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COMPARISON OF ESTIMATION METHODS OF THE POWER GENERALIZED WEIBULL DISTRIBUTION

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SUMMARY

This article aims to discuss different estimation methods for the power generalized Weibull distribution. An extensive simulation study is carried out to assess the effectiveness of the estimation of model parameters using numerous well known classical methods of estimation. Furthermore, the Bayes estimators of the unknown parameters are also obtained under different loss functions. Monte Carlo simulations are used to assess the performances of the proposed estimators. Besides, bootstrap/ credible intervals are obtained based on considered methods of estimation. Finally, the potentiality of the distribution is illustrated by means of re-analyzing one real data set.

Keywords: Power generalized Weibull distribution; Maximum product of spacings estimators; Percentile estimators; Order statistics, Bayesian estimation.

1. INTRODUCTION

Swedish physicist Waloddi (Weibull, 1951) generalized the exponential distribution, which came to be known as Weibull distribution after him. The Weibull distribution has been adopted as a successful model for modeling product failures because of its flexibility and wide range of applicability. The literature on Weibull models is vast,

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disjointed, and scattered across many different areas and journals. Many authors have applied the Weibull distribution to different areas. For greater details, one may refer to the works of Kadhem *et al.* (2017); Kumar and Dey (2017); Kumar and Jain (2018); Basheer (2019); Mahmood *et al.* (2020); Abd-EL-Baset and Ghazal (2000); Nassar *et al.* (2020); Ali *et al.* (2021); Yang *et al.* (2022) and many others. This model cannot accommodate non-monotone hazard rates such as bathtub shape, the unimodal (upside-down bathtub), or modified unimodal shape, common in human mortality machine life cycles, biological and medical studies. This has necessitated seeking generalizations of this distribution. Over the last 40 years, several new models have been proposed in the literature derived from the Weibull distribution. However, due to increased number of parameters in the extended model, the estimation problems become challenging, see the flexible Weibull extension by Bebbington *et al.* (2007), the Kumaraswamy Weibull distribution by Cordeiro *et al.* (2010), the beta modified Weibull distribution by Silva *et al.* (2010) and many others. The power generalized Weibull (PGW) distribution is another extension of the Weibull distribution, which Bagdonavičius and Nikulin (2001) first proposed, which was further studied by Nikulin and Haghighi (2006); Alloyarova *et al.* (2007); Nikulin and Haghighi (2009); Bagdonavičius and Nikulin (2011); Voinov *et al.* (2013); Mohie-EL-Din *et al.* (2015). Nikulin and Haghighi (2006) proposed chi-squared type statistic to test the validity of the power generalized Weibull family based on Head-and-Neck cancer censored data. Alloyarova *et al.* (2007) constructed the Hsuan-Robson-Mirvaliev (HRM) statistic for testing the hypothesis based on moment-type estimators and investigated its properties. Nikulin and Haghighi (2009) obtained maximum likelihood estimates of the parameters and the flexibility of the model was shown by using Efron (1988) head-and-neck cancer clinical trial data. Bagdonavičius and Nikulin (2011) proposed chi-squared goodness of fit test for right censored data. Voinov *et al.* (2013) constructed modified chi-squared tests based on MLEs. Further, they studied power of the tests against exponentiated Weibull, three-parameter Weibull, and generalized Weibull distributions using Monte Carlo simulations. Recently, Mohie-EL-Din *et al.* (2015) obtained maximum likelihood estimates and Bayes estimates based on progressive censoring, step-stress partially accelerated life tests. Further, they obtained the approximate and bootstrap confidence intervals of the estimators. Kumar and Dey (2017) obtained exact explicit expression and recurrence relations for order statistics from PGW distribution. Kumar and Jain (2018) obtained exact explicit expressions as well as recurrence relations for the single, product and conditional moments of generalized order statistics from the PGW and subsequently they obtained mean, variance of order statistics and record values. This extension of Weibull distribution has excellent properties. The hazard rate can be constant, increasing decreasing, bathtub, and unimodal shaped based on parameter values. One can find more details in Bagdonaviçius *et al.* (2006). Besides, it is a right-skewed heavy-tailed distribution which is not very common in the lifetime model. Thus PGW distribution can be considered as an alternative to the exponentiated Weibull distribution for lifetime data (Nikulin and Haghighi, 2009). The motivation of the paper is two-fold: first, to study the properties of the PGW distribution including moments, mean past lifetime (MPL), conditional moments, Bonferroni and Lorenz curves, stochastic ordering, entropies, order statistics and stress strength reliability. Secondly, to estimate the model parameters from both frequentist and Bayesian perspectives for different sample sizes and parameter values. Finally, to develop a guideline for choosing the best estimation method for the PGW distribution, which would be of great interest to applied statisticians. Hence efficient estimation of these parameters is essential. The study is unique because no previous study compared all of these estimators for the PGW distribution.

One indispensable fact for studying any probability distribution is the estimation of parameter(s). Usually, the maximum likelihood (ML) method is a highly popular method for estimation, although it does not always give the best estimates. In this paper, in addition to ML estimators (MLEs), we consider eight other estimators to estimate the parameters of the PGW model, viz, moment estimators (MEs), least squares estimators (LSEs), weighted least squares estimators (WLSEs), percentile estimators (PCEs), maximum product spacing estimators (MPSEs), Cramér-von-Mises estimators (CVMEs), Anderson-Darling estimators (ADEs) and Right-tail Anderson-Darling estimators (RTADEs). Several authors have emphasized the use of classical estimation methods in varied contexts to estimate the parameters of several well known distributions (see e.g. Kundu and Raqab, 2005; Ghitany *et al.*, 2005; Cordeiro *et al.*, 2010; Teimouri *et al.*, 2013; Mazucheli *et al.*, 2013; Bakouch *et al.*, 2017; Gui, 2017; Dey *et al.*, 2015, 2016, 2017a,b,c,d,e,f; Nassar *et al.*, 2018a,b; Khaoula *et al.*, 2021; Shakhatreh *et al.*, 2022, and many others). Further, the Bayes estimators of the unknown parameters are obtained using both symmetric and asymmetric loss functions, assuming gamma and inverse-gamma priors for both shape and scale parameters of PGW distribution. We carried out a simulation study to evaluate the performance of the estimators. Finally, one real-life data set is analyzed for illustrative purposes.

The paper is organized as follows. Various mathematical and statistical properties of the PGW distribution are presented in Section 2. Section 3 describes nine frequentist methods of estimation. In Section 4, Monte Carlo simulations are carried out to compare the performance of the different frequentist estimation techniques discussed in this article. In Section 5, Bayesian analysis is conducted, and Markov Chain Monte Carlo simulation is performed. In Section 6, the usefulness of the PGW distribution is illustrated by using one real data set. Finally, concluding remarks are provided in the same Section.

2. Statistical properties of the model

In this Section, we provide some statistical properties of the PGW distribution. The probability density function (pdf) of PGW distribution is given as

$$g(x) = \frac{\alpha}{\beta\sigma^{\alpha}} x^{\alpha-1} \left[1 + \left(\frac{x}{\sigma}\right)^{\alpha}\right]^{\frac{1}{\beta}-1} e^{1-\left[1+\left(\frac{x}{\sigma}\right)^{\alpha}\right]^{\frac{1}{\beta}}}, \quad x > 0, \quad \alpha, \beta, \sigma > 0, \tag{1}$$

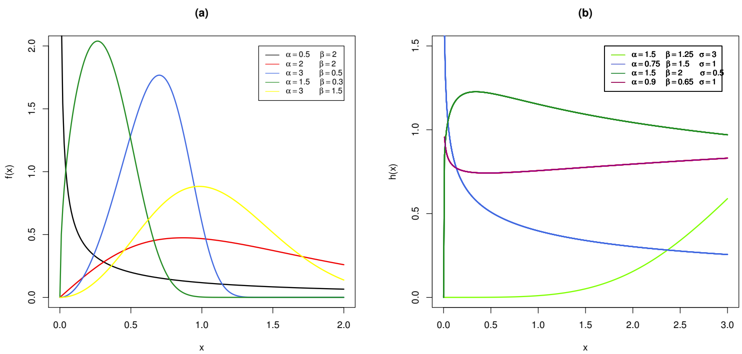

Figure 1 – (a) Density function and (b) hazard rate function of the PGW distribution with various parameters values.

the corresponding cumulative distribution function (cdf) is

$$G(x) = 1 - e^{1-[1+(\frac{x}{\sigma})^{\alpha}]^{\frac{1}{\beta}}}. \tag{2}$$

From Eq. (2), it follows that the quantile function $G^{-1}(p)$ is

$$x_p = \sigma \left\{ [1 - \log(1-p)]^{\beta} - 1 \right\}^{\frac{1}{\alpha}}, \quad 0 < p < 1. \tag{3}$$

The median of the PGW distribution is geven as:

$$\text{Median}(X) = \sigma [(1 - \log(1 - 0.5))^{\beta} - 1]^{\frac{1}{\alpha}}. \tag{4}$$

The hazard rate function is given by

$$h(x) = \frac{\alpha}{\beta \sigma^{\alpha}} x^{\alpha-1} \left[1 + \left(\frac{x}{\sigma} \right)^{\alpha} \right]^{\frac{1}{\beta}-1}.$$

Plot (a) of Figure 1 shows the density function plots of the PGW distribution for $\sigma = 1$ and different values of β and α, while plot (b) shows the hazard rate plots of the PGW distribution for different values of σ, β and α.

2.1. *Shape*

It follows from Eq. (1) that

$$g'(x) = \left[\frac{\alpha-1}{x} + \frac{\alpha}{\beta\,\sigma}\left(\frac{x}{\sigma}\right)^{\alpha-1}\left\{\frac{1-\beta-(1+(\frac{x}{\sigma})^{\alpha})^{\alpha-1}}{1+(\frac{x}{\sigma})^{\alpha}}\right\}\right] g(x), \quad x>0, \quad \alpha,\beta,\sigma>0. \tag{5}$$

PROPOSITION 1. *The shape of the density function corresponding to the PGW distribution may be characterized as follows (Nikulin and Haghighi, 2009).*

1. If $\alpha > 1$, $g(x) \to 0$ as $x \to 0$ and $x \to \infty$.
2. If $\alpha = 1$, then $g(x) \to \frac{1}{\beta\sigma}$ as $x \to 0$ and $g(x) \to 0$ as $x \to \infty$, hence the pdf is hight-tailed at the left end, similar to that of the exponential distribution.
3. If $\alpha < 1$, then $g(x) \to \infty$ as $x \to 0$ and $g(x) \to 0$ as $x \to \infty$, hence the family has a hight left tail asymptote at $x = 0$.

2.2. *Moments*

Moments perform a critical mission in statistical theory and many important aspects of any probability distribution. Let X be a random variable having the PGW distribution. It is easy to obtain the $n-th$ moment of X as the following form

$$\begin{aligned} \mu'_n &= E[X^n] = \int_0^\infty x^n g(x) dx \\ &= \sigma^n \sum_{p=0}^{\infty}\sum_{u=0}^{\beta p}(-1)^{p+(n/\alpha)}\binom{\beta p}{u}\frac{\Gamma\left(\frac{n}{\alpha}+1\right)\Gamma(u+1)}{p!\Gamma\left(\frac{n}{\alpha}+1-p\right)}. \end{aligned} \tag{6}$$

The central moments μ_r and cumulants k_r of X can be determined from Eq. (6) as

$$\mu_r = \sum_{k=0}^{r}(-1)^k \binom{r}{k} \mu_1'^{r} \mu'_{r-k},$$

and

$$\Delta_r = \mu'_r - \sum_{k=1}^{r-1}\binom{r-1}{k-1}\mu'_{r-k},$$

where $\Delta_1 = \mu'_1$. Thus $\Delta_2 = \mu'_2 - (\mu'_1)^2$, $\Delta_3 = \mu'_3 - 3\mu'_2\mu'_1 + 2(\mu'_1)^3$, $\Delta_4 = \mu'_4 - 4\mu'_3\mu'_1 - 3(\mu'_2)^2 + 12\mu'_2(\mu'_1)^2 - 6(\mu'_1)^4$, etc. The skewness $\gamma_1 = \Delta_3/\Delta_2^{3/2}$

and kurtosis $\gamma_2 = \Delta_4/\Delta_2^2$ can be calculated from the second, third and fourth standardized cumulants. Equations (4) and (6) are used to obtain the values of median, mean, variance, skewness and kurtosis of the PGW distribution for $\sigma = 1$ and some selected values for β and α. These values are displayed in Table 1. From Table 1, it is observed that for fixed σ and α, the median, mean, variance and the skewness are increasing functions of β. Also for fixed σ and β as α increases the variance and the skewness decrease.

TABLE 1
Median, mean, variance, skewness and kurtosis of PGW distribution for $\sigma = 1$ and various values of β and α.

β	α	Median	Mean	Varience	Skewness	Kurtosis
0.5	0.5	0.0907	0.2421	0.1520	3.5031	22.2858
	2.5	0.6188	0.6171	0.0545	0.0235	2.5260
	5	0.7866	0.7694	0.0252	-0.5276	3.0839
	10	0.8869	0.8719	0.0091	-0.8927	4.0748
0.75	0.5	0.2345	0.7692	2.0877	4.7289	41.7441
	2.5	0.7482	0.7568	0.0925	0.1874	2.6468
	5	0.8650	0.8501	0.0341	-0.3913	2.9598
	10	0.9301	0.9160	0.0110	-0.7677	3.7404
1	0.5	0.4805	2.0000	20.000	6.6191	87.7194
	2.5	0.8636	0.8873	0.1441	0.3586	2.8568
	5	0.9293	0.9182	0.0442	-0.2541	2.8803
	10	0.9640	0.9514	0.0131	-0.6376	3.6402
1.25	0.5	0.8675	4.7526	167.270	9.6722	208.965
	2.5	0.9720	1.0161	0.2153	0.5374	3.1698
	5	0.9859	0.9798	0.0561	-0.1169	2.8511
	10	0.9929	0.9820	0.0154	-0.5127	3.3047
1.5	0.5	1.4475	10.8632	1355.5	14.8049	563.79
	2.5	1.0768	1.1475	0.3137	0.7240	3.6016
	5	1.0377	1.0379	0.0703	0.0198	2.8725
	10	1.0187	1.0099	0.0180	-0.3908	3.3025

2.3. *Mean past lifetime*

The MPL is a very common tool in reliability analysis for estimating the average time passed after the existence of an event, under the condition that the event has happened before a distinct time. The MPL of the component can be defined as

$$\begin{aligned} k(x) &= E[x-X|X\leq x]=\frac{\int_0^x G(t)dt}{G(x)}=x-\frac{\int_0^x t\,g(t)\mathrm{d}t}{G(x)} \\ &= x-\frac{\sigma}{\left(1-e^{1-\left[1+\left(\frac{x}{\sigma}\right)^{\alpha}\right]^{1/\beta}}\right)}\sum_{p=0}^{\infty}\sum_{u=0}^{\beta p}\sum_{v=0}^{\infty}(-1)^{p+(1/\alpha)}\binom{\beta p}{u} \\ &\times \frac{\Gamma\left(\frac{1}{\alpha}+1\right)\left[1-\left[1+\left(\frac{x}{\sigma}\right)^{\alpha}\right]^{1/\beta}\right]^{v}}{p!v!\Gamma\left(\frac{1}{\alpha}+1-p\right)\Gamma(u+1)(u+v+1)}. \end{aligned} \tag{7}$$

2.4. *Conditional moments*

The conditional moments of PGW distribution is defined as $E(X^n|X>x)=\frac{1}{S(x)}J_n(x)$, where

$$\begin{aligned} J_n(x) &= \int_x^{\infty} t^n g(t)\mathrm{d}t=\sigma^n\sum_{p=0}^{\infty}\sum_{u=0}^{\beta p}(-1)^{p+(n/\alpha)}\binom{\beta p}{u}\frac{\Gamma\left(\frac{n}{\alpha}+1\right)\Gamma(u+1)}{p!\Gamma\left(\frac{n}{\alpha}+1-p\right)} \\ &\times \left[1-(-\log\bar{F}(x))^{u+1}\bar{F}(x)\sum_{q=0}^{\infty}\frac{[-\log\bar{F}(x)]^q}{\Gamma(u+q+2)}\right]. \end{aligned} \tag{8}$$

The mean residual life (MRL) function in terms of the first conditional moment is defined as

$$m_X(x)=E(X-x|X>x)=\frac{1}{S(x)}J_1(x)-x,$$

where $J_1(x)$ can be obtained from Eq. (8) with $n=1$. For the PGW distribution, the mean deviation about the mean (μ) can be obtained as

$$\alpha_{\mu} = \int_0^{\infty}|x-\mu|\,g(x)\mathrm{d}x=2\mu G(\mu)-2\mu+2J_1(\mu)$$

and the mean deviations about the median (M) is

$$\alpha_M = \int_0^{\infty}|x-M|\,g(x)\mathrm{d}x=2J_1(M)-\mu,$$

respectively, where $J_1(\mu)$ and $J_1(M)$ can be obtained from Eq. (8). Also, $G(\mu)$ and $G(M)$ are easily calculated from Eq. (2).

2.5. Bonferroni and Lorenz curves

Bonferroni and Lorenz curves are extensively used tools for interpreting and reflecting income imbalance. The Lorenz curve can be viewed as the proportion of total income amount gained by those members with income lower than or equal to the amount q. On the other hand, the Bonferroni curve is the scaled conditional mean curve, that is, the ratio of group mean income of the population. They are defined as

$$B(p)=\frac{1}{p\mu}\int_0^q x\,g(x)\mathrm{d}x \tag{9}$$

and

$$L(p)=\frac{1}{\mu}\int_0^q x\,g(x)\mathrm{d}x, \tag{10}$$

respectively, where $\mu=E(X)$ and $q=G^{-1}(p)$. By using Eq. (1), one can reduce Eq. (9) and Eq. (10), respectively, to

$$\begin{aligned} B(p) &= \frac{\sigma}{p\ \mu}\sum_{p=0}^{\infty}\sum_{u=0}^{\beta p}\sum_{v=0}^{\infty}(-1)^{p+(1/\alpha)}\binom{\beta p}{u} \\ &\times \frac{\Gamma\left(\frac{1}{\alpha}+1\right)\left[1-\left[1+\left(\frac{q}{\sigma}\right)^{\alpha}\right]^{1/\beta}\right]^{v}}{p!v!\Gamma\left(\frac{1}{\alpha}+1-p\right)\Gamma(u+1)(u+v+1)} \end{aligned} \tag{11}$$

and

$$\begin{aligned} L(p) &= \frac{\sigma}{\mu}\sum_{p=0}^{\infty}\sum_{u=0}^{\beta p}\sum_{v=0}^{\infty}(-1)^{p+(1/\alpha)}\binom{\beta p}{u} \\ &\times \frac{\Gamma\left(\frac{1}{\alpha}+1\right)\left[1-\left[1+\left(\frac{q}{\sigma}\right)^{\alpha}\right]^{1/\beta}\right]^{v}}{p!v!\Gamma\left(\frac{1}{\alpha}+1-p\right)\Gamma(u+1)(u+v+1)}. \end{aligned} \tag{12}$$

2.6. Stochastic ordering

Let $X\sim \mathrm{PGW}(\alpha_1,\beta_1,\sigma)$ and $Y\sim \mathrm{PGW}(\alpha_2,\beta_2,\sigma)$. If $\alpha_1=\alpha_2=\alpha$, $\beta_1\geq\beta_2$ and if $\beta_1=\beta_2=\beta$, $\alpha_1\geq\alpha_2$, then $X\leq_{\mathrm{lr}} Y$, $X\leq_{\mathrm{hr}} Y$, $X\leq_{\mathrm{mrl}} Y$ and $X\leq_{\mathrm{st}} Y$.

PROOF. The likelihood ratio is

$$\frac{g_X(x)}{g_Y(x)}=\frac{\beta_2\,\sigma^{\alpha_2}\alpha_1\,x^{\alpha_1-1}[1+(\frac{x}{\sigma})^{\alpha_1}]^{\frac{1}{\beta_1}-1}e^{1-[1+(\frac{x}{\sigma})^{\alpha_1}]^{\frac{1}{\beta_1}}}}{\beta_1\,\sigma^{\alpha_1}\alpha_2\,x^{\alpha_2-1}[1+(\frac{x}{\sigma})^{\alpha_2}]^{\frac{1}{\beta_2}-1}e^{1-[1+(\frac{x}{\sigma})^{\alpha_2}]^{\frac{1}{\beta_2}}}},$$

thus,

$$\begin{aligned}\frac{d}{x}\log\frac{g_X(x)}{g_Y(x)} &= \frac{(\alpha_1-1)}{x}+\frac{\alpha_1\, x^{\alpha_1-1}\left(\frac{1}{\beta_1}-1\right)}{\sigma^{\alpha_1}[1+(\frac{x}{\sigma})^{\alpha_1}]}-\frac{\alpha_1\, x^{\alpha_1-1}[1+(\frac{x}{\sigma})^{\alpha_1}]^{\frac{1}{\beta_1}-1}}{\beta_1\,\sigma^{\alpha_1}[1+(\frac{x}{\sigma})^{\alpha_1}]}\\ &- \frac{(\alpha_2-1)}{x}-\frac{\alpha_2\, x^{\alpha_2-1}\left(\frac{1}{\beta_2}-1\right)}{\sigma^{\alpha_2}[1+(\frac{x}{\sigma})^{\alpha_2}]}+\frac{\alpha_2\, x^{\alpha_2-1}[1+(\frac{x}{\sigma})^{\alpha_2}]^{\frac{1}{\beta_2}-1}}{\beta_2\,\sigma^{\alpha_2}[1+(\frac{x}{\sigma})^{\alpha_2}]}.\end{aligned}$$

Case (i): If $\alpha_1=\alpha_2=\alpha$, $\beta_1\geq\beta_2$ then $\frac{d}{dx}\log\frac{g_X(x)}{g_Y(x)}\leq 0$, which implies that $X\leq_{\text{lr}} Y$ and hence $X\leq_{\text{hr}} Y$, $X\leq_{\text{mrl}} Y$ and $X\leq_{\text{st}} Y$.

Case (ii): $\beta_1=\beta_2=\beta$, $\alpha_1\geq\alpha_2$ then $\frac{d}{dx}\log\frac{g_X(x)}{g_Y(x)}\leq 0$, which implies that $X\leq_{\text{lr}} Y$ and hence $X\leq_{\text{hr}} Y$,$X\leq_{\text{mrl}} Y$ and $X\leq_{\text{st}} Y$.

Hence, $X\leq_{\text{lr}} Y$ and $X\leq_{\text{hr}} Y$, $X\leq_{\text{mrl}} Y$ and $X\leq_{\text{st}} Y$. □

2.7. *Entropies*

Entropy has been used in various situations and has several utilizations in different disciplines such as statistics, physics and information theory. If X has the probability distribution function $f(\cdot)$, then the Rényi entropy (Rényi, 1961) can be expressed as

$$\begin{aligned}H_\rho(x) &= \frac{1}{1-\rho}\log\left(\int_0^\infty g^\rho(x)\mathrm{d}x\right),\ \ \rho>0,\ \ \rho\neq 1\\ &= \frac{1}{1-\rho}\left[\log\frac{\beta}{\rho}+\rho\log\frac{\alpha e}{\beta\sigma^\alpha}+\log\sigma\right]+\frac{1}{1-\rho}\sum_{p=0}^{\infty}\frac{(-1)^{p-\frac{\rho(\alpha-1)}{\alpha}}}{p!}\\ &\times \frac{\Gamma\left(\frac{\rho(\alpha-1)}{\alpha}+1\right)}{\Gamma\left(\frac{\rho(\alpha-1)}{\alpha}+1-p\right)}\Gamma\left(\frac{\beta\left[p+\rho\left(\frac{1}{\beta}-1\right)\right]}{\rho}+2,\ 1\right),\end{aligned} \tag{13}$$

when $\alpha\to 1$, the Rényi entropy converges to the Shannon entropy.

2.8. *Order statistics*

Moments of order statistics perform a critical role in reliability to predict the breakdown of future units based on the times of few initial failures. Let $X_{1:n}\leq\cdots\leq X_{n:n}$ denotes the order statistic of a random sample $X_1,\ldots,X_n$ from a continuous population with cdf $G_X(x)$ and pdf $g_X(x)$ then the pdf of $X_{j:n}$ is given by

$$g_{X_{j:n}}(x)=\frac{n!}{(j-1)!(n-j)!}g_X(x)(G_X(x))^{j-1}(1-G_X(x))^{n-j}, \tag{14}$$

for $j = 1, \ldots, n$. The pdf and cdf of the j^{th} order statistic for a PGW distribution is given by

$$\begin{aligned} g_{X_{j:n}}(x) &= \frac{\alpha x^{\alpha-1} n!}{\beta \sigma^{\alpha}(j-1)!(n-j)!} \sum_{u=0}^{j-1} (-1)^u \binom{j-1}{u} \\ &\times \left[1 + \left(\frac{x}{\sigma}\right)^{\alpha}\right]^{\frac{1}{\beta}-1} \left[e^{1-[1+(\frac{x}{\sigma})^{\alpha}]^{\frac{1}{\beta}}}\right]^{u+n-j+1} \end{aligned} \tag{15}$$

and

$$G_{j:n}(x) = \sum_{l=k}^{n} \sum_{l=k}^{n} \sum_{u=0}^{l} (-1)^u \binom{n}{l} \binom{l}{u} \left[e^{1-[1+(\frac{x}{\sigma})^{\alpha}]^{\frac{1}{\beta}}}\right]^{u+n-l}. \tag{16}$$

The k^{th} moments of $X_{j:n}$ can be expressed as

$$\begin{aligned} E[X_{j:n}^k] &= \alpha\beta \frac{\sigma^k n!}{(j-1)!(n-j)!} \sum_{p=0}^{\infty} \sum_{v=0}^{\beta p} \sum_{u=0}^{r-1} (-1)^{u+p+(k/\alpha)} \binom{r-1}{u} \binom{\beta p}{v} \\ &\times \frac{\Gamma\left(\frac{k}{\alpha}+1\right)\Gamma(v+1)}{p!\Gamma\left(\frac{k}{\alpha}+1-p\right)(u+n-j+1)^{v+1}}. \end{aligned} \tag{17}$$

2.9. *Stress strength reliability*

Let X_1 and X_2 be two independent random variables revealing the strength of a system and the stress implemented to it, respectively, then $R = P(X_2 < X_1)$ is the measure of the system performance which appears quite commonly in the mechanical reliability of a system. Here, we derive the reliability R when X_1 and X_2 are independent random variables distributed with parameters $(\alpha_1, \beta_1, \sigma_1)$ and $(\alpha_2, \beta_2, \sigma_2)$, then

$$\begin{aligned} R &= \int_0^{\infty} g_1(x) G_2(x) \mathrm{d}x \\ &= \frac{\alpha_1}{\beta_1 \sigma_1^{\alpha_1}} \int_0^{\infty} x^{\alpha 1-1} \left[1 + \left(\frac{x}{\sigma_1}\right)^{\alpha_1}\right]^{\frac{1}{\beta_1}-1} e^{1-\left[1+\left(\frac{x}{\sigma_1}\right)^{\alpha_1}\right]^{\frac{1}{\beta_1}}} \\ &\times \left\{1 - e^{1-\left[1+\left(\frac{x}{\sigma_2}\right)^{\alpha_2}\right]^{\frac{1}{\beta_2}}}\right\} \mathrm{d}x. \end{aligned}$$

If $\alpha_1 = \alpha_2 = \alpha$ and $\sigma_1 = \sigma_2 = \sigma$, then

$$\begin{aligned} R &= \sigma e \sum_{p=0}^{\infty} \frac{(-1)^{p-1/\alpha}\Gamma\left(\frac{1}{\alpha}+1\right)\Gamma(\beta_1 p+1,\ 1)}{p!\Gamma\left(\frac{1}{\alpha}+1-p\right)} \\ &\times -\frac{e^2\beta_2}{\beta_1+\beta_2}\Gamma\left(1-\frac{\beta_1}{\beta_1+\beta_2},\ 1\right). \end{aligned} \tag{18}$$

3. Frequentist methods of estimation

In this Section, we obtain the estimators of the unknown parameters of the PGW distribution based on different methods of estimation. The methods which we employ are as follows.

3.1. *Method of maximum likelihood*

Based on Eq. (1) the likelihood and log-likelihood function, respectively, are

$$L(\alpha,\beta,\sigma) = \left(\frac{\alpha}{\beta\sigma^{\alpha}}\right)^n \prod_{i=1}^{n} {x_i}^{\alpha-1}\left[1+\left(\frac{x_i}{\sigma}\right)^{\alpha}\right]^{\frac{1}{\beta}-1} e^{1-\left[1+\left(\frac{x_i}{\sigma}\right)^{\alpha}\right]^{\frac{1}{\beta}}} \tag{19}$$

and

$$\begin{aligned} l(\alpha,\beta,\sigma) &= n(\log\alpha-\log\beta-\alpha\log\sigma)+(\alpha-1)\sum_{i=1}^{n}\log x_i+\left(\frac{1}{\beta}-1\right) \\ &\times \sum_{i=1}^{n}\log\left[1+\left(\frac{x_i}{\sigma}\right)^{\alpha}\right]+n-\sum_{i=1}^{n}\left[1+\left(\frac{x_i}{\sigma}\right)^{\alpha}\right]^{\frac{1}{\beta}}. \end{aligned} \tag{20}$$

The MLEs $\widehat{\alpha}_{MLE}$, $\widehat{\beta}_{MLE}$ and $\widehat{\sigma}_{MLE}$ of the parameters α, β and σ, can be obtained numerically by maximizing the log-likelihood function in Eq. (20) with respect to α, β and σ. In this case, the log-likelihood function is maximized by solving in α, β and σ, the non-linear equations are:

$$\begin{aligned} \frac{\partial l(\alpha,\beta,\sigma)}{\partial\alpha} &= \frac{n}{\alpha}-n\log\sigma+\sum_{i=1}^{n}\log x_i+\left(\frac{1}{\beta}-1\right)\sum_{i=1}^{n}\frac{\left(\frac{x_i}{\sigma}\right)^{\alpha}\log\left(\frac{x_i}{\sigma}\right)}{\left[1+\left(\frac{x_i}{\sigma}\right)^{\alpha}\right]} \\ &- \frac{1}{\beta}\sum_{i=1}^{n}\left[1+\left(\frac{x_i}{\sigma}\right)^{\alpha}\right]^{\frac{1}{\beta}-1}\left(\frac{x_i}{\sigma}\right)^{\alpha}\log\left(\frac{x_i}{\sigma}\right)=0, \end{aligned} \tag{21}$$

$$\begin{aligned}\frac{\partial l(\alpha,\beta,\sigma)}{\partial\beta} &= -\frac{n}{\beta}-\frac{1}{\beta^2}\sum_{i=1}^{n}\log\left[1+\left(\frac{x_i}{\sigma}\right)^{\alpha}\right]+\frac{1}{\beta^2}\sum_{i=1}^{n}\left[1+\left(\frac{x_i}{\sigma}\right)^{\alpha}\right]^{\frac{1}{\beta}} \\ &\times \log\left[1+\left(\frac{x_i}{\sigma}\right)^{\alpha}\right]=0 \end{aligned} \tag{22}$$

and

$$\begin{aligned}\frac{\partial l(\alpha,\beta,\sigma)}{\partial\sigma} &= \frac{n\alpha}{\sigma}+\left(1-\frac{1}{\beta}\right)\frac{\alpha}{\sigma}\sum_{i=1}^{n}\frac{\left(\frac{x_i}{\sigma}\right)^{\alpha}}{1+\left(\frac{x_i}{\sigma}\right)^{\alpha}} \\ &+ \frac{\alpha}{\beta\sigma}\sum_{i=1}^{n}\left[1+\left(\frac{x_i}{\sigma}\right)^{\alpha}\right]^{\frac{1}{\beta}-1}\left(\frac{x_i}{\sigma}\right)^{\alpha}=0. \end{aligned} \tag{23}$$

3.2. *Method of moments*

The MEs of the three-parameter PGW distribution can be obtained by equating the first three theoretical moments of Eq. (1) with the sample moments $\frac{1}{n}\sum_{i=1}^{n}x_i$, $\frac{1}{n}\sum_{i=1}^{n}x_i^2$ and $\frac{1}{n}\sum_{i=1}^{n}x_i^3$, respectively,

$$\frac{1}{n}\sum_{i=1}^{n}x_i=\sigma\sum_{p=0}^{\infty}\sum_{u=0}^{\beta p}(-1)^{p+(1/\alpha)}\binom{\beta p}{u}\frac{\Gamma\left(\frac{1}{\alpha}+1\right)\Gamma(u+1)}{p!\Gamma\left(\frac{1}{\alpha}+1-p\right)}, \tag{24}$$

$$\frac{1}{n}\sum_{i=1}^{n}x_i^2=\sigma^2\sum_{p=0}^{\infty}\sum_{u=0}^{\beta p}(-1)^{p+(2/\alpha)}\binom{\beta p}{u}\frac{\Gamma\left(\frac{2}{\alpha}+1\right)\Gamma(u+1)}{p!\Gamma\left(\frac{2}{\alpha}+1-p\right)} \tag{25}$$

and

$$\frac{1}{n}\sum_{i=1}^{n}x_i^3=\sigma^3\sum_{p=0}^{\infty}\sum_{u=0}^{\beta p}(-1)^{p+(3/\alpha)}\binom{\beta p}{u}\frac{\Gamma\left(\frac{3}{\alpha}+1\right)\Gamma(u+1)}{p!\Gamma\left(\frac{3}{\alpha}+1-p\right)}. \tag{26}$$

3.3. *Method of maximum product of spacing*

Maximum product of spacing technique was developed as an alternative method to the maximum likelihood approach using the Kullback-Leibler information measure (Cheng and Amin, 1983). Suppose the uniform spacing

$$D_i(\alpha,\beta,\sigma)=G(x_{i:n}\mid\alpha,\beta,\sigma)-G\left(x_{i-1:n}\mid\alpha,\beta,\sigma\right),$$

where $i = 1,,\ldots,n$, $G(x_{0:n} \mid \alpha,\beta,\sigma) = 0$ and $G(x_{n+1:n} \mid \alpha,\beta,\sigma) = 1$. Clearly, $\sum_{i=1}^{n+1} D_i(\alpha,\beta,\sigma) = 1$. The MPSEs of the parameters α, β and σ are obtained by maximizing the following geometric mean of the spacings

$$G(\alpha,\beta,\sigma) = \left[\prod_{i=1}^{n+1} D_i(\alpha,\beta,\sigma)\right]^{\frac{1}{n+1}}, \tag{27}$$

or, equivalently, by maximizing the function

$$H(\alpha,\beta,\sigma) = \frac{1}{n+1}\sum_{i=1}^{n+1} \log D_i(\alpha,\beta,\sigma). \tag{28}$$

The MPSEs can also be obtained by solving the nonlinear equations:

$$\begin{aligned}
\frac{\partial}{\partial\alpha}H(\alpha,\beta,\sigma) &= \frac{1}{n+1}\sum_{i=1}^{n+1}\frac{\zeta_1(x_{i:n}|\alpha,\beta,\sigma) - \zeta_1(x_{i-1:n}|\alpha,\beta,\sigma)}{D_i(\alpha,\beta,\sigma)} = 0,\\
\frac{\partial}{\partial\beta}H(\alpha,\beta,\sigma) &= \frac{1}{n+1}\sum_{i=1}^{n+1}\frac{\zeta_2(x_{i:n}|\alpha,\beta,\sigma) - \zeta_2(x_{i-1:n}|\alpha,\beta,\sigma}{D_i(\alpha,\beta,\sigma)} = 0
\end{aligned}$$

and

$$\frac{\partial}{\partial\sigma}H(\alpha,\beta,\sigma) = \frac{1}{n+1}\sum_{i=1}^{n+1}\frac{\zeta_3(x_{i:n}|\alpha,\beta,\sigma) - \zeta_3(x_{i-1:n}|\alpha,\beta,\sigma)}{D_i(\alpha,\beta,\sigma)} = 0,$$

where

$$\zeta_1(x_{i:n} \mid \alpha,\beta,\sigma) = \frac{x_i^\alpha}{\beta\sigma^\alpha}\log\left(\frac{x_i}{\sigma}\right)\left[1+\left(\frac{x_i}{\sigma}\right)^\alpha\right]^{\frac{1}{\beta}-1} e^{1-\left[1+\left(\frac{x_i}{\sigma}\right)^\alpha\right]^{\frac{1}{\beta}}}, \tag{29}$$

$$\zeta_2(x_{i:n} \mid \alpha,\beta,\sigma) = -\frac{1}{\beta^2}\left[1+\left(\frac{x_i}{\sigma}\right)^\alpha\right]^{\frac{1}{\beta}} e^{1-\left[1+\left(\frac{x_i}{\sigma}\right)^\alpha\right]^{\frac{1}{\beta}}} \log\left[1+\left(\frac{x_i}{\sigma}\right)^\alpha\right] \tag{30}$$

and

$$\zeta_3(x_{i:n} \mid \alpha,\beta,\sigma) = -\frac{\alpha x_i^\alpha}{\beta\sigma^{\alpha+1}}\left[1+\left(\frac{x_i}{\sigma}\right)^\alpha\right]^{\frac{1}{\beta}-1} e^{1-\left[1+\left(\frac{x_i}{\sigma}\right)^\alpha\right]^{\frac{1}{\beta}}}. \tag{31}$$

3.4. *Method of ordinary and weighted least-squares*

The LSEs and the WLSEs were introduced by Swain *et al.* (1988) for estimating the parameters of a model. Using the same notations in Subsection 3.3, the LSEs are obtained by minimizing the following function:

$$S(\alpha,\beta,\sigma) = \sum_{i=1}^{n}\left[G(x_{i:n} \mid \alpha,\beta,\sigma) - \frac{i}{n+1}\right]^2, \tag{32}$$

with respect to the parameters α, β and σ. These estimates can also be obtained by solving the following non-linear equations:

$$\begin{aligned}\sum_{i=1}^{n}\left[G(x_{i:n}\,|\,\alpha,\beta,\sigma)-\frac{i}{n+1}\right]\zeta_1(x_{i:n}\,|\,\alpha,\beta,\sigma) &= 0,\\ \sum_{i=1}^{n}\left[G(x_{i:n}\,|\,\alpha,\beta,\sigma)-\frac{i}{n+1}\right]\zeta_2(x_{i:n}\,|\,\alpha,\beta,\sigma) &= 0\end{aligned}$$

and

$$\sum_{i=1}^{n}\left[G(x_{i:n}\,|\,\alpha,\beta,\sigma)-\frac{i}{n+1}\right]\zeta_3(x_{i:n}\,|\,\alpha,\beta,\sigma)=0.$$

where $\zeta_1(.|\alpha,\beta,\sigma)$, $\zeta_2(.|\alpha,\beta,\sigma)$ and $\zeta_3(.|\alpha,\beta,\sigma)$ are given by Eq. (29), Eq. (30) and Eq. (31), respectively. The WLSEs of the parameters α , β and σ can be obtained by minimising the function:

$$W(\alpha,\beta,\sigma)=\sum_{i=1}^{n}w_i\left[G(x_{i:n}\,|\,\alpha,\beta,\sigma)-\frac{i}{n+1}\right]^2, \tag{33}$$

with respect to the parameters α, β and σ. These estimators can also be obtained by solving the following non-linear equations:

$$\begin{aligned}\sum_{i=1}^{n}w_i\left[G(x_{i:n}\,|\,\alpha,\beta,\sigma)-\frac{i}{n+1}\right]\zeta_1(x_{i:n}\,|\,\alpha,\beta,\sigma) &= 0,\\ \sum_{i=1}^{n}w_i\left[G(x_{i:n}\,|\,\alpha,\beta,\sigma)-\frac{i}{n+1}\right]\zeta_2(x_{i:n}\,|\,\alpha,\beta,\sigma) &= 0\end{aligned}$$

and

$$\sum_{i=1}^{n}w_i\left[G(x_{i:n}\,|\,\alpha,\beta,\sigma)-\frac{i}{n+1}\right]\zeta_3(x_{i:n}\,|\,\alpha,\beta,\sigma)=0,$$

where $w_i=\frac{(n+1)^2(n+2)}{i(n-i+1)}$ and $\zeta_1(.|\alpha,\beta,\sigma)$, $\zeta_2(.|\alpha,\beta,\sigma)$ and $\zeta_3(.|\alpha,\beta,\sigma)$ are given by Eq. (29), Eq. (30) and Eq. (31), respectively.

3.5. Method of percentiles

The percentile method is also another approach for estimating the parameters of a model. $p_i=\frac{i}{n+1}$ is an unbiased estimator of $F(x_{i:n}\,|\,\alpha,\beta,\sigma)$. The PCEs of α, β and σ are obtained by minimizing the function

$$P(\alpha,\beta,\sigma)=\sum_{i=1}^{n}\left\{x_{i:n}-\sigma\left[(1-\log(1-p_i))^{\beta}-1\right]^{\frac{1}{\alpha}}\right\}^2,$$

with respect to α , β and σ. The PCEs of α, β and σ can be obtained by solving the following nonlinear equations

$$\sum_{i=1}^{n}\Psi_{i:n}\frac{\sigma}{\alpha^2}\left[(1-\log(1-p_i))^{\beta}-1\right]^{\frac{1}{\alpha}}\log\left[(1-\log(1-p_i))^{\beta}-1\right]=0,$$

$$\sum_{i=1}^{n}\Psi_{i:n}\frac{\sigma}{\alpha}\left[(1-\log(1-p_i))^{\beta}-1\right]^{\frac{1}{\alpha}-1}[1-\log(1-p_i)]^{\beta}\log[(1-\log(1-p_i))]=0$$

and

$$\sum_{i=1}^{n}\Psi_{i:n}[(1-\log(1-p_i))^{\beta}-1]^{\frac{1}{\alpha}}=0,$$

where $\Psi_{i:n}=\left\{x_{i:n}-\sigma\left[(1-\log(1-p_i))^{\beta}-1\right]^{\frac{1}{\alpha}}\right\}$.

3.6. *Method of Cramér-von-Mises*

The CMEs $\widehat{\alpha}_{CME}$, $\widehat{\beta}_{CME}$ and $\widehat{\sigma}_{CME}$ of the parameters can be obtained by minimizing the given function with respect to α , β and σ:

$$C(\alpha,\beta,\sigma)=\frac{1}{12n}+\sum_{i=1}^{n}\left[G(x_{i:n}\mid\alpha,\beta,\sigma)-\frac{2i-1}{2n}\right]^2. \tag{34}$$

The CMEs can be obtained as the solution of the following non-linear equations:

$$\begin{aligned}\sum_{i=1}^{n}\left[G(x_{i:n}\mid\alpha,\beta,\sigma)-\frac{2i-1}{2n}\right]\zeta_1(x_{i:n}\mid\alpha,\beta,\sigma) &= 0,\\ \sum_{i=1}^{n}\left[G(x_{i:n}\mid\alpha,\beta,\sigma)-\frac{2i-1}{2n}\right]\zeta_2(x_{i:n}\mid\alpha,\beta,\sigma) &= 0\end{aligned}$$

and

$$\sum_{i=1}^{n}\left[G(x_{i:n}\mid\alpha,\beta,\sigma)-\frac{2i-1}{2n}\right]\zeta_3(x_{i:n}\mid\alpha,\beta,\sigma)=0,$$

where $\zeta_1(.\mid\alpha,\beta,\sigma)$, $\zeta_2(.\mid\alpha,\beta,\sigma)$ and $\zeta_3(.\mid\alpha,\beta,\sigma)$ are given by Eq. (29), Eq. (30) and Eq. (31), respectively.

3.7. *Methods of Anderson-Darling and right-tail Anderson-Darling*

To obtain the ADEs, we minimize the following function in order to get the ADEs of the unknown parameters α , β and σ, denoted by $\widehat{\alpha}_{\mathrm{ADE}}$, $\widehat{\beta}_{\mathrm{ADE}}$ and $\widehat{\sigma}_{\mathrm{ADE}}$

$$A(\alpha,\beta,\sigma)=-n-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left\{\log G(x_{i:n}\mid\alpha,\beta,\sigma)+\log\overline{G}\left(x_{n+1-i:n}\mid\alpha,\beta,\sigma\right)\right\}, \tag{35}$$

where where $\overline{G}(.) = 1 - G(.)$. The ADEs can be obtained by solving following the non-linear equations:

$$\begin{aligned} \sum_{i=1}^{n}(2i-1)\left[\frac{\zeta_1(x_{i:n}\mid\alpha,\beta,\sigma)}{G(x_{i:n}\mid\alpha,\beta,\sigma)}-\frac{\zeta_1\left(x_{n+1-i:n}\mid\alpha,\beta,\sigma\right)}{\overline{G}\left(x_{n+1-i:n}\mid\alpha,\beta,\sigma\right)}\right] &= 0,\\ \sum_{i=1}^{n}(2i-1)\left[\frac{\zeta_2(x_{i:n}\mid\alpha,\beta,\sigma)}{G(x_{i:n}\mid\alpha,\beta,\sigma)}-\frac{\zeta_2\left(x_{n+1-i:n}\mid\alpha,\beta,\sigma\right)}{\overline{G}\left(x_{n+1-i:n}\mid\alpha,\beta,\sigma\right)}\right] &= 0 \end{aligned}$$

and

$$\sum_{i=1}^{n}(2i-1)\left[\frac{\zeta_2(x_{i:n}\mid\alpha,\beta,\sigma)}{G(x_{i:n}\mid\alpha,\beta,\sigma)}-\frac{\zeta_3\left(x_{n+1-i:n}\mid\alpha,\beta,\sigma\right)}{\overline{G}\left(x_{n+1-i:n}\mid\alpha,\beta,\sigma\right)}\right]=0,$$

where $\zeta_1(\cdot\mid\alpha,\beta,\sigma)$, $\zeta_2(\cdot\mid\alpha,\beta,\sigma)$ and $\zeta_3(\cdot\mid\alpha,\beta,\sigma)$ are given by Eq. (29), Eq. (30) and Eq. (31), respectively. The RTADEs $\widehat{\alpha}_{\text{RTADE}}$, $\widehat{\beta}_{\text{RTADE}}$ and $\widehat{\alpha}_{\text{RTADE}}$ of the parameters α , β and σ are obtained by minimizing $R(\alpha,\beta,\sigma)$ with respect to α , β and σ, respectively

$$R(\alpha,\beta,\sigma)=\frac{n}{2}-2\sum_{i=1}^{n}G(x_{i:n}\mid\alpha,\beta,\sigma)-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\log\overline{G}\left(x_{n+1-i:n}\mid\alpha,\beta,\sigma\right). \quad (36)$$

The RTADEs can also be obtained by solving the following non-linear equations:

$$\begin{aligned} -2\sum_{i=1}^{n}\frac{\zeta_1(x_{i:n}\mid\alpha,\beta,\sigma)}{G(x_{i:n}\mid\alpha,\beta,\sigma)}+\frac{1}{n}\sum_{i=1}^{n}(2i-1)\frac{\zeta_1\left(x_{n+1-i:n}\mid\alpha,\beta,\sigma\right)}{\overline{G}\left(x_{n+1-i:n}\mid\alpha,\beta,\sigma\right)} &= 0,\\ -2\sum_{i=1}^{n}\frac{\zeta_2(x_{i:n}\mid\alpha,\beta,\sigma)}{G(x_{i:n}\mid\alpha,\beta,\sigma)}+\frac{1}{n}\sum_{i=1}^{n}(2i-1)\frac{\zeta_2\left(x_{n+1-i:n}\mid\alpha,\beta,\sigma\right)}{\overline{G}\left(x_{n+1-i:n}\mid\alpha,\beta,\sigma\right)} &= 0 \end{aligned}$$

and

$$-2\sum_{i=1}^{n}\frac{\zeta_3(x_{i:n}\mid\alpha,\beta,\sigma)}{G(x_{i:n}\mid\alpha,\beta,\sigma)}+\frac{1}{n}\sum_{i=1}^{n}(2i-1)\frac{\zeta_3\left(x_{n+1-i:n}\mid\alpha,\beta,\sigma\right)}{\overline{G}\left(x_{n+1-i:n}\mid\alpha,\beta,\sigma\right)}=0.$$

where $\zeta_1(\cdot\mid\alpha,\beta,\sigma)$, $\zeta_2(\cdot\mid\alpha,\beta,\sigma)$ and $\zeta_3(\cdot\mid\alpha,\beta,\sigma)$ are given by Eq. (29), Eq. (30) and Eq. (31), respectively.

4. Monte Carlo simulation

We perform simulation study to compare the different estimators discussed here. Data were generated from PGW distribution with various combination of the three parameters of the distribution. All the methods were studied for sample sizes $n = 25$, 50, 100, and 150. 10,000 independent samples of size n are first generated from PGW distribution with four different parameter configurations (Confs)– Conf 1 ($\alpha = 1, \beta = 2, \sigma = 5$), Conf 2 ($\alpha = 1.5, \beta = 2, \sigma = 5$), Conf 3 ($\alpha = 10, \beta = 2, \sigma = 5$), Conf 4 ($\alpha = 1, \beta = 5, \sigma = 5$). We used R statistical software version 3.2.3 for Windows (2015) for all the computations. To compare the performance of the different estimates, in each setting, we obtain the average biases, root mean squared errors (RMSEs), D_{abs} and D_{max} using the following formulae (we only present the formula for α, whereas the details can be seen in Tahir *et al.* (2018). $\text{Bias}(\hat{\alpha}) = \frac{1}{M}\sum_{i=1}^{M}(\hat{\alpha}_i - \alpha)$, $\text{RMSE}(\hat{\alpha}) = \sqrt{\frac{1}{M}\sum_{i=1}^{M}(\hat{\alpha}_i - \alpha)^2}$, $D_{\text{abs}} = \frac{1}{(nM)}\sum_{i=1}^{M}\sum_{j=1}^{n}\left|F(x_{ij}|\alpha,\beta,\sigma) - F(x_{ij}|\hat{\alpha},\hat{\beta},\hat{\sigma})\right|$ and $D_{\text{max}} = \frac{1}{M}\sum_{i=1}^{M}\max_j\left|F(x_{ij}|\alpha,\beta,\sigma) - F(x_{ij}|\hat{\alpha},\hat{\beta},\hat{\sigma})\right|$, where M is the number of replications, D_{abs} denotes the absolute distance between the theoretical and estimated cumulative distribution, and D_{max} denotes the maximum distance between the theoretical and estimated cumulative distribution. These results are tabulated in Tables from 2 to 5. The ranks of the bias, RMSE, D_{abs}, and D_{max} of different estimators are also reported in the superscript. For example, in Table 2, for $n = 25$, the bias of MLE for $\hat{\alpha}$ is 0.448 and its rank among the estimation methods is 4^{th}. The following observations can be drawn from these Tables. Similarly, the sum of ranks row for each sample size denotes the column-wise rank sum of each estimation method.

1. All the estimators show the property of consistency, i.e., the RMSE decreases as the sample size increases.
2. The bias of α decreases with increasing n for all the methods of estimations.
3. The bias of β decreases with increasing n for all the methods of estimations.
4. The bias and RMSE of α generally increase with increasing α.
5. The bias and RMSE of β generally increase with increasing β.
6. D_{abs} is smaller than D_{max} for all the estimation techniques. Again, these statistics get smaller with the increase of sample size.
7. Overall, we found that the method of maximum likelihood, MPS and AD estimators uniformly provides superior estimates than their counter parts in terms of bias and RMSEs in our studies.

5. Bayesian estimation

In this Section, we discuss the Bayesian inference of the unknown parameters of the PGW distribution. It is needless to mention that if all the model parameters are unknown, a joint conjugate prior for the parameters may not exist. For this, we assume

TABLE 2
Simulation results for Conf 1: $(\alpha,\beta,\sigma)=(1,2,5)$.

n	Est.	MLE	ME	LSE	WLS	PCE	MPS	CVM	AD	RAD
25	Bias($\hat{\alpha}$)	0.448^{4}	0.536^{6}	0.410^{3}	0.332^{2}	2.152^{9}	0.154^{1}	0.552^{7}	0.452^{5}	1.137^{8}
	RMSE($\hat{\alpha}$)	1.432^{5}	0.580^{1}	1.387^{4}	1.269^{3}	14.034^{9}	0.944^{2}	1.555^{6}	4.313^{7}	5.531^{8}
	Bias($\hat{\beta}$)	1.226^{4}	0.493^{1}	1.363^{5}	1.080^{3}	7.696^{9}	0.684^{2}	1.561^{7}	1.378^{6}	3.289^{8}
	RMSE($\hat{\beta}$)	4.656^{5}	0.577^{1}	4.609^{4}	4.129^{3}	47.767^{9}	3.340^{2}	4.937^{6}	14.195^{7}	17.284^{8}
	Bias($\hat{\sigma}$)	5.498^{5}	1.337^{1}	5.644^{6}	5.193^{3}	12.114^{9}	4.948^{2}	5.468^{4}	6.446^{8}	5.804^{7}
	RMSE($\hat{\sigma}$)	10.705^{4}	1.410^{1}	11.143^{6}	10.540^{3}	21.173^{9}	10.242^{2}	10.903^{5}	12.522^{8}	12.471^{7}
	D_{abs}	0.056^{1}	0.083^{9}	0.058^{5}	0.056^{3}	0.079^{8}	0.078^{7}	0.058^{6}	0.056^{2}	0.058^{4}
	D_{max}	0.097^{3}	0.150^{9}	0.100^{4}	0.096^{2}	0.138^{8}	0.134^{7}	0.103^{6}	0.095^{1}	0.102^{5}
	$\sum$Ranks	31^{4}	29^{3}	37^{5}	22^{1}	70^{9}	25^{2}	47^{7}	46^{6}	55^{8}
50	Bias($\hat{\alpha}$)	0.135^{4}	0.493^{8}	0.150^{5}	0.103^{2}	2.060^{9}	0.028^{1}	0.202^{6}	0.110^{3}	0.216^{7}
	RMSE($\hat{\alpha}$)	0.580^{5}	0.507^{4}	0.663^{6}	0.476^{3}	12.606^{9}	0.423^{1}	0.735^{7}	0.448^{2}	0.780^{8}
	Bias($\hat{\beta}$)	0.344^{4}	0.535^{6}	0.520^{5}	0.342^{3}	7.839^{9}	0.218^{1}	0.584^{7}	0.288^{2}	0.589^{8}
	RMSE($\hat{\beta}$)	2.075^{5}	0.556^{1}	2.329^{6}	1.718^{4}	47.223^{9}	1.604^{3}	2.489^{8}	1.576^{2}	2.454^{7}
	Bias($\hat{\sigma}$)	2.072^{5}	1.289^{1}	3.089^{8}	2.284^{6}	7.595^{9}	1.688^{2}	2.875^{7}	1.833^{4}	1.739^{3}
	RMSE($\hat{\sigma}$)	4.842^{5}	1.333^{1}	6.844^{8}	5.391^{6}	14.839^{9}	4.448^{2}	6.439^{7}	4.533^{3}	4.560^{4}
	D_{abs}	0.038^{1}	0.078^{9}	0.039^{6}	0.038^{3}	0.063^{8}	0.045^{7}	0.039^{5}	0.038^{2}	0.039^{4}
	D_{max}	0.065^{1}	0.137^{9}	0.068^{4}	0.065^{2}	0.113^{8}	0.078^{7}	0.069^{6}	0.066^{3}	0.069^{5}
	$\sum$Ranks	30^{4}	39^{5}	48^{7}	29^{3}	70^{9}	24^{2}	53^{8}	21^{1}	46^{6}
100	Bias($\hat{\alpha}$)	0.042^{4}	0.459^{8}	0.048^{5}	0.032^{2}	1.923^{9}	-0.001^{1}	0.069^{6}	0.039^{3}	0.081^{7}
	RMSE($\hat{\alpha}$)	0.207^{2}	0.471^{8}	0.278^{5}	0.217^{4}	11.503^{9}	0.193^{1}	0.292^{6}	0.215^{3}	0.313^{7}
	Bias($\hat{\beta}$)	0.080^{2}	0.574^{8}	0.163^{5}	0.103^{3}	6.716^{9}	0.080^{1}	0.183^{6}	0.107^{4}	0.217^{7}
	RMSE($\hat{\beta}$)	0.768^{3}	0.595^{1}	1.092^{7}	0.837^{5}	38.807^{9}	0.740^{2}	1.114^{8}	0.828^{4}	1.062^{6}
	Bias($\hat{\sigma}$)	0.662^{3}	1.254^{8}	1.194^{7}	0.734^{5}	4.371^{9}	0.388^{1}	1.180^{6}	0.597^{2}	0.728^{4}
	RMSE($\hat{\sigma}$)	2.306^{3}	1.282^{1}	3.415^{8}	2.582^{5}	9.791^{9}	2.148^{2}	3.379^{7}	2.372^{4}	2.750^{6}
	D_{abs}	0.026^{1}	0.078^{9}	0.028^{6}	0.027^{3}	0.048^{8}	0.028^{7}	0.028^{5}	0.027^{2}	0.028^{4}
	D_{max}	0.045^{1}	0.135^{9}	0.049^{6}	0.046^{3}	0.090^{8}	0.048^{4}	0.049^{7}	0.046^{2}	0.049^{5}
	$\sum$Ranks	19^{1}	52^{8}	49^{6}	30^{4}	70^{9}	19^{1}	51^{7}	24^{3}	46^{5}
150	Bias($\hat{\alpha}$)	0.035^{5}	0.447^{8}	0.022^{4}	0.013^{2}	1.554^{9}	-0.008^{1}	0.036^{6}	0.018^{3}	0.049^{7}
	RMSE($\hat{\alpha}$)	0.172^{2}	0.458^{8}	0.223^{5}	0.179^{4}	9.881^{9}	0.157^{1}	0.229^{6}	0.175^{3}	0.244^{7}
	Bias($\hat{\beta}$)	0.078^{6}	0.585^{8}	0.062^{4}	0.028^{1}	5.511^{9}	0.033^{2}	0.074^{5}	0.035^{3}	0.120^{7}
	RMSE($\hat{\beta}$)	0.638^{3}	0.609^{2}	0.889^{7}	0.700^{5}	34.969^{9}	0.607^{1}	0.898^{8}	0.682^{4}	0.854^{6}
	Bias($\hat{\sigma}$)	0.374^{1}	1.257^{7}	1.261^{8}	0.677^{5}	2.625^{9}	0.387^{2}	1.232^{6}	0.663^{4}	0.645^{3}
	RMSE($\hat{\sigma}$)	1.853^{2}	1.286^{1}	3.266^{8}	2.196^{5}	7.236^{9}	1.854^{3}	3.214^{7}	2.150^{4}	2.316^{6}
	D_{abs}	0.022^{4}	0.078^{9}	0.023^{6}	0.022^{2}	0.043^{8}	0.022^{3}	0.023^{7}	0.022^{1}	0.023^{5}
	D_{max}	0.038^{2}	0.135^{9}	0.041^{6}	0.038^{4}	0.082^{8}	0.038^{1}	0.041^{7}	0.038^{3}	0.041^{5}
	$\sum$Ranks	25^{2}	52^{7}	48^{6}	28^{4}	70^{9}	14^{1}	52^{7}	25^{2}	46^{5}

TABLE 3
Simulation results for Conf 2: $(\alpha, \beta, \sigma) = (1.5, 2, 5)$.

n	Est.	MLE	ME	LSE	WLS	PCE	MPS	CVM	AD	RAD
25	Bias($\hat{\alpha}$)	0.453^{2}	0.607^{5}	0.597^{4}	0.481^{3}	2.999^{9}	0.207^{1}	0.810^{6}	0.997^{7}	1.968^{8}
	RMSE($\hat{\alpha}$)	1.166^{2}	0.671^{1}	2.084^{5}	1.906^{4}	17.805^{9}	1.424^{3}	2.335^{6}	10.332^{7}	11.280^{8}
	Bias($\hat{\beta}$)	0.781^{3}	0.410^{1}	1.298^{5}	1.024^{4}	8.247^{9}	0.631^{2}	1.499^{6}	2.004^{7}	3.817^{8}
	RMSE($\hat{\beta}$)	2.538^{2}	0.474^{1}	4.627^{5}	4.147^{4}	49.132^{9}	3.354^{3}	4.954^{6}	21.250^{7}	23.231^{8}
	Bias($\hat{\sigma}$)	1.419^{2}	0.724^{1}	4.844^{8}	3.205^{6}	4.888^{9}	2.095^{4}	4.264^{7}	1.914^{3}	2.559^{5}
	RMSE($\hat{\sigma}$)	3.405^{2}	1.235^{1}	9.919^{8}	6.890^{6}	10.210^{9}	5.030^{4}	8.867^{7}	4.588^{3}	6.072^{5}
	D_{abs}	0.052^{1}	0.075^{8}	0.058^{5}	0.057^{3}	0.063^{7}	0.084^{9}	0.058^{6}	0.056^{2}	0.058^{4}
	D_{max}	0.090^{1}	0.130^{8}	0.100^{4}	0.097^{3}	0.107^{7}	0.144^{9}	0.104^{6}	0.096^{2}	0.102^{5}
	$\sum$Ranks	15^{1}	26^{2}	44^{6}	33^{3}	68^{9}	35^{4}	50^{7}	38^{5}	51^{8}
50	Bias($\hat{\alpha}$)	0.188^{3}	0.519^{8}	0.213^{4}	0.145^{2}	0.929^{9}	0.032^{1}	0.292^{5}	0.297^{6}	0.457^{7}
	RMSE($\hat{\alpha}$)	0.656^{3}	0.562^{1}	1.000^{5}	0.718^{4}	6.017^{9}	0.643^{2}	1.106^{6}	5.146^{8}	4.275^{7}
	Bias($\hat{\beta}$)	0.319^{3}	0.451^{4}	0.478^{5}	0.312^{2}	2.430^{9}	0.197^{1}	0.544^{6}	0.603^{7}	0.870^{8}
	RMSE($\hat{\beta}$)	1.505^{2}	0.498^{1}	2.353^{5}	1.740^{4}	16.271^{9}	1.622^{3}	2.510^{6}	11.632^{8}	9.671^{7}
	Bias($\hat{\sigma}$)	1.056^{4}	0.397^{1}	1.484^{8}	1.130^{5}	1.602^{9}	0.775^{2}	1.393^{7}	0.825^{3}	1.263^{6}
	RMSE($\hat{\sigma}$)	2.790^{4}	0.928^{1}	3.733^{8}	3.070^{5}	4.828^{9}	2.517^{3}	3.553^{7}	2.435^{2}	3.401^{6}
	D_{abs}	0.037^{1}	0.060^{9}	0.039^{5}	0.038^{2}	0.044^{7}	0.045^{8}	0.039^{4}	0.039^{3}	0.041^{6}
	D_{max}	0.064^{1}	0.103^{9}	0.069^{4}	0.066^{2}	0.078^{8}	0.077^{7}	0.070^{5}	0.067^{3}	0.072^{6}
	$\sum$Ranks	21^{1}	34^{4}	44^{6}	26^{2}	69^{9}	27^{3}	46^{7}	40^{5}	53^{8}
100	Bias($\hat{\alpha}$)	0.068^{5}	0.445^{9}	0.068^{4}	0.046^{2}	0.439^{8}	-0.002^{1}	0.099^{6}	0.058^{3}	0.120^{7}
	RMSE($\hat{\alpha}$)	0.305^{2}	0.481^{7}	0.422^{5}	0.328^{3}	1.387^{9}	0.289^{1}	0.442^{6}	0.338^{4}	0.495^{8}
	Bias($\hat{\beta}$)	0.097^{3}	0.489^{8}	0.146^{5}	0.096^{2}	1.103^{9}	0.073^{1}	0.166^{6}	0.107^{4}	0.207^{7}
	RMSE($\hat{\beta}$)	0.746^{2}	0.536^{1}	1.112^{6}	0.848^{4}	4.038^{9}	0.753^{3}	1.134^{8}	0.863^{5}	1.112^{7}
	Bias($\hat{\sigma}$)	0.354^{5}	0.113^{1}	0.612^{9}	0.380^{6}	0.272^{3}	0.185^{2}	0.606^{8}	0.282^{4}	0.398^{7}
	RMSE($\hat{\sigma}$)	1.515^{3}	0.736^{1}	2.109^{8}	1.667^{5}	2.409^{9}	1.448^{2}	2.078^{7}	1.621^{4}	1.823^{6}
	D_{abs}	0.026^{1}	0.048^{9}	0.028^{5}	0.027^{2}	0.032^{8}	0.030^{7}	0.028^{4}	0.028^{3}	0.029^{6}
	D_{max}	0.045^{1}	0.083^{9}	0.049^{4}	0.046^{2}	0.058^{8}	0.050^{7}	0.050^{5}	0.047^{3}	0.050^{6}
	$\sum$Ranks	22^{1}	45^{5}	46^{6}	26^{3}	63^{9}	24^{2}	50^{7}	30^{4}	54^{8}
150	Bias($\hat{\alpha}$)	0.031^{3}	0.411^{9}	0.031^{4}	0.019^{2}	0.374^{8}	-0.014^{1}	0.051^{6}	0.032^{5}	0.069^{7}
	RMSE($\hat{\alpha}$)	0.242^{1}	0.451^{8}	0.338^{5}	0.270^{4}	0.976^{9}	0.243^{2}	0.347^{6}	0.251^{3}	0.366^{7}
	Bias($\hat{\beta}$)	0.027^{2}	0.486^{8}	0.053^{4}	0.025^{1}	0.886^{9}	0.029^{3}	0.065^{6}	0.060^{5}	0.120^{7}
	RMSE($\hat{\beta}$)	0.610^{2}	0.547^{1}	0.903^{7}	0.705^{5}	2.366^{9}	0.617^{3}	0.912^{8}	0.648^{4}	0.849^{6}
	Bias($\hat{\sigma}$)	0.352^{6}	-0.042^{2}	0.675^{9}	0.372^{7}	0.001^{1}	0.203^{4}	0.661^{8}	0.199^{3}	0.278^{5}
	RMSE($\hat{\sigma}$)	1.313^{3}	0.685^{1}	1.993^{8}	1.459^{5}	2.045^{9}	1.271^{2}	1.963^{7}	1.316^{4}	1.468^{6}
	D_{abs}	0.022^{1}	0.041^{9}	0.023^{5}	0.022^{3}	0.027^{8}	0.024^{7}	0.023^{6}	0.022^{2}	0.023^{4}
	D_{max}	0.037^{1}	0.071^{9}	0.041^{6}	0.038^{3}	0.049^{8}	0.041^{5}	0.041^{7}	0.037^{2}	0.040^{4}
	$\sum$Ranks	19^{1}	47^{6}	48^{7}	30^{4}	61^{9}	27^{2}	54^{8}	28^{3}	46^{5}

TABLE 4
Simulation results for Conf 3: $(\alpha, \beta, \sigma) = (10, 2, 5)$.

n	Est.	MLE	ME	LSE	WLS	PCE	MPS	CVM	AD	RAD
25	Bias($\hat{\alpha}$)	2.124^{3}	0.892^{2}	9.321^{7}	8.652^{6}	3.719^{4}	0.164^{1}	12.317^{8}	3.999^{5}	14.452^{9}
	RMSE($\hat{\alpha}$)	5.995^{3}	0.971^{1}	50.425^{5}	56.903^{7}	55.673^{6}	5.078^{2}	66.447^{8}	33.894^{4}	95.045^{9}
	Bias($\hat{\beta}$)	0.434^{3}	0.426^{2}	2.981^{7}	2.659^{6}	1.338^{5}	0.138^{1}	3.673^{8}	1.218^{4}	4.152^{9}
	RMSE($\hat{\beta}$)	2.128^{3}	0.429^{1}	16.235^{5}	17.303^{6}	18.920^{7}	2.014^{2}	21.573^{8}	10.921^{4}	27.711^{9}
	Bias($\hat{\sigma}$)	0.242^{2}	0.461^{6}	0.625^{9}	0.612^{8}	0.245^{3}	0.204^{1}	0.580^{7}	0.311^{4}	0.359^{5}
	RMSE($\hat{\sigma}$)	0.634^{3}	0.464^{1}	1.626^{8}	1.647^{9}	0.660^{4}	0.592^{2}	1.529^{7}	1.036^{5}	1.104^{6}
	D_{abs}	0.056^{3}	0.213^{9}	0.058^{6}	0.053^{2}	0.056^{4}	0.122^{8}	0.058^{7}	0.053^{1}	0.058^{5}
	D_{max}	0.098^{4}	0.308^{9}	0.101^{5}	0.092^{2}	0.094^{3}	0.228^{8}	0.105^{7}	0.092^{1}	0.102^{6}
	$\sum$Ranks	24^{1}	31^{4}	52^{7}	46^{6}	36^{5}	25^{2}	60^{9}	28^{3}	58^{8}
50	Bias($\hat{\alpha}$)	0.924^{4}	0.919^{3}	1.633^{7}	0.924^{5}	0.032^{1}	-0.084^{2}	2.204^{9}	0.953^{6}	2.123^{8}
	RMSE($\hat{\alpha}$)	3.814^{4}	0.924^{1}	12.831^{8}	4.817^{6}	3.421^{2}	3.631^{3}	14.022^{9}	4.046^{5}	7.830^{7}
	Bias($\hat{\beta}$)	0.183^{3}	0.432^{6}	0.522^{7}	0.289^{5}	0.072^{2}	0.051^{1}	0.602^{9}	0.270^{4}	0.574^{8}
	RMSE($\hat{\beta}$)	1.397^{3}	0.433^{1}	3.932^{8}	1.762^{6}	1.321^{2}	1.412^{4}	4.191^{9}	1.494^{5}	2.469^{7}
	Bias($\hat{\sigma}$)	0.094^{1}	0.467^{9}	0.354^{8}	0.277^{5}	0.303^{6}	0.095^{2}	0.338^{7}	0.180^{3}	0.216^{4}
	RMSE($\hat{\sigma}$)	0.372^{1}	0.467^{3}	1.108^{9}	0.976^{7}	0.965^{6}	0.399^{2}	1.068^{8}	0.707^{4}	0.786^{5}
	D_{abs}	0.038^{1}	0.214^{9}	0.040^{7}	0.039^{4}	0.038^{3}	0.078^{8}	0.040^{6}	0.038^{2}	0.039^{5}
	D_{max}	0.066^{3}	0.311^{9}	0.069^{6}	0.066^{4}	0.065^{1}	0.147^{8}	0.070^{7}	0.065^{2}	0.069^{5}
	$\sum$Ranks	209^{1}	41^{5}	60^{8}	42^{6}	23^{2}	30^{3}	64^{9}	31^{4}	49^{7}
100	Bias($\hat{\alpha}$)	0.405^{5}	0.913^{9}	0.436^{6}	0.304^{3}	-0.062^{1}	-0.192^{2}	0.646^{7}	0.388^{4}	0.803^{8}
	RMSE($\hat{\alpha}$)	2.085^{3}	0.918^{1}	2.831^{7}	2.198^{5}	1.923^{2}	2.306^{6}	2.960^{8}	2.153^{4}	3.145^{9}
	Bias($\hat{\beta}$)	0.073^{3}	0.433^{9}	0.137^{6}	0.093^{4}	0.025^{2}	-0.004^{1}	0.158^{7}	0.107^{5}	0.212^{8}
	RMSE($\hat{\beta}$)	0.780^{3}	0.433^{1}	1.124^{8}	0.853^{5}	0.761^{2}	0.870^{6}	1.145^{9}	0.829^{4}	1.069^{7}
	Bias($\hat{\sigma}$)	0.033^{2}	0.467^{9}	0.154^{8}	0.089^{4}	0.094^{5}	0.024^{1}	0.151^{7}	0.069^{3}	0.094^{6}
	RMSE($\hat{\sigma}$)	0.231^{1}	0.467^{7}	0.617^{9}	0.457^{5}	0.437^{4}	0.243^{2}	0.605^{8}	0.374^{3}	0.463^{6}
	D_{abs}	0.026^{1}	0.214^{9}	0.028^{7}	0.027^{4}	0.027^{2}	0.055^{8}	0.028^{6}	0.027^{3}	0.028^{5}
	D_{max}	0.045^{2}	0.312^{9}	0.049^{6}	0.046^{4}	0.045^{1}	0.102^{8}	0.050^{7}	0.046^{3}	0.049^{5}
	$\sum$Ranks	20^{2}	54^{6}	57^{8}	34^{4}	19^{1}	34^{4}	59^{9}	29^{3}	504^{6}
150	Bias($\hat{\alpha}$)	0.168^{3}	0.901^{9}	0.200^{5}	0.125^{1}	-0.146^{2}	-0.227^{6}	0.333^{7}	0.175^{4}	0.480^{8}
	RMSE($\hat{\alpha}$)	1.596^{3}	0.905^{1}	2.263^{7}	1.803^{5}	1.579^{2}	1.987^{6}	2.321^{8}	1.754^{4}	2.444^{9}
	Bias($\hat{\beta}$)	0.011^{1}	0.435^{9}	0.049^{6}	0.024^{4}	-0.017^{2}	-0.023^{3}	0.061^{7}	0.035^{5}	0.117^{8}
	RMSE($\hat{\beta}$)	0.614^{2}	0.435^{1}	0.910^{8}	0.707^{6}	0.627^{3}	0.705^{5}	0.918^{9}	0.683^{4}	0.858^{7}
	Bias($\hat{\sigma}$)	0.064^{3}	0.468^{9}	0.122^{8}	0.074^{5}	0.071^{4}	0.020^{1}	0.120^{7}	0.063^{2}	0.076^{6}
	RMSE($\hat{\sigma}$)	0.288^{2}	0.468^{8}	0.473^{9}	0.328^{5}	0.299^{3}	0.200^{1}	0.468^{7}	0.301^{4}	0.369^{6}
	D_{abs}	0.022^{1}	0.215^{9}	0.023^{6}	0.022^{4}	0.022^{2}	0.039^{8}	0.023^{7}	0.022^{3}	0.023^{5}
	D_{max}	0.037^{2}	0.313^{9}	0.041^{6}	0.039^{4}	0.037^{1}	0.072^{8}	0.041^{7}	0.038^{3}	0.041^{5}
	$\sum$Ranks	17^{1}	55^{7}	55^{7}	34^{4}	19^{2}	38^{5}	59^{9}	29^{3}	54^{6}

TABLE 5
Simulation results for Conf 4: $(\alpha, \beta, \sigma) = (1, 5, 5)$.

n	Est.	MLE	ME	LSE	WLS	PCE	MPS	CVM	AD	RAD
25	Bias($\hat{\alpha}$)	0.721^{5}	-0.447^{3}	0.425^{2}	0.341^{1}	3.441^{9}	1.035^{6}	0.598^{4}	2.304^{7}	2.836^{8}
	RMSE($\hat{\alpha}$)	1.528^{5}	0.792^{1}	1.192^{3}	1.090^{2}	15.357^{9}	6.478^{6}	1.387^{4}	10.952^{8}	10.163^{7}
	Bias($\hat{\beta}$)	3.843^{5}	3.971^{6}	2.453^{3}	1.936^{2}	16.382^{9}	0.245^{1}	2.966^{4}	12.836^{7}	15.701^{8}
	RMSE($\hat{\beta}$)	8.761^{6}	4.638^{1}	7.166^{4}	6.414^{3}	73.280^{9}	6.052^{2}	7.776^{5}	60.836^{8}	57.845^{7}
	Bias($\hat{\sigma}$)	2.661^{2}	22.885^{9}	4.961^{5}	3.311^{3}	19.993^{8}	0.463^{1}	5.605^{7}	4.115^{4}	5.026^{6}
	RMSE($\hat{\sigma}$)	6.005^{2}	27.184^{8}	9.932^{6}	6.937^{3}	34.286^{9}	1.872^{1}	10.338^{7}	8.456^{4}	9.864^{5}
	D_{abs}	0.054^{2}	0.321^{9}	0.058^{6}	0.056^{3}	0.153^{8}	0.093^{7}	0.058^{5}	0.053^{1}	0.057^{4}
	D_{max}	0.094^{2}	0.622^{9}	0.098^{4}	0.094^{3}	0.281^{8}	0.146^{7}	0.101^{6}	0.091^{1}	0.100^{5}
	$\sum$Ranks	29^{2}	46^{7}	33^{4}	20^{1}	69^{9}	31^{3}	42^{6}	40^{5}	50^{8}
50	Bias($\hat{\alpha}$)	0.294^{3}	-0.674^{6}	0.261^{2}	0.183^{1}	2.483^{8}	4.054^{9}	0.341^{4}	0.377^{5}	0.849^{7}
	RMSE($\hat{\alpha}$)	0.877^{4}	0.859^{2}	0.876^{3}	0.735^{1}	11.593^{9}	8.594^{8}	0.974^{5}	3.727^{6}	4.411^{7}
	Bias($\hat{\beta}$)	1.596^{3}	4.803^{7}	1.534^{2}	1.051^{1}	12.491^{9}	6.782^{8}	1.784^{4}	2.202^{5}	4.640^{6}
	RMSE($\hat{\beta}$)	5.223^{3}	5.186^{2}	5.284^{4}	4.394^{1}	60.448^{9}	12.913^{6}	5.628^{5}	23.410^{7}	24.002^{8}
	Bias($\hat{\sigma}$)	1.179^{1}	29.438^{9}	1.826^{5}	1.532^{4}	13.293^{8}	1.267^{2}	2.182^{7}	1.428^{3}	2.096^{6}
	RMSE($\hat{\sigma}$)	3.354^{2}	30.875^{9}	4.458^{5}	3.892^{4}	24.390^{8}	2.981^{1}	4.745^{6}	3.654^{3}	4.786^{7}
	D_{abs}	0.037^{1}	0.371^{9}	0.039^{6}	0.038^{3}	0.152^{8}	0.108^{7}	0.039^{5}	0.038^{2}	0.039^{4}
	D_{max}	0.064^{1}	0.740^{9}	0.068^{4}	0.065^{3}	0.283^{8}	0.210^{7}	0.069^{6}	0.065^{2}	0.069^{5}
	$\sum$Ranks	18$^{1.5}$	53^{8}	31^{3}	18$^{1.5}$	67^{9}	48^{6}	42^{5}	33^{4}	50^{7}
100	Bias($\hat{\alpha}$)	0.106^{4}	-0.888^{7}	0.100^{3}	0.059^{1}	2.489^{8}	9.568^{9}	0.130^{5}	0.063^{2}	0.164^{6}
	RMSE($\hat{\alpha}$)	0.367^{3}	0.945^{7}	0.454^{4}	0.316^{2}	10.688^{8}	13.838^{9}	0.487^{5}	0.301^{1}	0.555^{6}
	Bias($\hat{\beta}$)	0.551^{3}	5.591^{7}	0.596^{4}	0.349^{2}	13.237^{9}	10.112^{8}	0.679^{5}	0.347^{1}	0.885^{6}
	RMSE($\hat{\beta}$)	2.276^{3}	5.713^{7}	2.912^{4}	2.042^{2}	60.407^{9}	15.082^{8}	3.055^{5}	1.954^{1}	3.326^{6}
	Bias($\hat{\sigma}$)	0.445^{1}	35.386^{9}	0.802^{5}	0.519^{2}	10.095^{8}	0.663^{4}	0.978^{7}	0.520^{3}	0.815^{6}
	RMSE($\hat{\sigma}$)	2.088^{1}	35.386^{9}	2.557^{6}	2.133^{3}	19.191^{8}	2.144^{4}	2.673^{7}	2.108^{2}	2.502^{5}
	D_{abs}	0.027^{3}	0.420^{9}	0.028^{6}	0.027^{2}	0.149^{7}	0.155^{8}	0.028^{5}	0.027^{1}	0.028^{4}
	D_{max}	0.046^{3}	0.865^{9}	0.049^{4}	0.046^{2}	0.276^{8}	0.235^{7}	0.049^{6}	0.046^{1}	0.049^{5}
	$\sum$Ranks	21^{3}	64^{8}	36^{4}	16^{2}	65^{9}	57^{7}	45^{6}	12^{1}	44^{5}
150	Bias($\hat{\alpha}$)	0.047^{4}	-0.962^{7}	0.046^{3}	0.025^{1}	3.348^{8}	4.448^{9}	0.063^{5}	0.027^{2}	0.101^{6}
	RMSE($\hat{\alpha}$)	0.231^{1}	0.981^{7}	0.340^{4}	0.250^{3}	13.488^{9}	8.897^{8}	0.354^{5}	0.237^{2}	0.442^{6}
	Bias($\hat{\beta}$)	0.244^{3}	5.861^{8}	0.257^{4}	0.137^{2}	17.724^{9}	5.722^{7}	0.298^{5}	0.129^{1}	0.531^{6}
	RMSE($\hat{\beta}$)	1.484^{1}	5.902^{7}	2.146^{4}	1.598^{3}	72.197^{9}	11.404^{8}	2.202^{5}	1.518^{2}	2.649^{6}
	Bias($\hat{\sigma}$)	0.414^{1}	35.386^{9}	0.745^{6}	0.563^{2}	6.178^{8}	0.608^{4}	0.856^{7}	0.583^{3}	0.731^{5}
	RMSE($\hat{\sigma}$)	1.761^{2}	35.386^{9}	2.213^{6}	1.912^{4}	13.997^{8}	1.702^{1}	2.278^{7}	1.893^{3}	2.157^{5}
	D_{abs}	0.021^{1}	0.435^{9}	0.023^{5}	0.022^{3}	0.153^{8}	0.132^{7}	0.023^{6}	0.022^{2}	0.023^{4}
	D_{max}	0.036^{1}	0.909^{9}	0.041^{4}	0.038^{3}	0.283^{8}	0.244^{7}	0.041^{5}	0.038^{2}	0.041^{6}
	$\sum$Ranks	14^{1}	65^{8}	36^{4}	21^{3}	67^{9}	51^{7}	45^{6}	17^{2}	44^{5}

piecewise independent priors. Thus the proposed priors for the parameters α, β, and σ may be taken as: $\alpha \sim \text{Gamma}(a,b)$, $\beta \sim \text{Inverse-Gamma}(c,d)$ and $\sigma \sim \text{Inverse-Gamma}(e,f)$. The joint prior distribution of α, β, and σ can be written as

$$p(\alpha,\beta,\sigma) \propto \alpha^{a-1}\beta^{-c-1}\sigma^{-e-1}\exp(-b\alpha)\exp(-d/\beta - f/\sigma),$$

where the hyperparameters a, b, c, d, e and f are known and non-negative. We assume $\boldsymbol{\sigma} = (\alpha,\beta,\sigma), \boldsymbol{x} = (x_1, x_2, \cdots, x_n)$ and $P(\boldsymbol{\sigma})$ denotes the joint posterior. Therefore, we write the joint posterior as

$$\begin{aligned} P(\alpha,\beta,\sigma|\boldsymbol{x}) &\propto \alpha^{n+a-1}\exp(-\alpha(b+\sum_{i=1}^{n}\log x_i^{-1}))\beta^{-n-c-1} \\ &\times \exp\Big(-\beta^{-1}(d-\sum_{i=1}^{n}\log[1+(x_i/\sigma)^{\alpha}])\Big)\sigma^{-n\alpha-e-1} \\ &\times \exp\Big(-f/\sigma+\sum_{i=1}^{n}[1+(x_i/\sigma)^{\alpha}]^{1/\beta} \\ &- \sum_{i=1}^{n}\log[1+(x_i/\sigma)^{\alpha}]\Big), \end{aligned} \tag{37}$$

then, we can write $P(\alpha,\beta,\sigma|\boldsymbol{x})$ as

$$\begin{aligned} P(\alpha,\beta,\sigma|\boldsymbol{x}) &\propto P_{\alpha}\Big(n+a, b+\sum_{i=1}^{n}\log x_i^{-1}\Big|\boldsymbol{x}\Big) \\ &\times P_{\beta}\Big(n+c, d-\sum_{i=1}^{n}\log[1+(x_i/\sigma)^{\alpha}]\Big|\alpha,\sigma,\boldsymbol{x}\Big) \\ &\times P_{\sigma}\Big(\sigma^{-n\alpha-e-1}\exp\Big(-f/\sigma+\sum_{i=1}^{n}[1+(x_i/\sigma)^{\alpha}]^{1/\beta} \\ &- \sum_{i=1}^{n}\log[1+(x_i/\sigma)^{\alpha}]\Big)\Big|\boldsymbol{x},\alpha,\beta\Big), \end{aligned} \tag{38}$$

where P_{α} and P_{β} are the gamma and inverted-gamma densities, and $P(\sigma|\boldsymbol{x},\alpha,\beta) = \sigma^{-n\alpha-e-1}\exp\big(-f/\sigma+\sum_{i=1}^{n}[1+(x_i/\sigma)^{\alpha}]^{1/\beta}-\sum_{i=1}^{n}\log[1+(x_i/\sigma)^{\alpha}]\big)$, which can be generated by Metropolis Hastings (MH) sampling. For this, we assume inverse-gamma density as transition kernel $q(\sigma^{(i)}|\sigma^{(*)})$ for sampling value of σ. The choice of inverse-gamma distribution has been done purely for illustration purpose, and other suitable distributions could be used. After generating the marginal densities, the next step is to calculate the posterior summaries, $\mathbb{E}(\boldsymbol{\sigma}|\boldsymbol{x}) = \int_{\boldsymbol{\sigma}} \boldsymbol{\sigma}\mathbb{P}(\boldsymbol{\sigma}|\boldsymbol{x})$. An important problem in Bayesian estimation is the choice of an appropriate loss function (see Ali, 2013; Dey *et al.*, 2015). In this study, we obtain the posterior summaries under the loss functions resumed in Table 6, where SELF is squared error loss function, WSELF is weighted squared error loss function, MSELF is modified squared error loss function, PLF is precautionary loss function and KLF is K-loss function.

TABLE 6
Loss Functions.

Loss Function	SELF	WSELF	MSELF	PLF	KLF
Math. Form	$(\sigma-d)^2$	$(\sigma-d)^2/\sigma$	$(1-d/\sigma)^2$	$(\sigma-d)^2/d$	$(\sqrt{d/\sigma}-\sqrt{\sigma/d})^2$
Bayes estimate	$E(\sigma\|\boldsymbol{x})$	$(E(\sigma^{-1}\|\boldsymbol{x}))^{-1}$	$\frac{E(\sigma^{-1}\|\boldsymbol{x})}{E(\sigma^{-2}\|\boldsymbol{x})}$	$\sqrt{E(\sigma^2\|\boldsymbol{x})}$	$\frac{E(\sigma\|\boldsymbol{x})}{E(\sigma^{-1}\|\boldsymbol{x})}$

The steps to calculate the Bayes estimates are:

Step 1: take some initial guess values of β and σ, say β_0 and σ_0, respectively.

Step 2: generate α from P_α.

Step 3: generate β from P_β;

Step 4: generate σ from P_σ, where:

1. to generate σ, evaluate the acceptance probability by $k\left(\sigma^{(i)},\sigma^{(*)}\right)=\min\left(1,\frac{P(\sigma^{(*)}|\alpha,\beta,\boldsymbol{x})q(\sigma^{(i)}|\sigma^{(*)})}{P(\sigma^{(i)}|\alpha,\beta,\boldsymbol{x})q(\sigma^{(*)}|\alpha,\beta,\sigma^{(i)})}\right)$, where $P(\sigma|\alpha,\beta,\boldsymbol{x})$ has been defined above,
2. generate a random numbers u from Uniform$(0,1)$,
3. if $k\left(\sigma^{(i)},\sigma^{(*)}\right)\geq u$, $\sigma^{(i+1)}=\sigma^{(*)}$, otherwise $\sigma^{(i+1)}=\sigma^{(i)}$;

Step 5: suppose at the i^{th} step, α, β and σ take the values α_i,β_i and σ_i. Now we can generate $P(\sigma_{i+1}|\alpha_i,\beta_i,\boldsymbol{x})$, $P(\alpha_{i+1}|\boldsymbol{x})$ and $P(\beta_{i+1}|\alpha_i,\sigma_i,\boldsymbol{x})$;

Step 6: repeat the above step N times;

Step 7: calculate the Bayes estimate of $h(\alpha,\beta,\sigma)$ by $\frac{1}{N-Q}\sum_{i=Q+1}^{N}h(\alpha_i,\beta,\sigma_i)$, where Q denotes the number of burn-in sample. Also, obtain the standard deviation as $SD(\hat{\sigma}_B)=\sqrt{\frac{\sum_{i=Q+1}^{N}(\sigma_i-\hat{\sigma}_B)^2}{N-Q}}$.

For the Bayesian analysis, we generated $12,000$ samples of α,β and σ, and the Bayes estimates with other posterior summaries, like SD and MCMC error have been tabulated in Tables 7 and 8 for different sample sizes and three different parameter configurations: Conf 1 ($\alpha=2,\beta=2,\sigma=5$), Conf 2 ($\alpha=1.5,\beta=2,\sigma=5$), Conf 3 ($\alpha=10,\beta=2,\sigma=5$) and Conf 4 ($\alpha=1.5,\beta=2,\sigma=10$). We computed the summaries for parameter combinations and sample sizes as mentioned above. To compute the posterior summaries, we have selected the hyper-parameters so that the mean of the priors equal to the nominal parameter values with large variances. Moreover, we have used $Q = 2,000$ as a burn-in period for our calculations. From the Tables, we observe that as the sample sizes increase, the Bayes estimates approach to the nominal values, and the MCMC error decreases with the increase of sample size. We also observed that the WSELF has the least posterior standard derivation among the other loss functions.

6. Real data analysis

Here, we present one real data application related to bladder cancer data to demonstrate the flexibility of the PGW distribution. The data set describes the remission times (in months) of 128 patients with bladder cancer, (see Lee and Wang, 2003, for more details about the data). This data was analyzed by many authors (see e.g. Pena-Ramirez *et al.*, 2018; Nassar *et al.*, 2018a; Afify *et al.*, 2020). We first check the suitability of the PGW model to fit this data by using Kolmogorov-Smirnov(K-S) statistics and the associated p-value. Using the MLEs of the unknown parameters (the MLEs are displayed in Table 9, the K-S and p-value are 0.0321 and 0.9994, respectively. Based on these results, we can conclude that the PGW distribution is suitable to model the bladder cancer data. Figure 2 presents the TTT plot for the data set. It is clear from Figure 2 that the TTT plot is concave than convex, which indicates an up-side-down bathtub hazard rate, so it is reasonable to use the PGW distribution to analyze the bladder cancer data. We also compared the fit of the PGW model with some well-known models including, Weibull (We), exponentiated Weibull (EWe) and beta Weibull (BWe) models. Table 9 displays the MLEs of all fitted models as well as the K-S and the corresponding p-value for each model. The results in Table 9 show that the PGW model performs better fits to the bladder cancer data than other competitive models. Next, we use the different methods of estimation discussed in the previous Sections to estimate the unknown parameters of the PGW distribution

TABLE 7
Monte Carlo Markov Chain results for Bayesian analysis.

			Conf 1			Conf 2		
n	LF	Parameter	Estimate	SD	MC error	Estimate	SD	MC error
25	SELF	α	1.068	1.060	0.009464	1.540	0.8851	0.009555
		β	2.117	1.497	0.014180	2.095	1.4870	0.014430
		σ	5.288	2.384	0.023000	5.252	2.3490	0.021420
	WSELF	α	1.048	1.040	0.010980	1.537	0.8829	0.008660
		β	2.110	1.488	0.015320	2.132	1.5010	0.015420
		σ	5.274	2.380	0.024150	5.251	2.3640	0.022230
	MSELF	α	1.053	1.063	0.011890	1.530	0.8872	0.009367
		β	2.110	1.491	0.014610	2.111	1.4960	0.013910
		σ	5.246	2.332	0.019680	5.266	2.3620	0.025360
	PLF	α	1.056	1.045	0.010640	1.550	0.8998	0.009970
		β	2.106	1.497	0.014440	2.133	1.4990	0.013950
		σ	5.275	2.347	0.022790	5.279	2.3700	0.023810
	KLF	α	1.043	1.068	0.010310	1.536	0.8825	0.008998
		β	2.089	1.462	0.016060	2.124	1.5230	0.013001
		σ	5.268	2.371	0.023550	5.270	2.4030	0.023690
50	SELF	α	1.048	1.050	0.006972	1.531	0.8846	0.006490
		β	2.090	1.474	0.009489	2.087	1.4730	0.009821
		σ	5.232	2.351	0.01666	5.189	2.3250	0.016740
	WSELF	α	1.048	1.050	0.006972	1.531	0.8828	0.006288
		β	2.090	1.474	0.009489	2.101	1.4790	0.009927
		σ	5.232	2.351	0.016660	5.211	2.3410	0.017700
	MSELF	α	1.034	1.039	0.007316	1.527	0.8874	0.006448
		β	2.092	1.469	0.009653	2.098	1.4920	0.010050
		σ	5.222	2.338	0.015010	5.210	2.3380	0.016960
	PLF	α	1.040	1.039	0.007868	1.532	0.8821	0.005888
		β	2.092	1.483	0.010740	2.106	1.4870	0.011150
		σ	5.213	2.344	0.016290	5.208	2.3490	0.015410
	KLF	α	1.036	1.049	0.007766	1.525	0.8777	0.006727
		β	2.083	1.469	0.010860	2.090	1.5010	0.009645
		σ	5.214	2.339	0.016290	5.211	2.3680	0.016840
100	SELF	α	1.032	1.033	0.005961	1.525	0.8811	0.005172
		β	2.068	1.468	0.008333	2.062	1.4570	0.008538
		σ	5.156	2.305	0.013300	5.138	2.2990	0.013400
	WSELF	α	1.026	1.016	0.005706	1.527	0.8789	0.005251
		β	2.066	1.460	0.008159	2.079	1.4720	0.008774
		σ	5.171	2.334	0.014370	5.156	2.3230	0.012850
	MSELF	α	1.025	1.034	0.005769	1.515	0.8779	0.005380
		β	2.076	1.465	0.007611	2.069	1.4680	0.008506
		σ	5.158	2.310	0.012610	5.150	2.3050	0.013730
	PLF	α	1.031	1.032	0.006003	1.531	0.8831	0.004924
		β	2.070	1.467	0.008199	2.075	1.4710	0.008767
		σ	5.145	2.310	0.012640	5.154	2.3120	0.012400
	KLF	α	1.033	1.039	0.005909	1.525	0.8796	0.005095
		β	2.059	1.451	0.009003	2.069	1.4770	0.008671
		σ	5.167	2.318	0.011850	5.163	2.3220	0.013640
200	SELF	α	1.017	1.018	0.005003	1.517	0.8755	0.004717
		β	2.048	1.449	0.006965	2.046	1.4460	0.007279
		σ	5.107	2.288	0.010740	5.095	2.2900	0.011710
	WSELF	α	1.015	1.011	0.004826	1.516	0.8728	0.004627
		β	2.043	1.446	0.007476	2.050	1.4480	0.007437
		σ	5.108	2.300	0.011210	5.099	2.2970	0.011140
	MSELF	α	1.015	1.026	0.004901	1.511	0.8737	0.004783
		β	2.055	1.450	0.007008	2.042	1.4470	0.007440
		σ	5.100	2.275	0.010780	5.097	2.2730	0.011330
	PLF	α	1.020	1.020	0.005365	1.528	0.8823	0.004259
		β	2.050	1.452	0.007344	2.054	1.4570	0.007597
		σ	5.097	2.285	0.010740	5.105	2.2860	0.010340
	KLF	α	1.025	1.027	0.005254	1.520	0.8760	0.004363
		β	2.042	1.439	0.008152	2.044	1.4550	0.007055
		σ	5.120	2.294	0.011180	5.110	2.3010	0.011000

TABLE 8
Monte Carlo Markov Chain results for Bayesian analysis.

			Conf 3			Conf 4		
n	LF	Parameter	Estimate	SD	MC error	Estimate	SD	MC error
25	SELF	α	10.550	3.341	0.033460	1.5270	0.8761	0.008765
		β	2.0990	1.490	0.014450	2.0930	1.4730	0.013280
		σ	5.2470	2.332	0.021880	10.520	3.3580	0.028410
	WSELF	α	10.520	3.339	0.031020	1.5250	0.8728	0.009352
		β	2.1080	1.488	0.014580	2.1130	1.4790	0.014250
		σ	5.2560	2.358	0.025710	10.530	3.3370	0.032700
	MSELF	α	10.560	3.309	0.031930	1.5290	0.8774	0.008371
		β	2.1070	1.521	0.015960	2.1110	1.4800	0.016890
		σ	5.2490	2.357	0.020070	10.520	3.3560	0.033210
	PLF	α	10.480	3.339	0.031200	1.5200	0.8697	0.008357
		β	2.1170	1.505	0.013440	2.1050	1.4850	0.015900
		σ	5.2530	2.363	0.024950	10.530	3.3560	0.030850
	KLF	α	10.500	3.298	0.037100	1.5310	0.8916	0.007701
		β	2.1150	1.497	0.013240	2.1050	1.4760	0.014250
		σ	5.2860	2.374	0.024530	10.480	3.3230	0.028260
50	SELF	α	10.430	3.302	0.022690	1.5230	0.8776	0.006221
		β	2.0920	1.486	0.010540	2.0810	1.4600	0.009970
		σ	5.1870	2.319	0.016010	10.400	3.3080	0.021830
	WSELF	α	10.400	3.291	0.021650	1.5230	0.8738	0.006244
		β	2.0860	1.476	0.010100	2.0840	1.4650	0.010920
		σ	5.2000	2.342	0.017380	10.420	3.3090	0.020990
	MSELF	α	10.430	3.272	0.027290	1.5340	0.8873	0.006008
		β	2.0850	1.489	0.011340	2.0840	1.4570	0.010210
		σ	5.2120	2.347	0.014750	10.420	3.3260	0.023290
	PLF	α	10.370	3.277	0.022780	1.5200	0.8734	0.005806
		β	2.0900	1.484	0.010820	2.0810	1.4720	0.011170
		σ	5.2000	2.321	0.016350	10.410	3.3080	0.022360
	KLF	α	10.410	3.284	0.022750	1.5320	0.8855	0.005686
		β	2.0720	1.465	0.009511	2.0830	1.4720	0.010080
		σ	5.2130	2.335	0.016140	10.380	3.2730	0.023400
100	SELF	α	10.330	3.275	0.017820	1.5160	0.8759	0.005006
		β	2.0660	1.470	0.008910	2.0630	1.4530	0.009293
		σ	5.1390	2.301	0.012920	10.310	3.2830	0.018050
	WSELF	α	10.270	3.243	0.01828	1.5180	0.8710	0.004976
		β	2.0580	1.456	0.008232	2.0650	1.4560	0.008876
		σ	5.1620	2.326	0.014480	10.300	3.2640	0.018510
	MSELF	α	10.300	3.241	0.019550	1.5260	0.8834	0.005104
		β	2.0640	1.472	0.008006	2.0580	1.4470	0.008555
		σ	5.1570	2.315	0.011940	10.330	3.2860	0.018540
	PLF	α	10.260	3.234	0.019820	1.5180	0.8734	0.004490
		β	2.0680	1.459	0.007952	2.0570	1.4530	0.009185
		σ	5.1480	2.310	0.012960	10.310	3.2490	0.018990
	KLF	α	10.310	3.261	0.019990	1.5270	0.8765	0.004858
		β	2.0540	1.458	0.008557	2.0690	1.4610	0.008564
		σ	5.1590	2.298	0.013150	10.290	3.2530	0.018150
200	SELF	α	10.230	3.236	0.015260	1.5130	0.8758	0.004373
		β	2.0430	1.452	0.007414	2.0380	1.43800	0.007754
		σ	5.0970	2.283	0.011030	10.200	3.2510	0.015330
	WSELF	α	10.160	3.213	0.017050	1.5120	0.8700	0.004208
		β	2.0400	1.442	0.006924	2.0430	1.4430	0.007404
		σ	5.1060	2.289	0.011910	10.190	3.2300	0.015150
	MSELF	α	10.190	3.206	0.016820	1.5200	0.8799	0.004689
		β	2.0400	1.447	0.007311	2.0360	1.4330	0.006840
		σ	5.1080	2.286	0.010730	10.220	3.2410	0.015530
	PLF	α	10.170	3.204	0.015260	1.5140	0.8729	0.003971
		β	2.0430	1.440	0.006612	2.0400	1.4430	0.007440
		σ	5.0880	2.279	0.010770	10.210	3.2230	0.017140
	KLF	α	10.220	3.225	0.016310	1.5210	0.8762	0.004170
		β	2.0340	1.439	0.007505	2.0410	1.4440	0.007536
		σ	5.1030	2.276	0.011810	10.200	3.2250	0.016800

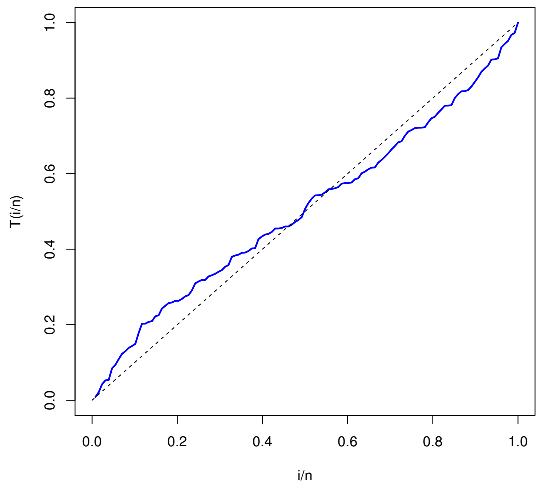

Figure 2 – The TTT plot for bladder cancer data.

TABLE 9
MLEs, K-S and the corresponding p-value for the real data set.

Model		Estimates			K-S	p-value
PGW(α,β,σ)	1.56	2.37	3.51		0.03209	0.99941
EWe(α,β,σ)	0.49	0.54	3.97		0.04830	0.92630
BWe(α,β,a,b)	0.39	25.46	6.45	8.33	0.05430	0.84480
We(α,β)	9.55	1.05			0.06950	0.56660

We use the different estimation methods discussed in Sections 3 and 5 to estimate the unknown parameters of the PGW model for the bladder cancer data. The estimates are displayed in Table 10. The Bayes estimates are obtained under different loss functions and based on non-informative prior and 12000 MCMC samples with 2000 as a burn-in period. The values of K-S and the corresponding p-value are also presented in Table 10 for all methods. The trace plot of 10000 chain values of α, β and σ are presented in Figure 3. From the results in Table 10, we can conclude that the WLSEs gives a better fit to the bladder cancer data than the other methods because it has the smallest K-S distance with largest p-value, i.e. K-S $=$ 0.025119 and p-value $=$ 0.999998. Also, Table 11 shows the 95% percentiles bootstrap confidence intervals for the classical methods of estimation and the credible intervals for the unknown parameters of the PGW distribution. As expected, the Bayes credible intervals have the shortest lengths when compared with those obtained based on frequentist methods of estimation.

TABLE 10
The parameters estimates, K-S and the corresponding p-value.

Method	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	K-S	p-value
MLE	1.56	2.37	3.51	0.032090	0.999405
MME	1.69	2.81	2.94	0.035014	0.997562
LSE	1.77	2.73	3.22	0.025619	0.999996
WLS	1.69	2.61	3.29	0.025119	0.999998
PCE	1.76	2.88	3.31	0.062299	0.703170
MPS	1.58	2.41	3.31	0.039635	0.987897
CVM	1.80	2.75	3.22	0.026308	0.999992
AD	1.66	2.50	3.42	0.025265	0.999998
RAD	1.66	2.45	3.53	0.028283	0.999954
SELF	1.56	2.38	3.51	0.034567	0.997988
WSELF	1.55	2.37	3.51	0.033061	0.999008
MSELF	1.54	2.36	3.50	0.034335	0.998184
PLF	1.56	2.39	3.52	0.035316	0.997237
KLF	1.55	2.38	3.51	0.033813	0.998570

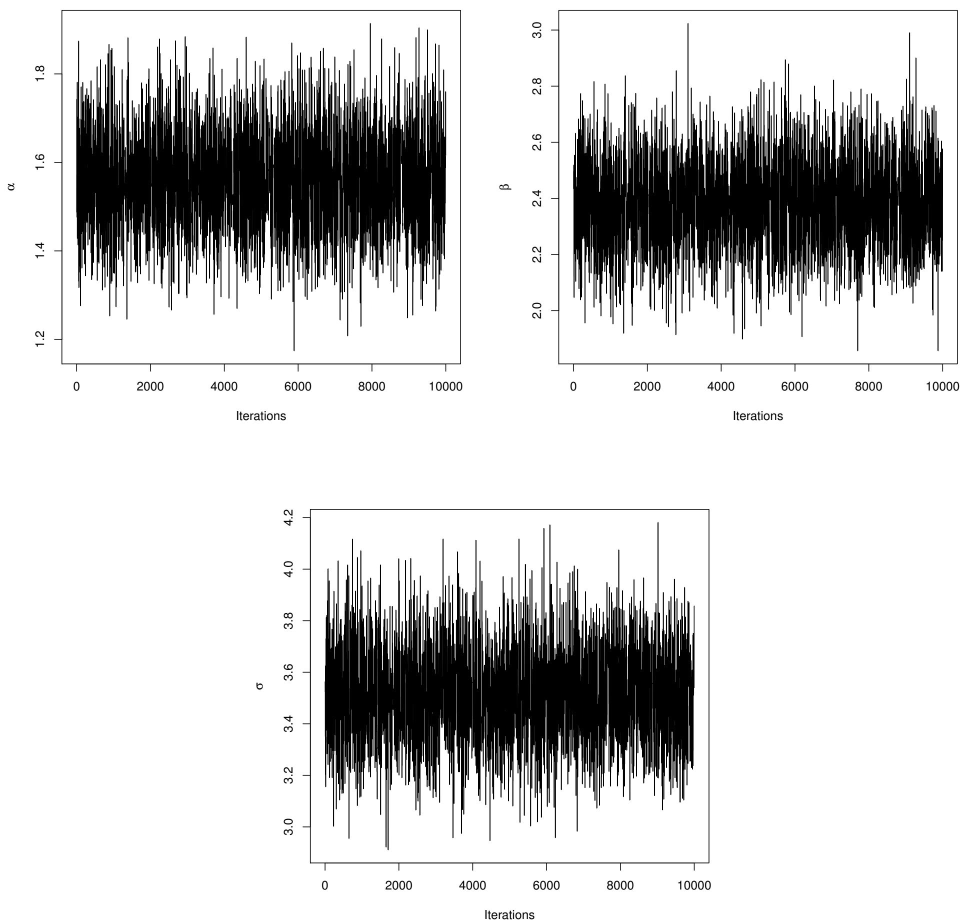

Figure 3 – MCMC outputs for α, β and σ.

TABLE 11
Bootstrap confidence\credible intervals of the unknown parameters of the PGW distribution.

Method	α	β	σ
MLE	(1.182, 2.335)	(1.230, 4.260)	(2.186, 7.244)
MME	(0.953, 2.142)	(1.026, 4.221)	(1.818, 7.677)
LSE	(1.213, 2.871)	(1.103, 5.542)	(1.951, 7.571)
WLS	(1.225, 2.541)	(1.299, 4.852)	(2.043, 6.726)
PCE	(1.577, 1.776)	(2.617, 3.223)	(1.707, 3.365)
MPS	(1.087, 2.125)	(0.972, 3.684)	(2.233, 8.175)
CVM	(1.253, 2.766)	(1.154, 4.968)	(1.985, 6.241)
AD	(1.225, 2.309)	(1.275, 4.053)	(2.105, 6.141)
RAD	(1.340, 2.836)	(1.734, 5.217)	(2.101, 5.248)
Bayes	(1.364, 1.769)	(2.086, 2.700)	(3.145, 3.868)

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