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A NOTE ON FIBONACCI SEQUENCES OF RANDOM VARIABLES

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1. INTRODUCTION

Let $\{\Omega, \mathcal{F}, P\}$ be a probability space and $X_i \equiv X_i(\omega), \omega \in \Omega, i = 0, 1$ be absolutely continuous random variables defined on this probability space with joint probability density function (pdf) $f_{X_0,X_1}(x, y)$. Consider a sequence of random variables $X_n \equiv X_n(\omega), n \geq 1$ given in $\{\Omega, \mathcal{F}, P\}$ defined as $\{X_0, X_1, X_n = X_{n-2} + X_{n-1}, n = 2, 3, ..\}$. We call this sequence "the Fibonacci Sequence of Random Variables". It is clear that $X_2 = X_0 + X_1$, $X_3 = X_0 + 2X_1, ...$ and for any $n = 0, 1, 2, ...$ we have $X_n = a_{n-1}X_0 + a_nX_1$, where $\{a_n = a_{n-2} + a_{n-1}, n = 2, 3, ...; a_0 = 0, a_1 = 1, a_2 = 1\}$ is the Fibonacci sequence $\mathbf{F} \equiv \{0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ...\}$. It is also clear that the Fibonacci Sequence of Random Variables (FSRV) $X_n, n = 0, 1, 2, ...$ is the sequence of dependent random variables based on initial random variables X_0 and X_1, which fully defined by the members of the Fibonacci sequence $\mathbf{F}$. We are interested in the behavior of FSRV, i.e. the distributional properties of X_n and joint distributions of X_n and X_{n+k} for any n and k.

This paper is organized as follows. In Section 2, the probability density function of X_n is considered, followed by a discussion of two cases where X_0 and X_1 have Exponential and Uniform distributions, respectively. Then, there is an investigation of limit behavior of ratios of some characteristics of pdf of X_n for large n. In the considered examples, the ratio of maximums of the pdfs, modes and expected values of consecutive elements of FSRV converge to golden ratio $\varphi \equiv \frac{1+\sqrt{5}}{2} = 1,6180339887...$. The ratio X_{n+1}/X_n and normalized sums of X_n's for large n are discussed in Section 3. In Section 4, the focus is on the joint distributions of X_n and X_{n+k}, for $2 \leq k \leq n$ and on the prediction of X_{n+k} given X_n.

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2. Distributions

Consider $X_n = a_{n-1}X_0 + a_n X_1$, $n = 0, 1, 2, \dots$, where X_0 and X_1 are absolutely continuous random variables with joint pdf $f_{X_0,X_1}(x,y)$, $(x,y) \in \mathbb{R}^2$ and $a_n, n = 0, 1, 2, \dots$ is the Fibonacci sequence. Denote by f_0 and f_1 the marginal pdf's of X_0 and X_1, respectively.

THEOREM 1. *The pdf of X_n is*

$$f_{X_n}(x) = \frac{1}{a_n a_{n-1}} \int_{-\infty}^{\infty} f_{X_0,X_1}\left(\frac{x-t}{a_{n-1}}, \frac{t}{a_n}\right) dt. \tag{1}$$

If X_0 and X_1 are independent, then

$$f_{X_n}(x) = \frac{1}{a_n a_{n-1}} \int_{-\infty}^{\infty} f_{X_0}\left(\frac{x-t}{a_{n-1}}\right) f_{X_1}\left(\frac{t}{a_n}\right) dt. \tag{2}$$

PROOF. Equations (1) and (2) are straightforward results of distributions of linear functions of random variables (see eg., (Feller, 1971), (Ross, 2016), (Gnedenko, 1978) , (Skorokhod, 2005)) □

CASE 1. *Exponential distribution. Let X_0 and X_1 be independent and identically distributed (iid) random variables having Exponential distribution with parameter $\lambda = 1$. Then the pdf of X_n is*

$$f_{X_n}(x) = \frac{1}{a_{n-2}} \left\{ \exp\left(\frac{x a_{n-2}}{a_{n-1} a_n} \right) - 1 \right\} \exp\left(-\frac{x}{a_{n-1}}\right), x \geq 0,\ n = 3, 4, \dots \tag{3}$$

$$f_{X_2}(x) = x \exp(-x), x \geq 0.$$

In Figure 1, the graphs of $f_{X_n}(x)$ for different values of n are presented.

The expected value of X_n is

$$E(X_n) = \frac{1}{a_{n-2}} \left(\int_0^{\infty} x \exp\left(-x \left(\frac{a_n - a_{n-2}}{a_{n-1} a_n} \right) \right) dx - \int_0^{\infty} x \exp\left(-\frac{x}{a_{n-1}}\right) \right) dx$$

$$= \frac{1}{a_{n-1}} \left(\frac{a_n^2 a_{n-1}^2}{(a_n - a_{n-2})^2} - a_{n-1}^2 \right) = a_{n+1}.$$

and variance is

$$Var(X_n) = \frac{1}{a_{n-2}} \left(\int_0^{\infty} x^2 \exp\left(-x \left(\frac{a_n - a_{n-2}}{a_{n-1} a_n} \right) \right) dx - \int_0^{\infty} x^2 \exp\left(-\frac{x}{a_{n-1}}\right) \right) dx$$

$$= a_{2n-1}.$$

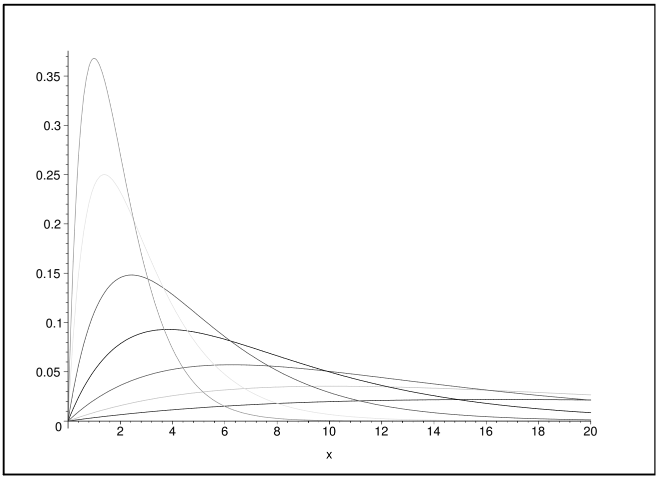

Figure 1 – Graphs of $f_{X_n}(x)$, $n = 2, 3, 4, 5, 6, 7, 8$, given in (3).

THEOREM 2. *Let X_0 and X_1 be iid random variables having Exponential distribution with parameter $\lambda = 1$. Let $M_n = \max_{0<x<\infty} f_{X_n}(x)$ and $x_n^* = \arg\max_{0<x<\infty} f_{X_n}(x)$ be the maximum of $f_{X_n}(x)$ and mode of X_n, $n = 2, 3, \ldots$, respectively. Then*

$$\lim_{n\to\infty} \frac{M_n}{M_{n+1}} = \lim_{n\to\infty} \frac{x_{n+1}^*}{x_n^*} = \lim_{n\to\infty} \frac{E(X_{n+1})}{E(X_n)} = \varphi$$

and

$$\lim_{n\to\infty} \frac{Var(X_{n+1})}{Var(X_n)} = \varphi^2,$$

where

$$\varphi \equiv \frac{1+\sqrt{5}}{2} = 1,6180339887\ldots$$

is the golden ratio.

PROOF. The following can easily be verified

$$\frac{d}{dx} f_{X_n}(x) = \frac{(-\frac{x}{a_{n-1}})(e^{\frac{x a_{n-2}}{a_{n-1}a_n}} - 1)}{a_{n-2}a_{n-1}} + \frac{e^{-\frac{x}{a_{n-1}}} e^{\frac{x a_{n-2}}{a_{n-1}a_n}}}{a_{n-1}a_n} = 0. \tag{4}$$

The equation (4) has unique solution

$$x_n^* = \frac{a_{n-1}a_n \ln\left(\frac{a_n}{a_n - a_{n-2}}\right)}{a_{n-2}}.$$

Therefore X_n is unimodal and we have

$$M_n = f_{X_n}(x_n^*) = \frac{1}{a_n - a_{n-2}}\left(\frac{a_n}{a_n - a_{n-2}}\right)^{-\frac{a_n}{a_{n-2}}}$$

$$M_{n+1} = f_{X_{n+1}}(x^*) = \frac{1}{a_{n+1} - a_{n-1}}\left(\frac{a_{n+1}}{a_{n+1} - a_{n-1}}\right)^{-\frac{a_{n+1}}{a_{n-1}}}$$

and using

$$\lim_{n\to\infty}\frac{a_{n+\alpha}}{a_n} = \varphi^{\alpha}$$

we obtain

$$\lim_{n\to\infty}\frac{M_n}{M_{n+1}} = \frac{x_{n+1}^*}{x_n^*} = \varphi.$$

□

CASE 2. *Uniform distribution. Let X_0 and X_1 be iid with $Uniform(0,1)$ distribution. Then from (2) we obtain*

$$\begin{aligned} f_{X_n}(x) &= \frac{1}{a_n a_{n-1}}\int_0^{a_n} f_{X_0}\left(\frac{x-t}{a_{n-1}}\right)dt = \frac{a_{n-1}}{a_n a_{n-1}}\int_0^{a_n} dF_{X_0}\left(\frac{x-t}{a_{n-1}}\right) \\ &= \frac{1}{a_n}\left\{F_{X_0}\left(\frac{x}{a_{n-1}}\right) - F_{X_0}\left(\frac{x-a_n}{a_{n-1}}\right)\right\} \\ &= \begin{cases} 0, & x < 0 \text{ and } x > a_n + a_{n-1} \\ \frac{x}{a_n a_{n-1}} & 0 \le x \le a_{n-1} \\ \frac{1}{a_n} & a_{n-1} \le x \le a_n \\ \frac{1}{a_n}\left(1 - \frac{x-a_n}{a_{n-1}}\right) & a_n \le x \le a_n + a_{n-1} \end{cases}. \end{aligned} \tag{5}$$

In Figure 2, the graphs of $f_{X_n}(x)$ for different values of n are presented. It can be easily verified that $E(X_n) = \frac{a_{n-1}+a_n}{2} = \frac{a_{n+1}}{2}$ and $varX_n) = \frac{a_{n-1}^2 + a_n^2}{12}$.

One can observe that $f_{X_n}(x)$ is not unimodal, $f_{X_n}(x)$ is constant in the interval (a_{n-1}, a_n) and $M_n = \max_{0<x<1} f_{X_n}(x) = \frac{1}{a_{n-1}}$, $\inf \arg\min_{0<x<1} f_{X_n}(x) = a_{n-1}$, $\sup \arg\min_{0<x<1} f_{X_n}(x) = a_n$.

It is not difficult to observe that the similar to Theorem 1 results hold also in this case.

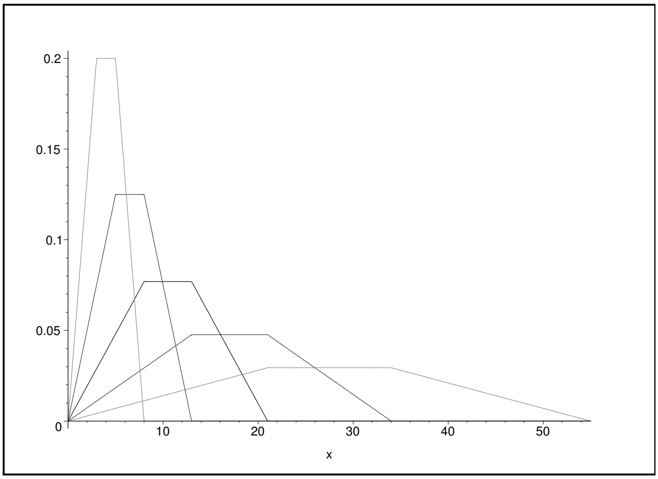

Figure 2 – Graphs of $f_{X_n}(x), n = 5, 6, 7, 8, 9$, given in (5).

3. Large n and normalized Fibonacci sequence of random variables

Let $X_n = a_{n-1}X_0 + a_nX_1$, $n = 0, 1, 2, ...$ be FSRV, where X_0 and X_1 are absolutely continuous random variables with joint pdf $f_{X_0,X_1}(x, y)$, $(x, y) \in \mathbb{R}^2$. Consider the sequence of random variables $Z_n \equiv \frac{X_{n+1}}{X_n}, n = 1, 2, ...$. One has

$$\begin{aligned} Z_n(\omega) &= \frac{X_{n+1}(\omega)}{X_n(\omega)} \\ &= \frac{a_{n+1}X_1(\omega) + a_nX_0(\omega)}{a_nX_1(\omega) + a_{n-1}X_0(\omega)} \\ &= \frac{\frac{a_{n+1}}{a_n}X_1(\omega) + X_0(\omega)}{X_1(\omega) + \frac{a_{n-1}}{a_n}X_0(\omega)} \\ &= \frac{\frac{a_{n+1}}{a_n}X_1(\omega) + X_0(\omega)}{X_1(\omega) + \frac{1}{\frac{a_n}{a_{n-1}}}X_0(\omega)}. \end{aligned}$$

Since $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \varphi$, ($\varphi = \frac{1+\sqrt{5}}{2} = 1,6180339887...$ is the golden ratio), it follows that

$$\begin{aligned} &Z_n(\omega) \to \varphi, \text{ pointwise in} \\ &\quad \Omega_1 = \{\omega : \varphi X_1 + X_0 \neq 0 \text{ and } \varphi X_1 + X_0 \neq \infty\} \subset \Omega \end{aligned}$$

For the normalized FSRV, the following limit relationship is valid. (Here we use "sure convergence or pointwise convergence" as follows: to say that the sequence of random variables Z_n,$n = 1.2,...$ defined over the same probability space converges surely or everywhere or pointwise towards Z means

$$\lim_{n\to\infty} Z_n(\omega) = Z(\omega) \text{ for all } \omega \in \Omega$$

where Ω is the sample space of the underlying probability space where the random variables are defined. This is the notion of pointwise convergence of a sequence of functions extended to a sequence of random variables. Sure convergence (pointwise convergence) of a random variable implies all the other kinds of convergence of sequences of random variables.)

THEOREM 3. *Let* $E(X_i) = \mu_i, Var(X_i) = \sigma_i^2, i = 0, 1$ *and*

$$Y_n(\omega) \equiv Y_n = \frac{X_n - E(X_n)}{\sqrt{Var(X_n)}} = \frac{X_0 + \frac{a_n}{a_{n-1}} X_1 - (\mu_0 + \frac{a_n}{a_{n-1}}\mu_1)}{\sqrt{\sigma_0^2 + \frac{a_n^2}{a_{n-1}^2}\sigma_1^2}}, \ \omega \in \Omega.$$

Then,

$$Y_n \to Y \equiv \frac{X_0 + \varphi X_1 - (\mu_0 + \varphi\mu_1)}{\sqrt{\sigma_0^2 + \varphi^2\sigma_1^2}}, \text{ as } n \to \infty \text{ for all } \omega \in \Omega.$$

The limiting random variable $Y \equiv Y(\omega)$ *has distribution function (cdf)*

$$P\{Y \leq x\} = P\{X_0 + \varphi X_1 \leq x\sqrt{\sigma_0^2 + \varphi^2\sigma_1^2} + (\mu_0 + \varphi\mu_1)\}. \tag{6}$$

It is clear that the pdf of $X_0 + \varphi X_1$ is

$$f_{X_0+\varphi X_1}(x) = \frac{1}{\varphi}\int_0^\infty f_{X_0}(x-t) f_{X_1}(\frac{t}{\varphi})dt. \tag{7}$$

and the pdf of Y is then

$$f_Y(x) = \sqrt{\sigma_0^2 + \varphi^2\sigma_1^2} f_{X_0+\varphi X_1}(x\sqrt{\sigma_0^2 + \varphi^2\sigma_1^2} + (\mu_0 + \varphi\mu_1)) \tag{8}$$

EXAMPLE 4. *Let* X_0 *and* X_1 *be iid random variables having Exponential distribution with parameter* $\lambda = 1$, *then from (7) we have*

$$\begin{aligned} f_{X_0+\varphi X_1}(x) &= \frac{1}{\varphi}\int_0^x \exp(-x-t)\exp(t - \frac{t}{\varphi})dt \\ &= \frac{\exp(-x)}{\varphi - 1}[\exp(x(1-\frac{1}{\varphi})) - 1]. \end{aligned}$$

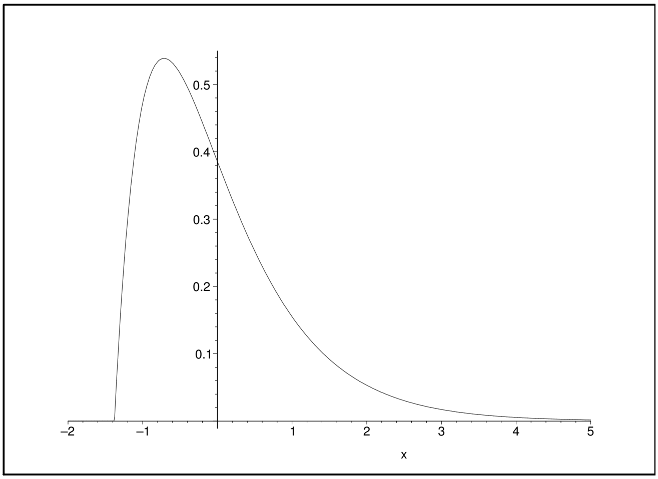

Figure 3 – The graph of pdf $f_Y(x)$.

Therefore,

$$P\{Y \le x\} = P\{X_0 + \varphi X_1 \le x\sqrt{\sigma_0^2 + \varphi^2\sigma_1^2} + (\mu_0 + \varphi\mu_1)\}$$
$$= \int_0^{c(x)} \left\{ \frac{\exp(-t)}{\varphi - 1}(\exp(t(1 - \frac{1}{\varphi})) - 1) \right\} dt,$$

where $c(x) = x\sqrt{\sigma_0^2 + \varphi^2\sigma_1^2} + (\mu_0 + \varphi\mu_1)$. *And the pdf is*

$$f_Y(x) = \begin{cases} \sqrt{\sigma_0^2 + \varphi^2\sigma_1^2}\left\{\frac{\exp(-c(x))}{\varphi-1}\left[\exp\left(c(x)(1-\frac{1}{\varphi})\right)-1\right]\right\}, & x \ge -\frac{(\mu_0+\varphi\mu_1)}{\sqrt{\sigma_0^2+\varphi^2\sigma_1^2}} \\ 0 & Otherwise \end{cases}$$
$$= \begin{cases} \sqrt{1+\varphi^2}\left\{\frac{\exp(-c(x))}{\varphi-1}\left[\exp\left(c(x)(1-\frac{1}{\varphi})\right)-1\right]\right\}, & x \ge -\frac{1+\varphi}{\sqrt{1+\varphi^2}} \\ 0 & Otherwise \end{cases}.$$

Figure 3 shows the graph of pdf $f_Y(y)$*.*

EXAMPLE 5. *Let* X_0 *and* X_1 *be independent random variables with* $Uniform(0,1)$ *distribution. Then from (7) we have*

$$f_{X_0+\varphi X_1}(x) = \begin{cases} \frac{x}{\varphi}, & 0 \le x \le 1 \\ \frac{1}{\varphi}, & 1 \le x \le \varphi \\ \frac{1-x}{\varphi} + 1, & \varphi \le x \le 1+\varphi \\ 0, & elsehwere \end{cases}.$$

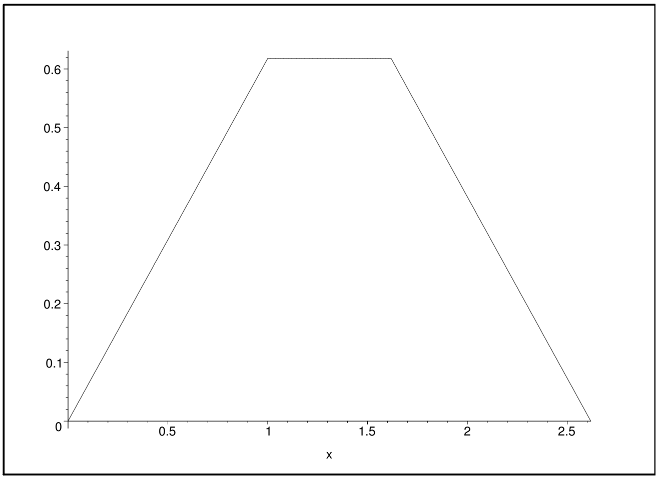

Figure 4 – The graph of $f_{X_0+\varphi X_1}(x)$.

This is a trapezoidal pdf with graph given in Figure 4.

To find the distribution of limiting random variable Y, we consider

$$P\{Y \le x\} = P\{X_0 + \varphi X_1 \le x\sqrt{\sigma_0^2 + \varphi^2\sigma_1^2} + (\mu_0 + \varphi\mu_1)\}.$$

It is clear that

$$\mu_0 = \mu_1 = 1/2,\ \sigma_0^2 = \sigma_1^2 = 1/12,$$
$$a = \sqrt{\sigma_0^2 + \varphi^2\sigma_1^2} = \sqrt{\frac{1+\varphi^2}{12}}, b = \mu_0 + \varphi\mu_1 = \frac{1+\varphi}{2}$$

and the cdf of Y is

$$F_Y(x) = P\{Y \le x\} = P\{X_0 + \varphi X_1 \le ax + b\} =$$
$$\begin{cases} 0 & x \le -\frac{b}{a} \\ \frac{1}{\varphi}\int_0^{ax+b} u\,du = \frac{(ax+b)^2}{2\varphi}, & -\frac{b}{a} \le x \le \frac{1-b}{a} \\ \frac{1}{2\varphi} + \frac{1}{\varphi}\int_1^{ax+b} du = \frac{1}{2\varphi} + \frac{ax+b-1}{\varphi}, & \frac{1-b}{a} \le x \le \frac{\varphi-b}{a} \\ \quad \frac{1}{2\varphi} + \frac{1}{\varphi} + \frac{1}{\varphi}\int_\varphi^{ax+b} (\frac{1-u}{\varphi} + 1)du & \\ = \frac{2ax+2b+2ax\varphi+2b\varphi-a^2x^2-2axb-b^2-\varphi^2-1}{2\varphi} & \frac{\varphi-b}{a} \le x \le \frac{1+\varphi-b}{a} \\ 1 & x \ge \frac{1+\varphi-b}{a}. \end{cases}$$

The pdf of Y is

$$f_Y(x)=\begin{cases} 0, & x<-\frac{b}{a} \text{ or } x>\frac{1+\varphi-b}{a} \\ \frac{(ax+b)a}{\varphi}, & -\frac{b}{a}<x\le\frac{1-b}{a} \\ \frac{a}{\varphi} & \frac{1-b}{a}<x\le\frac{\varphi-b}{a} \\ \frac{a(1+\varphi-b-ax)}{\varphi} & \frac{\varphi-b}{a}<x\le\frac{1+\varphi-b}{a}. \end{cases}$$

3.1. Limits of normalized sums of Fibonacci sequence of random variables

Here we are interested in the limiting behavior of sums of members of FSRV. Consider $S_n=\sum_{i=0}^{n}X_i$. We have

$$\begin{aligned} S_n &= X_0+X_1+\cdots+X_n = X_0+X_1+\sum_{i=2}^{n}X_i \\ &= X_0+X_1+\sum_{i=2}^{n}(a_{i-1}X_0+a_iX_1) \\ &= X_0+X_1+X_0\sum_{i=2}^{n}a_{i-1}+X_1\sum_{i=2}^{n}a_i \\ &= X_0+X_1+X_0\sum_{i=1}^{n-1}a_i+X_1(\sum_{i=1}^{n}a_i-a_1) \\ &= X_0+X_1+X_0(a_{n+1}-1)+X_1(a_{n+2}-1-a_1) \\ &= a_{n+1}X_0+(a_{n+2}-1)X. \end{aligned}$$

Since

$$\sum_{i=1}^{n}a_i=a_{n+2}-1.$$

Therefore

$$\begin{aligned} S_n &= X_0+X_1+\cdots+X_n \\ &= a_{n+1}X_0+(a_{n+2}-1)X_1. \end{aligned}$$

The pdf of S_n is

$$f_{S_n}(x)=\frac{1}{a_{n+1}(a_{n+1}-1)}\int_{-\infty}^{\infty}f_{X_0}(\frac{x-t}{a_{n+1}})f_{X_1}(\frac{t}{a_{n+2}-1})dt. \tag{9}$$

THEOREM 6. *Under conditions of Theorem 3 for a sequence X_0, X_1, $X_n = a_{n-1}X_0 + a_nX_1$, $n = 2,3,...$ we have*

$$E(S_n) = a_{n+1}\mu_0 + (a_{n+2} - 1)\mu_1$$
$$Var(S_n) = a_{n+1}^2\sigma_0^2 + (a_{n+2} - 1)^2\sigma_1^2$$

$$\frac{S_n - E(S_n)}{\sqrt{Var(S_n)}} \to Y \text{ as } n \to \infty, \text{ for all } \omega \in \Omega,$$

where Y has cdf (6).

PROOF. Indeed,

$$\frac{S_n - E(S_n)}{\sqrt{\mathrm{Var}(S_n)}} = \frac{X_0 + (\frac{a_{n+2}}{a_{n+1}} - \frac{1}{a_{n+1}})X_1 - (\mu_0 + (\frac{a_{n+2}}{a_{n+1}} - \frac{1}{a_{n+1}})\mu_1}{\sqrt{\sigma_0^2 + (\frac{a_{n+2}}{a_{n+1}} - \frac{1}{a_{n+1}})^2\sigma_1^2},} \to \frac{X_0 + \varphi X_1 - (\mu_0 + \varphi\mu_1)}{\sqrt{\sigma_0^2 + \varphi^2\sigma_1^2}} = Y, \text{ as } n \to \infty.$$

EXAMPLE 7. *Let X_0 and X_1 be iid Exponential(1) random variables. Then the pdf of S_n is*

$$f_{S_n}(x) = \frac{1}{a_{n+1}(a_{n+2} - 1)} \int_0^x \exp(\frac{x - t}{a_{n+1}}) \exp(\frac{t}{a_{n+2} - 1}) dt$$
$$= \frac{\exp(-\frac{x}{a_{n+1}})}{a_{n+1} - a_{n+2} + 1} \left(1 - \exp(-x\left(\frac{1}{a_{n+2} - 1} - \frac{1}{a_{n+1}}\right)\right). \tag{10}$$

4. JOINT DISTRIBUTIONS OF X_n AND X_{n+k}

Next, we focus on the joint distributions of $X_n = a_{n-1}X_0 + a_nX_1$ and $X_{n+k} = a_{n+k-1}X_0 + a_{n+k}X_1$, for $k \geq 1$.

THEOREM 8. *The joint pdf of X_n and X_{n+k} is*

$$f_{X_n, X_{n+k}}(y_0, y_1) = \frac{1}{a_k} f_{X_0, X_1}\left(\frac{a_{n+k}y_0 - y_1 a_n}{(-1)^n a_k}, \frac{a_{n-1}y_1 - a_{n+k-1}y_0}{(-1)^n a_k}\right). \tag{11}$$

PROOF. Let

$$\begin{cases} y_0 = a_{n-1}x_0 + a_n x_1 \\ y_1 = a_{n+k-1}x_0 + a_{n+k}x_1 \end{cases}. \tag{12}$$

The Jacobian of this linear transformation is $J = a_{n-1}a_{n+k} - a_n a_{n+k-1}$ and the solution of the system of equations (12) is

$$\begin{cases} x_0 = (a_{n+k}y_0 - y_1 a_n)/(a_{n-1}a_{n+k} - a_n a_{n+k-1}) \\ x_1 = (a_{n-1}y_1 - a_{n+k-1}y_0)/(a_{n-1}a_{n+k} - a_n a_{n+k-1}) \end{cases}.$$

Therefore, the joint pdf of X_n and X_{n+k} is

$$\begin{aligned} &f_{X_n,X_{n+k}}(y_0,y_1) \\ &= \frac{1}{|a_{n-1}a_{n+k} - a_n a_{n+k-1}|} f_{X_0,X_1}(\frac{a_{n+k}y_0 - y_1 a_n}{a_{n-1}a_{n+k} - a_n a_{n+k-1}}, \\ &\frac{a_{n-1}y_1 - a_{n+k-1}y_0}{a_{n-1}a_{n+k} - a_n a_{n+k-1}}). \end{aligned} \tag{13}$$

Using the d'Ocagne's identity (Dickson, 1966) $a_m a_{n+1} - a_{m+1}a_n = (-1)^n a_{m-n}$ we have $J = a_{n-1}a_{n+k} - a_n a_{n+k-1} = -(a_{n+k-1}a_n - a_{n+k}a_{n-1}) = (-1)^n a_k$. Therefore,

$$\begin{aligned} &f_{X_n,X_{n+k}}(y_0,y_1) \\ &= \frac{1}{a_k} f_{X_0,X_1}\left(\frac{a_{n+k}y_0 - y_1 a_n}{(-1)^n a_k}, \frac{a_{n-1}y_1 - a_{n+k-1}y_0}{(-1)^n a_k}\right). \end{aligned}$$

□

COROLLARY 9. *If X_0 and X_1 are independent then*

$$\begin{aligned} &f_{X_n,X_{n+k}}(x,y) \\ &= \frac{1}{a_k} f_{X_0}\left(\frac{a_{n+k}x - y a_n}{(-1)^n a_k}\right) f_{X_1}\left(\frac{a_{n-1}y - a_{n+k-1}x}{(-1)^n a_k}\right). \end{aligned} \tag{14}$$

EXAMPLE 10. *Let X_0 and X_1 be iid Exponential(1) random variables, $n = 4, k = 3$. Then $a_{n+k} = a_7 = 13$, $a_{n+k-1} = a_6 = 8$, $a_{n-1} = a_3 = 2$, $a_n = a_4 = 3$ and $a_k = a_3 = 2$. Then from (14)*

$$\begin{aligned} &f_{X_4,X_7}(x,y) \\ &= \begin{cases} \frac{1}{2}\exp(-(13/2)x + (3/2)y) \\ \quad \times \exp(-y + 4x), & x \geq 0 \text{ and } 4x \leq y \leq 13/3x \\ 0 & \text{otherwise} \end{cases}. \\ &= \begin{cases} \frac{1}{2}\exp(-(5/2)x)\exp(y/2) & x \geq 0 \text{ and } 4x \leq y \leq 13/3x \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{15}$$

The marginal pdf's are

$$f_{X_4}(x) = \begin{cases} e^{-\frac{x}{3}} - e^{-\frac{x}{2}}, & x \geq 0 \\ 0, & otherwise \end{cases} \tag{16}$$

and

$$f_{X_7}(x) = \begin{cases} \frac{1}{5}\left(e^{-\frac{x}{13}} - e^{-\frac{x}{8}}\right), & x \geq 0 \\ 0, & otherwise \end{cases} .$$

EXAMPLE 11. *Let X_0 and X_1 be independent Uniform(0,1) random variables. Again, let $n = 4, k = 3$. Then $a_{n+k} = a_7 = 13$, $a_{n+k-1} = a_6 = 8$, $a_{n-1} = a_3 = 2$, $a_n = a_4 = 3$ and $a_k = a_3 = 2$. Then*

$$\begin{aligned} & f_{X_n,X_{n+k}}(x,y) \\ &= \frac{1}{a_k} f_{X_0}\left(\frac{a_{n+k}x - ya_n}{(-1)^n a_k}\right) f_{X_1}\left(\frac{a_{n-1}y - a_{n+k-1}x}{(-1)^n a_k}\right) \\ &= \begin{cases} \frac{1}{a_k} & 0 \leq \frac{a_{n+k}x - ya_n}{(-1)^n a_k} \leq 1, 0 \leq \frac{a_{n-1}y - a_{n+k-1}x}{(-1)^n a_k} \leq 1 \\ 0, & otherwise \end{cases} \end{aligned} \tag{17}$$

(To check whether (17) is a pdf, we need to show $\int_0^1 \int_0^1 f_{X_n,X_{n+k}}(x,y)dxdy = 1$. *Indeed,*

$$\begin{aligned} & \int_0^1 \int_0^1 f_{X_n,X_{n+k}}(x,y)dxdy \\ &= \frac{1}{a_k} \iint\limits_{0 \leq \frac{a_{n+k}x - ya_n}{(-1)^n a_k} \leq 1, 0 \leq \frac{a_{n-1}y - a_{n+k-1}x}{(-1)^n a_k} \leq 1} dxdy \\ &= \left\{ \begin{array}{c} a_{n+k}x - ya_n = t, a_{n-1}y - a_{n+k-1}x = s \\ x = \frac{ta_{n-1} + sa_n}{(-1)^n a_k}, y = \frac{sa_{n+k} + ta_{n+k-1}}{(-1)^n a_k} \\ t \leq (-1)^n a_k, s \leq (-1)^n a_k \\ J = \begin{vmatrix} \frac{a_{n-1}}{(-1)^n a_k} & \frac{a_n}{(-1)^n a_k} \\ \frac{a_{n+k-1}}{(-1)^n a_k} & \frac{a_{n+k}}{(-1)^n a_k} \end{vmatrix} = \frac{a_{n-1}a_{n+k} - a_n a_{n+k-1}}{(-1)^{2n} a_k^2} = \frac{(-1)^n a_k}{(-1)^{2n} a_k^2} \end{array} \right\} \\ &= \frac{1}{a_k} \int_0^{(-1)^n a_k} \int_0^{(-1)^n a_k} \frac{1}{|(-1)^n a_k|} dxdy = 1.) \end{aligned}$$

For $n=4$ and $k=3$, the

$$f_{X_4,X_7}(x,y) = \frac{1}{2} f_{X_0}\left(\frac{13x-3y}{2}\right) f_{X_1}\left(\frac{2y-8x}{2}\right) = \begin{cases} \frac{1}{2}, & 0 \le \frac{13x-3y}{2} \le 1, 0 \le \frac{2y-8x}{2} \le 1 \\ 0, & otherwise \end{cases}. \tag{18}$$

5. Prediction of future values

It is well known that with respect to squared error loss, the best unbiased predictor of X_{n+k}, given X_n is

$$E\{X_{n+k} \mid X_n\}.$$

Let

$$g(x) = E\{X_{n+k} \mid X_n = x\} = \frac{1}{f_{X_n}(x)} \int_{-\infty}^{\infty} y f_{X_n,X_{n+k}}(x,y) dy, \tag{19}$$

then $E\{X_{n+k} \mid X_n\} = g(X_n)$. Using (1) and (11) from (19) one can easily calculate the best predictor of X_{n+k}, given X_n.

EXAMPLE 12. *Let X_0 and X_1 be independent Exponential(1) random variables. Let $n=4, k=3$. Then $a_{n+k} = a_7 = 13$, $a_{n+k-1} = a_6 = 8$, $a_{n-1} = a_3 = 2$, $a_n = a_4 = 3$ and $a_k = a_3 = 2$ as in Example (4). Then from (15) we can write*

$$g(x) = \frac{1}{e^{-\frac{x}{3}} - e^{-\frac{x}{2}}} \int_{4x}^{13/3x} y \frac{1}{2} \exp(-(5/2)x) \exp(y/2) dy = \frac{1}{3} \frac{12e^{-x/2} - 6e^{-x/2} + 6e^{-x/3} - 13e^{-x/3}}{e^{-x/2} - e^{-x/3}} = 4x - 2 - \frac{x}{3(e^{-x/6}-1)},$$

Therefore,

$$X_7 \simeq 4X_4 - 2 - \frac{X_4}{3(e^{-X_4/6}-1)}.$$

6. Conclusion

In this note, we considered the sequence of random variables $\{X_0, X_1, X_n = X_{n-2} + X_{n-1}, n = 2,3,..\}$ which is equivalent to $\{X_0, X_1, X_n = a_{n-1}X_0 + a_n X_1, n = 2,3,...\}$, where X_0 and X_1 are absolutely continuous random variables with joint pdf f_{X_0,X_1}, and $a_n = a_{n-1} + a_{n-2}$, $n = 2,3,...$ ($a_0 = 0$, $a_1 = 1$) is the Fibonacci sequence. In the paper, the sequence $X_n, n = 0,1,2,...$ is referred to as the Fibonacci Sequence of Random Variables. We investigated the limiting properties of some ratios and normalizing sums of this sequence. For Exponential and Uniform distribution cases, we derived some interesting limiting properties that reduce to the golden ratio and also investigated the joint distributions of X_n and X_{n+k}. The considered random sequence has benefical properties and may be worthy of attention associated with random sequences and autoregressive models.

The FSRV can also find wide applications in many areas including biology, economy, finance. Recently, Fibonacci sequence and Golden Ratio aroused interest of many researchers in fields of science including high energy physics, quantum mechanics, cryptography, and coding. In these areas the randomness is a reality of the nature and the FSRV may play an important role.

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References

L. E. Dickson (1966). *History of the Theory of Numbers*. Chelsea Publishing Company, New York.

W. Feller (1971). *An Introduction to Probability Theory and Its Applications*. John Wiley & Sons Inc., New York, London, Sydney.

B. Gnedenko (1978). *The Theory of Probability*. Mir Publishers, Moscow.

S. Ross (2016). *A First Course in Probability*. Prentice-Hall Inc, New Jersey.

A. Skorokhod (2005). *Basic Principles and Applications of Probability Theory*. Springer, Berlin.

Summary

The aim of this paper is to introduce and investigate the new random sequence in the form $\{X_0, X_1, X_n = X_{n-2} + X_{n-1}, n = 2,3,..\}$, referred to as Fibonacci Sequence of Random Variables (FSRV).

The initial random variables X_0 and X_1 are assumed to be absolutely continuous with joint probability density function (pdf) f_{X_0,X_1}. The FSRV is completely determined by X_0 and X_1 and the members of Fibonacci sequence $\mathbf{F} \equiv \{0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots\}$. We examine the distributional and limit properties of the random sequence $X_n, n = 0, 1, 2, \ldots$.

Keywords: Random variable; Distribution function; Probability density function; Sequence of random variables.