206	0.0007411	-0.0005060	0.0020992	-0.0056005	0.0110844
-0.0009192	0.0019014	-0.0042364	0.0146814	-0.0002242	0.0024545
-0.0003083	-0.0917876	0.0149552	0.0520705	0.0117482	0.0087458

One can easily observe that the number of observations in the above data with positive sign is 17 and those with negative sign is 13. To test the null hypothesis H_0 that the median m of the population from which the above data is obtained is equal to zero against the alternative $H_1 : m \neq 0$ we consider the test statistic of a sign test (assuming large n) $Z = \left| \frac{17-\frac{n}{2}}{\sqrt{\frac{n}{4}}} \right|$. Since $n = 30$, we have $Z = 0.7303$. For a two sided test the p-value is 0.4653.

Similarly, if we use the Wilcoxon signed-rank test, then we rank the absolute values of the observations and write $T+$ and $T-$ as the sum of ranks of positive and negative observations. If $T = \text{Min}(T+, T-)$, then $E(T) = \frac{n(n+1)}{4}$, $\text{Var}(T) = \frac{n(n+1)(2n+1)}{24}$. Then assuming n large we use the statistic $Z = \left| \frac{T-E(T)}{\sqrt{V(T)}} \right|$, and in this case we have Z=0.8536 with a p-value equal to 0.3934. Thus from both of the above tests there are no reasons against believing that the given data arises from a distribution belonging to $\mathscr{F}_1$.

We follow the claims of Nolan (2014) that the Cauchy distribution is a suitable model to describe financial return data. Mahdizadeh and Zamanzade (2019) have used the Cauchy distribution to deal with some similar inference problems. For the above data we also propose a Cauchy distribution with pdf in Eq. (1). To estimate the parameter σ of the Cauchy distribution, we use the estimator $\hat{\sigma}$ as given in Eq. (5) and its variance as given in Eq. (6). We have evaluated the means, variances and covariances of order statistics $Y_{1:30}, Y_{2:30}, ..., Y_{28:30}$ of a random sample of size 30 arising from the half-Cauchy distribution with pdf as given in Eq. (4). We have used those values in Eq. (5) to get the estimate $\hat{\sigma}$ based on the AOS of the sample. The estimate $\hat{\sigma}$ is given in Table 7. Mahdizadeh and Zamanzade (2019) estimated σ by an estimate $\hat{\hat{\sigma}}$ based on the half interquartile range, and this also is included in Table 7. We have computed the Kolmogrov-Smirnov (K-S) goodness-of-fit statistic for the fitted Cauchy distributions based on $\hat{\sigma}$ and $\hat{\hat{\sigma}}$ together with the respective p-values, and those values as well are included in Table 7. Further, we obtained the χ^2 statistics and their p-values with respect to the Cauchy distributions based on the estimates $\hat{\sigma}$ and $\hat{\hat{\sigma}}$. These are also given in the Table 7. The AIC and BIC values corresponding to each of the fitted Cauchy distributions based on $\hat{\sigma}$ and $\hat{\hat{\sigma}}$ have been computed as well, and they are provided in Table 7.

TABLE 7
Estimates $\hat{\sigma}$ and $\hat{\hat{\sigma}}$, K-S statistics and chi-square test statistics with respect to $\hat{\sigma}$, $\hat{\hat{\sigma}}$, the associated p-values, AIC and BIC values for modelling by a Cauchy distribution of the German stock market data on annual return scores.

Statistic	Estimate	K-S Statistic (p-value)	Chi-Square Test Statistic (p-value)	AIC	BIC
AOS based	$\hat{\sigma}=0.003385$	0.1194 (0.7414)	3.6000 (0.6083)	-192.153	-190.751
Half interquartile range based	$\hat{\hat{\sigma}}=0.003658$	0.1263 (0.67822)	4.0000 (0.5494)	-192.137	-190.736

From the Table 7, we see that the K-S goodness-of-fit statistic and the chi-square goodness-of-fit statistic are both smaller for the fitted Cauchy distribution with $\hat{\sigma}=0.003385$. The AIC and BIC values are also least for the Cauchy distribution with $\hat{\sigma}=0.003385$. This illustrates the advantage of using the minimal sufficient statistic based on AOS in estimating σ of the Cauchy distribution.

8. Conclusions

From the results generated in the previous sections, we conclude that the usual perception that the "vector of order statistics of a sample of size n drawn from the scale dependent Cauchy distribution forms a minimal sufficient statistic" needed the modification that the recently defined "vector of Absolved Order Statistics (AOS) of the sample forms the minimal sufficient statistic". We further conclude that the best linear unbiased estimate of the scale parameter σ of Cauchy distribution based on AOS is a more efficient estimate than that based on order statistics. We also conclude that the estimate of σ based on censoring on AOS is more realistic and efficient than that involving order statistics. A new method of RSS proposed in this paper known as Adjusted Ranked Set Sampling for the Cauchy distribution, which results in observations having distributions of AOS. This was found to be useful for estimating σ efficiently when the sampling involved is either costly or strenuous.

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APPENDIX

A. PROOFS

A.1. *Proof of Theorem 3*

PROOF. Let $X_1, X_2, ..., X_n$ be a random sample of size n drawn from the Cauchy distribution with pdf $f(x, \sigma)$. Now the absolute values $|X_1|, |X_2|, ..., |X_n|$ of the above observations are distributed as that of a random sample of size n drawn from the half-Cauchy distribution with pdf $g(x, \sigma) = 2f(x, \sigma), x \geq 0$, which is obtained by folding the density $f(x, \sigma)$ about $x = 0$. If we write the ordered values of $|X_1|, |X_2|, ..., |X_n|$ as $X_{(1:n)} \leq X_{(2:n)} \leq ... \leq X_{(n:n)}$, then clearly by definition $X_{(1:n)}, X_{(2:n)}, ..., X_{(n:n)}$ are the AOS of the given sample and they behave also as the order statistics of the random sample $|X_1|, |X_2|, ..., |X_n|$ arising from the half-Cauchy distribution. This proves that $(X_{(1:n)}, X_{(2:n)}, ..., X_{(n:n)})$ is distributed identically as the vector of order statistics of a random sample of size n arising from the half-Cauchy distribution. □

A.2. *Proof of Theorem 4.*

PROOF. The joint pdf of $\underline{X} = (X_1, X_2, ..., X_n)$ is $L(\underset{\sim}{x}, \theta) = \prod_{i=1}^{n} f_\theta(x_i) = \prod_{i=1}^{n} f_\theta(x_{i:n})$.

Since $f_\theta(x)$ is symmetric about zero, for any given set of reals $(x_1, x_2, ..., x_n)$, $\prod_{i=1}^{n} f_\theta(x_i)$ is a constant for all $n!$ permutations $(i_1, i_2, ..., i_n)$ of $(1, 2, ..., n)$ as well as 2^n ways of assigning a coefficient $(-1)^j$, $j = 1, 2$ with each of the components x_i in $(x_1, x_2, ..., x_n)$.

Thus if $T = (X_{(1:n)}, X_{(2:n)}, ..., X_{(n:n)})$ is the statistic defined by the AOS, then the conditional pdf of $\underline{X} = (X_1, X_2, ..., X_n)$ given $T = t = (x_{(1:n)}, x_{(2:n)}, ..., x_{(n:n)})$ is given by

$$L(\underline{X}|T = t) = \frac{1}{2^n n!},$$

which is a constant for Cauchy distribution with pdf in Eq. (1). This proves that the statistic $T = (X_{(1:n)}, X_{(2:n)}, ..., X_{(n:n)})$ based on AOS is sufficient for the Cauchy family of distributions. We now establish the minimal sufficient property of $(X_{(1:n)}, X_{(2:n)}, ..., X_{(n:n)})$.

Let $\mathscr{D}_1$ be the partition of the sample space Ω determined by the statistic $T = (X_{(1:n)}, X_{(2:n)}, ..., X_{(n:n)})$. Let $\underset{\sim}{x}$ be a sample point so that $\underset{\sim}{x} \in \Omega$. Since T is sufficient for the Cauchy family of distributions, using the representation given in Lindgren (1962), the partition set in $\mathscr{D}_1$, which contains $\underset{\sim}{x}$, is determined by

$$E = \{\underset{\sim}{y} / L(\underset{\sim}{y}, \theta) = h(\underset{\sim}{x}, \underset{\sim}{y}) L(\underset{\sim}{x}, \theta)\}, \tag{30}$$

where $h(\underset{\sim}{x}, \underset{\sim}{y})$ is independent of θ and is not equal to zero except for a set E_0 of probability measure zero. In particular, E is the set of all $\underset{\sim}{y} \in \Omega$ for which the ratio $\frac{L(\underset{\sim}{y},\theta)}{L(\underset{\sim}{x},\theta)}$ is independent of θ. Since T is sufficient for the Cauchy family of distributions, by the factorization theorem we write

$$L(\underset{\sim}{x},\theta) = w_1(\underset{\sim}{x})g_1(\underset{\sim}{t},\theta), \tag{31}$$

where w_1 is a function of $\underset{\sim}{x}$ alone, and the function $g_1(\underset{\sim}{t},\theta)$ depends on θ and $\underset{\sim}{x}$ only through $T = t$. Now suppose that $U(\underset{\sim}{X})$ is any other sufficient statistic that makes a partition $\mathscr{D}_2$ of the sample space Ω. Then the proof of the minimal sufficiency of T follows if we prove that each set $F \in \mathscr{D}_2$ is contained in some $E \in \mathscr{D}_1$ except possibly for a set of points with probability measure zero.

Now assume that $\underset{\sim}{x}, \underset{\sim}{y} \in F$, where $F \in \mathscr{D}_2$ so that we have $U(\underset{\sim}{x}) = U(\underset{\sim}{y})$. But $U(\underset{\sim}{x})$ is a sufficient statistic, and hence by factorization theorem we write

$$L(\underset{\sim}{x},\theta) = w_2(\underset{\sim}{x})g_2(U(\underset{\sim}{x}),\theta), \tag{32}$$

and

$$L(\underset{\sim}{y},\theta) = w_2(\underset{\sim}{y})g_2(U(\underset{\sim}{y}),\theta), \tag{33}$$

where $w_2(\underset{\sim}{x})$ and $w_2(\underset{\sim}{y})$ are not zero and independent of θ. As $U(\underset{\sim}{x}) = U(\underset{\sim}{y})$, we can further write

$$L(\underset{\sim}{x},\theta) = w_2(\underset{\sim}{x})g_2(U(\underset{\sim}{y}),\theta), \tag{34}$$

where w_2 is independent of θ. If $w_2(\underset{\sim}{x})$ is not zero, then from Equations (32),(33) and (34) we have

$$L(\underset{\sim}{y},\theta) = \frac{w_2(\underset{\sim}{y})}{w_2(\underset{\sim}{x})}L(\underset{\sim}{x},\theta). \tag{35}$$

Clearly, $\frac{w_2(\underset{\sim}{y})}{w_2(\underset{\sim}{x})}$ is not zero provided $w_2(\underset{\sim}{y})$ is not zero. Hence if $w_2(\underset{\sim}{y}) \neq 0$, then on comparing Eq. (35) with Eq. (30) we can deduce that $\underset{\sim}{x}, \underset{\sim}{y}$ belongs to the same partition set, say $E \in \mathscr{D}_1$. This establishes that $F \subset E$, except possibly for those points $\underset{\sim}{x}$ such that $w_2(\underset{\sim}{x}) = 0$, however for such points $L(\underset{\sim}{x},\theta) = 0$ for all θ, and the collection of all such points E_0 is null in the sense that it has zero probability measure. This establishes the theorem. □

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