

VARIANCE ESTIMATION USING MULTIAUXILIARY INFORMATION FOR RANDOM NON-RESPONSE IN SURVEY SAMPLING

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1. INTRODUCTION

Consider a finite population $U = \{U_1, U_2, \dots, U_N\}$ of N identifiable units taking values (y_1, y_2, \dots, y_N) on a study variable Y . Let (X_1, X_2, \dots, X_m) be the m auxiliary variables taking the corresponding values x_{ij} ($i = 1, 2, \dots, m; j = 1, 2, \dots, N$).

The problem of estimating the population variance $S_Y^2 = (N - 1)^{-1} \sum_{j=1}^N (y_j - \bar{Y})^2$

with $\bar{Y} = \frac{1}{N} \sum_{j=1}^N y_j$ using information on single auxiliary variable X_1 has been

discussed by various authors including Das and Tripathi (1978), Srivastava and Jhaji (1980), Isaki (1983), Upadhyaya and Singh (1986), Isaki (1983), Singh, Upadhyaya and Namjosh (1988), Biradar and Singh (1994), Prasad and Singh (1990, 1992), Garcia and Cebrian (1996). Quite often information on many auxiliary variables are available in the survey which can be utilized to increase the precision of the estimate. Following Olkin (1958), Isaki (1983) has considered the use of multiauxiliary variables in building up ratio and regression estimators for population variance. Later Srivastava and Jhaji (1980), Upadhyaya and Singh (1983), and Cebrian and Garcia (1997) have suggested estimators for population variance using information on several auxiliary variables.

Let $\bar{X}_i = \frac{1}{N} \sum_{j=1}^N X_{ij}$ and $S_{X_i}^2 = (N - 1)^{-1} \sum_{j=1}^N (X_{ij} - \bar{X}_i)^2$ be the known popula-

tion means and variances of the i -th auxiliary variable X_i , $i = 1, 2, \dots, m$. Assume that a simple random sample of size n is drawn from U . De-

fining $s_Y^2 = (n - 1)^{-1} \sum_{j=1}^n (y_j - \bar{y})^2$, $\bar{y} = \frac{1}{n} \sum_{j=1}^n y_j$, $s_{X_i}^2 = (n - 1)^{-1} \sum_{j=1}^n (x_{ij} - \bar{x}_i)^2$,

$\bar{x}_i = \frac{1}{n} \sum_{j=1}^n x_{ij}$, $u_i = s_{X_i}^2 / S_{X_i}^2$, $u_{m+i} = \bar{x}_i / \bar{X}_i$, and u denote the column vector of $2m$ elements u_1, u_2, \dots, u_{2m} . Srivastava and Jhajj (1980) suggested a class of estimators of S_Y^2 as

$$d_b = s_Y^2 b(u_1, u_2, \dots, u_{2m}) = s_Y^2 b(u) \quad (1)$$

where $b(u)$ is a function of u such that $b(e) = 1$, $e^T = (1, 1, \dots, 1)$ and such that it satisfies the following conditions:

(a) Whatever be the sample chosen, let $u^T = (u_1, u_2, \dots, u_{2m})$ assumes values in a bounded, closed convex subset Q of the $2m$ dimensional real space containing the point $e^T = (1, 1, \dots, 1)$.

(b) The function $b(u)$ is continuous and bounded in Q .

(c) The first and second order partial derivative of $b(u)$ exist and are continuous.

Using results from the Appendix (A), to the first degree of approximation, it can be easily shown that

$$E(d_b) = S_Y^2 + O(n^{-1})$$

and

$$\text{MSE}(d_b) = \theta S_Y^4 [(\beta_2(Y) - 1) + 2b^T b'(e) + (b'(e))^T D(b'(e))] \quad (2)$$

where $b'(e)$ denotes the column vector of first ordered partial derivatives of $b(u)$ at the point $u = e$. The $\text{MSE}(d_b)$ at (2) is minimized for

$$b'(e) = -D^{-1}b \quad (3)$$

and thus the resulting (minimum) MSE of d_b is given by

$$\min \text{MSE}(d_b) = \theta S_Y^4 [(\beta_2(y) - 1) - b^T D^{-1}b] \quad (4)$$

which is less than the variance of the usual unbiased estimator s_y^2 , since $b^T D^{-1}b > 0$. Tracy and Osahan (1994) and Singh, Joarder and Tracy (2000) studied the effect of random non-response:

(i) On the study as well as the auxiliary variable (situation 1), and

(ii) On the study variable only (situation 2), on the usual ratio and regression estimators of the population mean.

Singh and Joarder (1998) studied the effect of random non-response on the study and auxiliary variables on several estimators of variance.

In this paper, we study the effect of random non-response on the study and auxiliary variables on different classes of estimators of population variance S_Y^2 .

2. RANDOM NON-RESPONSE AND SOME EXPECTED VALUES

If r ($r=0, 1, 2, \dots, n-2$) denote the number of sampling units on which information could not be obtained due to random non-response, then the remaining $(n-r)$ units in the sample can be treated as SRSWOR sample from the population U . Since we are considering the problem of unbiased estimation of finite population variance, therefore we are assuming that r should be less than $(n-1)$, that is, $0 \leq r \leq (n-2)$. We assume that if p stands for the probability of non-response among the $(n-2)$ possible values of non-response, then r has the following distribution given by

$$P(r) = \frac{(n-r)}{nq+2p} {}^{n-2}C_r p^r q^{n-r-2} \quad (5)$$

where $q = (1-p)$, $r = 0, 1, 2, \dots, (n-2)$ and ${}^{n-2}C_r$ denote the total number of ways of r non-responses out of total possible $(n-2)$ responses.

3. PROPOSED STRATEGIES

3.1 Strategy I

We are considering the situation when random non-response exists on the study variable Y and several auxiliary variables ($m > 1$) X_1, X_2, \dots, X_m . We assume that the population means $\bar{X}_1, \bar{X}_2, \dots, \bar{X}_m$ and population variances $S_{X_1}^2, S_{X_2}^2, \dots, S_{X_m}^2$ of auxiliary variables X_1, X_2, \dots, X_m are known. Using results defined in Appendix (B), we define a class of estimators for population variance S_Y^2 as

$$d_1 = s_Y^{*2} f(u^*) \quad (6)$$

where $f(u^*)$ is a function of u^* such that $f(e) = 1$, $e^T = (1, 1, \dots, 1)$ and such that it satisfies the regularity conditions defined in section 1 for d_b .

Expanding $f(u^*)$ about the point $u^* = e$ in a second order Taylor's series, we get

$$d_1 = s_Y^{*2} \left[f(e) + (u^* - e)^T f'(e) + \frac{1}{2} (u^* - e)^T f''(u^{**}) (u^* - e) \right] \quad (7)$$

where $u^{**} = e + \phi(u^* - e)$, $0 < \phi < 1$ and $f'(e)$ denotes the column vector of first order partial derivatives of $f(u^*)$ at the point $u^* = e$, $f''(u^{**})$ denotes the matrix of second order partial derivatives of $f(u^*)$ at the point $u^* = u^{**}$. Substituting for s_Y^{*2} and u^* in (7) in terms of ϵ_0^* and ϵ^* , we have

$$\begin{aligned} d_1 &= S_Y^2 (1 + \epsilon_0) \left[1 + \epsilon^{*T} f'(e) + \frac{1}{2} \epsilon^{*T} f''(u^{**}) \epsilon^* \right] \\ &= S_Y^2 \left[1 + \epsilon_0 + \epsilon^{*T} f'(e) + \epsilon_0 \epsilon^{*T} f'(e) + \frac{1}{2} \epsilon^{*T} f''(u^{**}) \epsilon^* + \frac{1}{2} \epsilon_0 \epsilon^{*T} f''(u^{**}) \epsilon^* \right] \end{aligned} \quad (8)$$

Taking expectation and noting that $f''(u^{**})$ is bounded, it is observed that the bias,

$$B(d_1) = E(d_1) - S_Y^2,$$

is of order n^{-1} , and hence its contribution to the mean square error will be of the order of n^{-2} . The mean square error of d_1 to the first degree of approximation is

$$\begin{aligned} \text{MSE}(d_1) &= E[d_1 - S_Y^2]^2 = S_Y^4 E[\epsilon_0^2 + 2\epsilon_0 \epsilon^{*T} f'(e) + (f'(e))^T \epsilon^* \epsilon^{*T} f'(e)] \\ &= \theta^* S_Y^4 [\beta_2(y) - 1 + 2b^T f'(e) + (f'(e))^T D(f'(e))] \end{aligned} \quad (9)$$

which is minimized for

$$f'(e) = -D^{-1}b \quad (10)$$

Thus the resulting (minimum) MSE of d_1 is given by

$$\min \text{MSE}(d_1) = \theta^* S_Y^4 [\beta_2(y) - 1 - b^T D^{-1}b] \quad (11)$$

Now we state the following theorem:

Theorem 3.1. Up to the terms of order n^{-1} ,

$$\text{MSE}(d_1) \geq \theta^* S_Y^4 [\beta_2(y) - 1 - b^T D^{-1}b]$$

with equality holding if

$$f'(e) = -D^{-1}b.$$

Any parametric function $f(u^*)$ satisfying the regularity conditions can generate any asymptotically acceptable estimator. The class of such estimators are very large. The following are the examples:

$$d_1^{(1)} = s_Y^{*2} \exp \left\{ \sum_{i=1}^m (\alpha_i \log u_i^* + \alpha_{m+i} \log u_{m+i}^*) \right\}$$

$$d_1^{(2)} = s_Y^{*2} \exp \left\{ \sum_{i=1}^m (\alpha_i (u_i^* - 1) + \alpha_{m+i} (u_{m+i}^* - 1)) \right\}$$

and

$$d_1^{(3)} = s_Y^{*2} \left\{ \sum_{i=1}^m (w_i \exp(\alpha_i / w_i) \log u_i^* + w_{m+i} \exp(\alpha_{m+i} / w_{m+i}) \log u_{m+i}^*) \right\} \text{ etc.}$$

The optimum values of α_i 's which minimizes the MSE of $d_1^{(i)}, i = 1, 2, 3$ are obtained from the condition (10). For all the three estimators $d_1^{(i)}, i = 1, 2, 3$, the optimum value of α where $\alpha^T = (\alpha_1, \alpha_2, \dots, \alpha_{2m})$ is given by

$$\alpha = -D^{-1}b \tag{12}$$

The minimum MSE of d_1 is no longer than $\theta^* (\beta_2(y) - 1) S_Y^4$, the variance of the unbiased estimator s_Y^{*2} , since $b^T D^{-1} b > 0$.

It is easily shown that if we consider a wider class of estimators

$$d_2 = F(s_Y^{*2}, u^*) \tag{13}$$

of S_Y^2 , where the function $F(\bullet)$ satisfies $F(S_Y^2, e) = S_Y^2$, for all S_Y^{*2} , and $F'(S_Y^2, e) = 1$, $F'(\bullet)$ denoting the first partial derivative of $F(\bullet)$ with respect to s_Y^{*2} , the minimum MSE of d_2 is equal to (11) and is not reduced.

The difference estimator

$$d_2^{(1)} = s_Y^{*2} + \sum_{i=1}^{2m} \alpha_i (u_i^* - 1)$$

is a member of the class (13) and but not of the class (6). Before obtaining an estimator of $MSE(d_1)$, we state the following lemma from Singh and Joarder (1998) and Singh (2003).

Lemma 3.1. A maximum likelihood estimator of the probability of non-response, p , is given by

$$\hat{p} = \frac{(n-1+r) - \sqrt{(n-1+r)^2 - 4nr(n-3)/(n-2)}}{2(n-3)}$$

Now we state the following theorem

Theorem 3.2. An estimator of the minimum $\text{MSE}(d_1)$ is given by

$$\min \widehat{\text{MSE}}(d_1) = \hat{\theta}^* s_Y^{*4} [\hat{\beta}_2^*(Y) - 1 - \hat{b}^{*T} \hat{D}^{*-1} \hat{b}^*]$$

where different notation have the same meaning as defined in Appendix (C).

3.2 Estimators based on estimated optimum values

If optimum values of constants involved in the estimator are not known, then it is advisable to replace them with their consistent estimators. Let $\hat{\delta}^* = \hat{D}^{*-1} \hat{b}^*$ be the consistent estimators of $\delta = D^{-1}b$. Then we define a class of estimators for S_Y^2 as

$$d_1^* = s_Y^{*2} f^*(u^*, \hat{\delta}^*) \quad (14)$$

where $f^*(u^*, \hat{\delta}^*)$ is the function of $(u^*, \hat{\delta}^*)$ such that

$$\begin{aligned} f^*(u^*, \hat{\delta}^*) \\ f_1^*(e, \delta) &= \frac{\partial f^*(u^*, \hat{\delta}^*)}{\partial u^*} \Big|_{(e, \delta)} = -\delta \\ f_2^*(e, \delta) &= \frac{\partial f^*(u^*, \hat{\delta}^*)}{\partial \hat{\delta}^*} \Big|_{(e, \delta)} = 0 \end{aligned} \quad (15)$$

Thus under the conditions (15), it can be shown to the first degree of approximation that

$$\text{MSE}(d_1^*) = \min \text{MSE}(d_1) \quad (16)$$

where $\min \text{MSE}(d_1)$ is given in (11).

A class wider than (14) is defined as

$$d_2^* = F^*(s_Y^{*2}, u^*, \hat{\delta}^*)$$

where $F^*(s_Y^{*2}, u^*, \hat{\delta}^*)$ is a function of $(s_Y^{*2}, u^*, \hat{\delta}^*)$ such that $F^*(S_Y^2, e, \delta) = S_Y^2$ for all S_Y^{*2} , which in turn implies that

$$F_1^*(S_Y^2, e, \delta) = \frac{\partial F^*(\bullet)}{\partial s_Y^{*2}} \Big|_{(S_Y^2, e, \delta)} = 1$$

$$F_2^*(S_Y^2, e, \delta) = \frac{\partial F^*(\bullet)}{\partial u^*} \Big|_{(S_Y^2, e, \delta)} = -S_Y^2 \delta$$

and

$$F_3^*(S_Y^2, e, \delta) = \frac{\partial F^*(\bullet)}{\partial \hat{\delta}^*} \Big|_{(S_Y^2, e, \delta)} = 0$$

It can be easily shown that to the first degree of approximation that

$$\text{MSE}(d_1^*) = \text{MSE}(d_2^*) = \min \text{MSE}(d_1)$$

3.3 Strategy II

We consider the situation when information on the study variable Y can not be obtained for r units while information on $m > 1$ auxiliary variables X_1, X_2, \dots, X_m is available. The population means $\bar{X}_1, \bar{X}_2, \dots, \bar{X}_m$ and variances $S_{X_1}^2, S_{X_2}^2, \dots, S_{X_m}^2$ of the auxiliary variables X_1, X_2, \dots, X_m are known. We suggest the following class of estimators for population variance S_Y^2 as

$$d_3 = s_Y^{*2} g(u_1, u_2, \dots, u_{2m}) = s_Y^{*2} g(u) \tag{17}$$

where $g(u)$ is a function of u such that $g(e) = 1$, and such that it satisfies the regularity conditions defined in section 1 for d_b .

To the first degree of approximation, it can be shown that

$$E(d_3) = S_Y^2 + O(n^{-1}) \tag{18}$$

and

$$\text{MSE}(d_3) = S_Y^4 [\theta^* (\beta_2(Y) - 1) + \theta \{2b^T g'(e) + (g'(e))^T Dg'(e)\}] \tag{19}$$

where $g'(e)$ denotes the column vector of first order partial derivatives of $g(e)$ at the point $u = e$. The MSE at (19) is minimized for

$$g'(e) = -D^{-1}b \quad (20)$$

Thus the resulting (minimum) MSE of d_3 is given by

$$\min \text{MSE}(d_3) = S_Y^4 [\theta^* (\beta_2(y) - 1) - \theta b^T D^{-1}b] = \min \text{MSE}(d_1) + (\theta^* - \theta) b^T D^{-1}b S_Y^4 \quad (21)$$

Now we state the following theorem:

Theorem 3.3. Up to the terms of order n^{-1} ,

$$\text{MSE}(d_3) \geq S_Y^4 [\theta^* \{\beta_2(Y) - 1\} - \theta b^T D^{-1}b]$$

with equality holding if

$$g'(e) = -D^{-1}b \quad (22)$$

Any parametric function $g(u)$ satisfying the regularity conditions can generate any asymptotically acceptable estimator. The class of such estimators is very large. The following are the examples:

$$d_2^{(1)} = s_Y^{*2} \exp \left\{ \sum_{i=1}^m (\alpha_i \log u_i + \alpha_{m+i} \log u_{m+i}) \right\}$$

$$d_2^{(2)} = s_Y^{*2} \exp \left\{ \sum_{i=1}^m (\alpha_i (u_i - 1) + \alpha_{m+i} (u_{m+i} - 1)) \right\}$$

and

$$d_2^{(3)} = s_Y^{*2} \left\{ \sum_{i=1}^m (w_i \exp(\alpha_i / w_i) \log u_i + w_{m+i} \exp(\alpha_{m+i} / w_{m+i}) \log u_{m+i}) \right\} \text{ etc.}$$

The optimum values of α_i 's which minimizes the MSE of $d_2^{(i)}$, $i = 1, 2, 3$ are obtained from the condition (20). For all the three estimators $d_2^{(i)}$, $i = 1, 2, 3$, the optimum value of α where $\alpha^T = (\alpha_1, \alpha_2, \dots, \alpha_{2m})$ is given by

$$\alpha = -D^{-1}b \quad (23)$$

The minimum MSE of d_2 is no longer than $\theta^* (\beta_2(y) - 1) S_Y^4$, the variance of the unbiased estimator s_Y^{*2} , since $b^T D^{-1}b > 0$. The class of estimators (17) does not include even simple difference type estimators, such as:

$$d_4^{(1)} = s_Y^{*2} + \sum_{i=1}^m \alpha_i (u_i - 1) + \sum_{i=1}^m \alpha_{m+i} (u_{m+i} - 1) \quad (24)$$

$$d_4^{(2)} = s_Y^{*2} + \sum_{i=1}^m \alpha_i (u_i - 1) + \sum_{i=1}^m \alpha_{m+i} \left(\frac{u_i}{u_{m+i}} - 1 \right) \quad (25)$$

However, it is easily shown that if we consider a class of estimators wider than (17), defined by

$$d_4 = G(s_Y^{*2}, u) \quad (26)$$

of S_Y^2 , where $G(\bullet)$ is a function of s_Y^{*2} and u such that $G(S_Y^2, e) = S_Y^2$ for all S_Y^2 which implies $\frac{\partial G(\bullet)}{\partial s_Y^{*2}} \Big|_{(S_Y^2, e)} = 1$. The minimum asymptotic mean square error of d_4

is equal to (21) and is not reduced. The estimators $d_4^{(1)}$ and $d_4^{(2)}$ are members of the class (24) and attain the minimum MSE (21) for optimum values of parameters in (24) and (25).

Now we state the following theorems.

Theorem 3.4. The class of estimators of S_Y^2 based on estimated optimum values of constants defined by

$$d_3^* = s_Y^{*2} g^*(u, \hat{\delta}_{(1)}) \quad (27)$$

has the MSE (to the terms of order n^{-1}) equal to minimum MSE d_3 that is $\text{MSE}(d_3^*) = \min \text{MSE}(d_3)$, where $\min \text{MSE}(d_3)$ is given by (21), $\hat{\delta}_{(1)} = \hat{D}^{-1} \hat{b}^*$ is a consistent estimate of $\delta = D^{-1}b$, and other notation have same meaning as defined in Appendix (D)

Theorem 3.5. A wider class is defined by

$$d_4^* = G^*(s_Y^{*2}, u, \hat{\delta}_{(1)}) \quad (28)$$

has the MSE (to the first degree of approximation) equal to that of d_3^* , where $G^*(\bullet)$ is a function of $G^*(s_Y^{*2}, u, \hat{\delta}_{(1)})$ such that

$$G^*(S_Y^2, e, \delta) = S_Y^2 \text{ for all } S_Y^2,$$

$$G_1^*(S_Y^2, e, \delta) = \frac{\partial G^*(\bullet)}{\partial u} \Big|_{(S_Y^2, e, \delta)} = -S_Y^2 \delta$$

and

$$G_2^*(S_Y^2, e, \delta) = \frac{\partial G^*(\cdot)}{\partial \hat{\delta}} \Big|_{(S_Y^2, e, \delta)} = 0.$$

Theorem 3.6. An estimator of the $\min \text{MSE}(d_3)$ is given by

$$\min \text{MSE}(d_3) = s_Y^{*4} [\hat{\theta}^* (\hat{\beta}_2^*(Y) - 1) - \theta \hat{b}^{*T} \hat{D}^{-1} \hat{b}^*]$$

3.4 Strategy III

We again considering the situation when information on variable Y can not be obtained for r units while information on $m > 1$ auxiliary variables X_1, X_2, \dots, X_m is obtained for all sample units. But the difference is that the population means $\bar{X}_1, \bar{X}_2, \dots, \bar{X}_m$ and variances $S_{X_1}^2, S_{X_2}^2, \dots, S_{X_m}^2$ of the auxiliary variables X_1, X_2, \dots, X_m are not known. We suggest the following class of estimators for population variance S_Y^2 as

$$d_5 = s_Y^{*2} l(v) \tag{29}$$

where $v^T = (v_1, v_2, \dots, v_{2m})$, $v_i = s_{X_i}^{*2} / s_{X_i}^2$, $v_{m+i} = \bar{x}_i^* / \bar{x}_i$, $i = 1, 2, \dots, m$ and $l(v)$ is a function of v such that $l(e) = 1$, and such that it satisfies the regularity conditions defined in section 1 for d_b . To the first degree of approximation, it can be shown that

$$E(d_5) = S_Y^2 + O(n^{-1}) \tag{30}$$

and

$$\text{MSE}(d_5) = S_Y^4 [\theta^* (\beta_2(Y) - 1) + (\theta^* - \theta) \{2b^T l'(e) + (l'(e))^T D l'(e)\}] \tag{31}$$

where $l'(e)$ denotes the column vector of first order partial derivatives of $l(v)$ at the point $v = e$. The MSE at (31) is minimized for

$$l'(e) = -D^{-1}b \tag{32}$$

Substitution of (32) in (31) yields the minimum MSE of d_5 as

$$\min \text{MSE}(d_5) = S_Y^4 [\theta^* (\beta_2(Y) - 1) - (\theta^* - \theta) b^T D^{-1}b] = \min \text{MSE}(d_1) + \theta b^T D^{-1}b S_Y^4 \tag{33}$$

where $\min \text{MSE}(d_1)$ is cited in (11).

Any parametric function $l(v)$ satisfying the regularity conditions can generate an asymptotically acceptable estimator. The class of such estimators are very large and following are a few examples:

$$d_5^{(1)} = s_Y^{*2} \exp \left\{ \sum_{i=1}^m (\alpha_i \log v_i + \alpha_{m+i} \log v_{m+i}) \right\}$$

$$d_5^{(2)} = s_Y^{*2} \exp \left\{ \sum_{i=1}^m (\alpha_i (v_i - 1) + \alpha_{m+i} (v_{m+i} - 1)) \right\}$$

and

$$d_5^{(3)} = s_Y^{*2} \left\{ \sum_{i=1}^m (w_i \exp(\alpha_i / w_i) \log v_i + w_{m+i} \exp(\alpha_{m+i} / w_{m+i}) \log v_{m+i}) \right\} \text{ etc.}$$

may be identified as particular members of the suggested class of estimators d_5 . The optimum values of α_i 's which minimizes the MSE of $d_5^{(i)}, i = 1, 2, 3$ are obtained from the condition (31). For all the three estimators $d_5^{(i)}, i = 1, 2, 3$, the optimum value of α where $\alpha^T = (\alpha_1, \alpha_2, \dots, \alpha_{2m})$ is given by

$$\alpha = -D^{-1}b$$

The minimum MSE of d_5 is no longer than $\theta^* (\beta_2(Y) - 1) S_Y^4$, the variance of the unbiased estimator s_Y^{*2} , since $b^T D^{-1}b > 0$. Thus we have the following theorem:

Theorem 3.7. Up to terms of order n^{-1}

$$\text{MSE}(d_5) \geq S_Y^4 [\theta^* (\beta_2(Y) - 1) - (\theta^* - \theta) b^T D^{-1}b]$$

with equality holding if

$$l'(e) = -D^{-1}b.$$

It is to be noted that the following difference-type estimators:

$$d_5^{(1)} = s_Y^{*2} + \sum_{i=1}^m \alpha_i (v_i - 1) + \sum_{i=1}^m \alpha_{m+i} (v_{m+i} - 1) \tag{34}$$

$$d_5^{(2)} = s_Y^{*2} + \sum_{i=1}^m \alpha_i (v_i - 1) + \sum_{i=1}^m \alpha_{m+i} \left(\frac{v_i}{v_{m+i}} - 1 \right) \tag{35}$$

etc. of S_Y^2 are not members of the class (29).

To overcome this we define a wider class of estimators of S_Y^2 as

$$d_6 = L(s_Y^{*2}, \nu) \quad (36)$$

where $L(s_Y^{*2}, \nu)$ is the function of (s_Y^{*2}, ν) such that $L(S_Y^2, e) = S_Y^2$ for all S_Y^2 which implies $\frac{\partial L(\bullet)}{\partial s_Y^{*2}} \Big|_{(s_Y^2, e)} = 1$. The minimum asymptotic mean square error of d_6 is equal to (33) and is not reduced. The estimators $d_6^{(1)}$ and $d_6^{(2)}$ are members of the class (36) and attain the minimum MSE (33) for optimum values of parameters in (34) and (35).

Remark 3.1. Let $\hat{\delta}_2 = \hat{D}_{(1)}^{-1} b^*$ be the consistent estimate of $\delta = D^{-1}b$, where other notation have same meaning as defined in Appendix (E).

Then the class of estimators (based on estimated optimum values of constants) of S_Y^2 is defined by

$$d_5^* = s_Y^{*2} l^*(\nu, \hat{\delta}_{(2)}) \quad (37)$$

where $l^*(\bullet)$ is a function of $(\nu, \hat{\delta}_{(2)})$ such that

$$l^*(e, \delta) = 1, \quad l_1^*(e, \delta) = \frac{\partial l^*(\bullet)}{\partial \nu} \Big|_{(e, \delta)} = -D^{-1}b \quad \text{and} \quad l_2^*(e, \delta) = \frac{\partial l^*(\bullet)}{\partial \delta} \Big|_{(e, \delta)} = 0.$$

It can be easily shown to the first degree of approximation that

$$\text{MSE}(d_5^*) = \min \text{MSE}(d_5) \quad (38)$$

where $\min \text{MSE}(d_5)$ is given by (33).

Remark 3.2. A class wider than (37) is defined by

$$d_6^* = L^*(s_Y^{*2}, \nu, \hat{\delta}_{(2)}) \quad (39)$$

where $L^*(\bullet)$ is a function of $(s_Y^{*2}, \nu, \hat{\delta}_{(2)})$ such that

$$\begin{aligned}
 L^*(S_Y^2, e, \delta) &= S_Y^2 \text{ for all } S_Y^2, \\
 L_1^*(S_Y^2, e, \delta) &= \frac{\partial L^*(\bullet)}{\partial s_Y^{*2}} \Big|_{(S_Y^2, e, \delta)} = 1 \\
 L_2^*(S_Y^2, e, \delta) &= \frac{\partial L^*(\bullet)}{\partial v} \Big|_{(S_Y^2, e, \delta)} = -S_Y^2 D^{-1} b \\
 \text{and} \\
 L_3^*(S_Y^2, e, \delta) &= \frac{\partial L^*(\bullet)}{\partial \hat{\delta}_{(2)}} \Big|_{(S_Y^2, e, \delta)} = 0.
 \end{aligned} \tag{40}$$

It can be shown to the first degree of approximation that

$$\text{MSE}(d_6^*) = \text{MSE}(d_5^*) = \min \text{MSE}(d_6) = \min \text{MSE}(d_5) \tag{41}$$

Now we state the following theorem:

Theorem 3.8. An estimator of the $\min \text{MSE}(d_5)$ is given by

$$\min \hat{\text{MSE}}(d_5) = s_Y^{*4} [\hat{\theta}^* (\hat{\beta}_2^*(Y) - 1) - \theta \hat{b}^{*T} \hat{D}_{(1)}^{-1} \hat{b}^*]$$

4. EFFICIENCY COMPARISON

It is well known that

$$V(s_Y^{*2}) = \theta^* S_Y^4 (\beta_2(Y) - 1) \tag{42}$$

From (11), (21) and (37), we have

$$\min \text{MSE}(d_1) = \text{MSE}(d_1^*) = \theta^* S_Y^4 [(\beta_2(Y) - 1) - b^T D^{-1} b] \tag{43}$$

$$\min \text{MSE}(d_3) = \text{MSE}(d_3^*) = \min \text{MSE}(d_1) + (\theta^* - \theta) S_Y^4 [b^T D^{-1} b] \tag{44}$$

and

$$\min \text{MSE}(d_5) = \text{MSE}(d_5^*) = \min \text{MSE}(d_1) + \theta S_Y^4 [b^T D^{-1} b] \tag{45}$$

It follows from above expressions that the proposed estimators d_1, d_3, d_5 (or d_1^*, d_3^*, d_5^*) are more efficient than the unbiased estimator s_Y^{*2} . It is further ob-

served that the proposed estimator d_1 (or d_1^*) is more efficient than d_5 (or d_5^*) and d_3 (or d_3^*).

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APPENDIX

(A) Defining

$$\epsilon_0 = s_Y^2 / S_Y^2 - 1, \quad \epsilon_i = u_i - 1, \quad \text{and} \quad \epsilon^T = (\epsilon_1, \epsilon_2, \dots, \epsilon_{2m}),$$

we have

$$E(\epsilon_0) = E(\epsilon_i) = 0; \quad i = 1, 2, \dots, 2m,$$

$$E(\epsilon_0^2) = \theta(\beta_2(y) - 1), \quad E(\epsilon_0 \epsilon) = \theta b, \quad E(\epsilon \epsilon^T) = \theta D$$

where $\theta = \frac{N-n}{Nn}$ and $b^T = \{(l_1 - 1), (l_2 - 1), \dots, (l_m - 1), k_1, k_2, \dots, k_m\}$ and the

matrix $D = \begin{bmatrix} A & | & B \\ \hline B^T & | & C \end{bmatrix}$ is assumed to be positive definite. Here $A = [a_{ii'}]$,

$B = [b_{ii'}]$ and $C = [c_{ii'}]$ are $m \times m$ matrices with $a_{ii'} = (l_{ii'} - 1)$, $b_{ii'} = k_{ii'}$,

$c_{ii'} = \rho_{ii'} C_i C_{i'}$, $l_i = \mu_{22}(Y, X_i) / (S_Y^2 S_{X_i}^2)$, $\rho_{ii'} = \mu_{11}(X_i, X_{i'}) / (S_{X_i} S_{X_{i'}})$,

$k_{ii'} = \mu_{21}(X_i, X_{i'}) / (S_{X_i}^2 \bar{X}_{i'})$, $l_{ii'} = \mu_{22}(X_i, X_{i'}) / (S_{X_i}^2 S_{X_{i'}}^2)$,

$k_i = \mu_{21}(Y, X_i) / (S_Y^2 \bar{X}_i)$, $\lambda_i = k_{ii} = \mu_3(X_i) / (S_{X_i}^2 \bar{X}_i)$, $C_i^2 = S_{X_i}^2 / \bar{X}_i^2 = C_{ii}$,

$\beta_2(X_i) = l_{ii} = \mu_4(X_i) / S_{X_i}^4$, $\beta_2(Y) = \mu_4(Y) / S_Y^4$, $\mu_4(Y) = (N-1)^{-1} \sum_{j=1}^N (y_j - \bar{Y})^4$,

$$\mu_t(X_i) = (N-1)^{-1} \sum_{j=1}^N (x_{ij} - \bar{X}_i)^t, \quad t = 2, 3, 4 \quad \text{and}$$

$$\mu_{st}(Y, X_i) = (N-1)^{-1} \sum_{j=1}^N (y_j - \bar{Y})^s (x_{ij} - \bar{X}_i)^t, \quad (s, t) = 1, 2.$$

(B) Let us define

$$\epsilon_0^* = \frac{s_Y^{*2}}{S_Y^2} - 1, \quad \epsilon_i^* = u_i^* - 1, \quad i = 1, 2, \dots, 2m$$

$$\epsilon^{*T} = (\epsilon_1^*, \epsilon_2^*, \dots, \epsilon_{2m}^*), \quad u_i^* = s_{X_i}^{*2} / S_{X_i}^2, \quad u_{m+i}^* = \bar{x}_i^* / \bar{X}_i, \quad i = 1, 2, \dots, m$$

$$u^{*T} = (u_1^*, u_2^*, \dots, u_{2m}^*)$$

where $s_Y^{*2} = (n-r-1)^{-1} \sum_{j=1}^{n-r} (y_j - \bar{y}^*)^2$, $s_{X_i}^{*2} = (n-r-1)^{-1} \sum_{j=1}^{n-r} (x_{ij} - \bar{x}_i^*)^2$, $i = 1, 2, \dots, m$

are conditionally unbiased estimators of S_Y^2 and $S_{X_i}^2$, respectively and where

$\bar{y}^* = (n-r)^{-1} \sum_{j=1}^{n-r} y_j$ and $\bar{x}_i^* = (n-r)^{-1} \sum_{j=1}^{n-r} x_{ij}$. Thus under the probability model

given by (5), we have the following results:

$$E(\epsilon_0^*) = E(\epsilon_i^*) = 0 \quad \forall \quad i = 1, 2, \dots, 2m$$

$$E(\epsilon_0^{*2}) = \theta^* \{\beta_2(y) - 1\}, \quad E(\epsilon \epsilon^{*T}) = \theta D, \quad E(\epsilon_0^* \epsilon^*) = \theta^* b, \quad E(\epsilon^* \epsilon^{*T}) = \theta^* D$$

where $\theta^* = \left(\frac{1}{nq + 2p} - \frac{1}{N} \right)$. It may be observed that if $p = 0$ that is if there is

no non-response, the above expected values coincides with usual results used by Cebrian and Garcia (1997).

(C) $\hat{b}^{*T} = \{(\hat{l}_1^* - 1), (\hat{l}_2^* - 1), \dots, (\hat{l}_m^* - 1), \hat{k}_1^*, \hat{k}_2^*, \dots, \hat{k}_m^*\}$, $\hat{l}_i^* = \hat{\mu}_{22}^*(Y, X_i) / (s_Y^{*2} S_{X_i}^2)$,

$\hat{k}_i^* = \hat{\mu}_{21}^*(Y, X_i) / (s_Y^{*2} \bar{X}_i)$, $\hat{D}^* = \begin{bmatrix} \hat{A}^* & \hat{B}^* \\ \hat{B}^* & \hat{C}^* \end{bmatrix}$, $\hat{A}^* = [\hat{a}_{ii}^*]$, $\hat{B}^* = [\hat{b}_{ii}^*]$, $\hat{C}^* = [\hat{c}_{ii}^*]$ are

$m \times m$ matrices with $\hat{a}_{ii'} = (\hat{l}_{ii'}^* - 1)$, $\hat{b}_{ii'}^* = \hat{k}_{ii'}^*$, $\hat{c}_{ii'}^* = \hat{\rho}_{ii'}^* C_i C_{i'}$, $\hat{k}_{ii'}^* = \frac{\hat{\mu}_{21}^*(X_i, X_{i'})}{S_{X_i}^2 \bar{X}_i}$,

$$\hat{\rho}_{ii'}^* = \frac{\hat{\mu}_{11}^*(X_i, X_{i'})}{S_{X_i} S_{X_{i'}}}, \hat{l}_{ii'}^* = \frac{\hat{\mu}_{22}^*(X_i, X_{i'})}{S_{X_i}^2 S_{X_{i'}}^2}, \hat{\lambda}_i^* = \hat{k}_{ii}^* = \frac{\hat{\mu}_3^*(X_i)}{S_{X_i}^2 \bar{X}_i}, \hat{\beta}_2^*(X_i) = \hat{l}_{ii}^* = \frac{\hat{\mu}_4^*(X_i)}{S_{X_i}^4},$$

$$\hat{\beta}_2^*(Y) = \frac{\hat{\mu}_4^*(Y)}{s_Y^{*4}}, \hat{\mu}_t^*(X_i) = (n-r-1)^{-1} \sum_{j=1}^{n-r} (y_j - \bar{y}^*)^t, \quad t=2,3,4;$$

$$\hat{\mu}_{st}^*(Y, X_i) = (n-r-1)^{-1} \sum_{j=1}^{n-r} (y_j - \bar{y}^*)^s (x_{ij} - \bar{x}_i^*)^t, \quad (s, t) = 1, 2 \text{ and}$$

$$\hat{\theta}^* = \left(\frac{1}{n\hat{q} + 2\hat{p}} - \frac{1}{N} \right).$$

$$(D) \hat{D} = \begin{bmatrix} \hat{A} & | & \hat{B} \\ \hline & & \\ \hat{B} & | & \hat{C} \end{bmatrix}, \quad \hat{A} = [\hat{a}_{ii'}], \hat{B} = [\hat{b}_{ii'}], \hat{C} = [\hat{c}_{ii'}] \text{ are } m \times m \text{ matrices}$$

$$\hat{a}_{ii'} = (\hat{l}_{ii'}^* - 1), \hat{b}_{ii'}^* = \hat{k}_{ii'}^*, \hat{c}_{ii'}^* = \hat{\rho}_{ii'}^* C_i C_{i'}, \hat{k}_{ii'}^* = \frac{\hat{\mu}_{21}^*(X_i, X_{i'})}{S_{X_i}^2 \bar{X}_i}, \hat{\rho}_{ii'}^* = \frac{\hat{\mu}_{11}^*(X_i, X_{i'})}{S_{X_i} S_{X_{i'}}},$$

$$\hat{l}_{ii'}^* = \frac{\hat{\mu}_{22}^*(X_i, X_{i'})}{S_{X_i}^2 S_{X_{i'}}^2}, \hat{\lambda}_i^* = \hat{k}_{ii}^* = \frac{\hat{\mu}_3^*(X_i)}{S_{X_i}^2 \bar{X}_i}, \hat{\beta}_2^*(X_i) = \hat{l}_{ii}^* = \frac{\hat{\mu}_4^*(X_i)}{S_{X_i}^4},$$

$$\hat{\mu}_t^*(X_i) = (n-1)^{-1} \sum_{j=1}^n (x_{ij} - \bar{x}_i)^t, \quad t=2,3,4;$$

$$\hat{\mu}_{st}^*(X_i, X_{i'}) = (n-1)^{-1} \sum_{j=1}^n (x_{ij} - \bar{x}_i)^s (x_{i'j} - \bar{x}_{i'})^t, \quad (s, t) = 1, 2; \text{ and } g^*(u, \hat{\delta}_{(1)}) \text{ is a}$$

function of u and $\hat{\delta}_{(1)}$ such that $g^*(e, \delta) = 1$, $g_1^*(e, \delta) = \frac{\partial g^*(\bullet)}{\partial u} \Big|_{(e, \delta)} = -\delta$ and

$$g_2^*(e, \delta) = \frac{\partial g^*(\bullet)}{\partial \hat{\delta}} \Big|_{(e, \delta)} = 0.$$

$$(E) \hat{D}_{(1)} = \begin{bmatrix} \hat{A}_{(1)} & | & \hat{B}_{(1)} \\ \hline & & \\ \hat{B}_{(1)}^T & | & \hat{C}_{(1)} \end{bmatrix}, \quad \hat{A}_{(1)} = [\hat{a}_{ii'}^{(1)}], \hat{B}_{(1)} = [\hat{b}_{ii'}^{(1)}], \hat{C}_{(1)} = [\hat{c}_{ii'}^{(1)}] \text{ are } m \times m \text{ ma-}$$

$$\begin{aligned} \text{trices } \hat{a}_{ii'}^{(1)} &= (\hat{l}_{ii'}^{(1)} - 1), \hat{b}_{ii'}^{(1)} = \hat{k}_{ii'}^{(1)}, \hat{c}_{ii'}^{(1)} = \hat{\rho}_{ii'}^{(1)} \hat{C}_i \hat{C}_{i'}, \hat{k}_{ii'}^{(1)} = \hat{\mu}_{21}(X_i, X_{i'}) / (s_{X_i}^2 \bar{x}_i), \\ \hat{\rho}_{ii'}^{(1)} &= \hat{\mu}_{11}(X_i, X_{i'}) / (s_{X_i} s_{X_{i'}}), \hat{l}_{ii'}^{(1)} = \hat{\mu}_{22}(X_i, X_{i'}) / (s_{X_i}^2 s_{X_{i'}}^2), \hat{C}_i = s_{X_i} / \bar{x}_i, \\ \hat{C}_{i'} &= s_{X_{i'}} / \bar{x}_{i'}, \hat{\lambda}_i^{(1)} = \hat{k}_{ii}^{(1)} = \hat{\mu}_3(X_i) / (s_{X_i}^2 \bar{x}_i), \hat{\beta}_2^{(1)}(X_i) = \hat{l}_{ii}^{(1)} = \hat{\mu}_4(X_i) / s_{X_i}^4, \\ \hat{\mu}_t(X_i) &= (n-1)^{-1} \sum_{j=1}^n (y_j - \bar{y})^t, \quad t=2,3,4, \\ \hat{\mu}_{st}(X_i, X_{i'}) &= (n-1)^{-1} \sum_{j=1}^n (x_{ij} - \bar{x}_i)^s (x_{ij'} - \bar{x}_{i'})^t, \quad (s, t) = 1, 2. \end{aligned}$$

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RIASSUNTO

Stima della varianza nelle indagini campionarie con risposte mancanti casuali, sulla base di informazioni ausiliarie multiple

Lo scopo del presente contributo è di studiare le proprietà di una classe di stimatori della varianza di popolazione basati su informazioni ausiliarie multiple, proposta da Srivastava e Jhajj (1980), nelle due situazioni di mancate risposte casuali considerate da Tracy e Osahan (1994). I risultati sono stati ottenuti sotto la condizione che il numero di unità statistiche campionarie, per le quali si ha una mancata risposta accidentale, segna una particolare distribuzione (vedi Singh e Joarder (1998), Singh, Joarder e Tracy (2000), Singh (2003)).

SUMMARY

Variance estimation using multiauxiliary information for random non-response in survey sampling

The goal of this paper is to study the properties of a class of estimators of population variance based on multi-auxiliary variables proposed by Srivastava and Jhajj (1980), under the two different situations of random non-response advocated by Tracy and Osahan (1994). The results are obtained under the assumption that the number of sampling units on which information could not be obtained due to random non-response, follows some distribution; for instance, see Singh and Joarder (1998), Singh, Joarder and Tracy (2000), Singh (2003).