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MIXED LOSS AND DELAY RETRIAL QUEUEING SYSTEM WITH TWO CLASSES OF CUSTOMERS

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1. INTRODUCTION

Retrial queues are described by the feature that arriving customers (or calls, requests) who find the server busy join the retrial group/orbit to try again for their requests in a random order and at random intervals . Retrial queues are widely used as mathematical models of several computer systems: packet switching networks, shared bus local area networks operating under the Carrier Sense Multiple Access protocol and collision avoidance star local area networks. For recent bibliographies on retrial queues, see Yang and Templeton (1987), Falin (1990), Falin and Templeton (1997) and Artalejo (1999).

In modern communication networks, the multiplexers are expected to handle a variety of data traffic types. The communication services required may be carried out using packet-switched technology, circuit-switched technology or hybrid techniques. Integration of these services and traffic types corresponding to them, for transmission over networks, has to be carried out in an effective manner. A single system with a number of data inputs coming into it may be used, or the outputs of a number of stations may be multiplexed together for transmission over a network or a series of networks. These types of networks are termed as integrated services digital networks.

Here, we consider the integrated communication systems and focus on multiplexing of two types of traffic (customers) only. One type requires circuit-switched service and the other type consists of packet-switched traffic, typically voice and data traffic, respectively, which will hence-forth be referred to as class-1 and class-2 customers. The class-1 customers are blocked if transmission channel is not available and leave the system once for all while class-2 customers may be obliged to leave the service area and join the retrial group/orbit, to try again for their service after a random interval of time. These studies have been applied to the study of packet-switched data, circuit-switched voice, circuit-switched wide band video and other types of traffic (see Hammond and O'Reilly (1986)). The usage of these networks differ in their bandwidth requirements, their holding and service times and their arrival rates.

Moreover, queueing networks with batch arrivals are common in a number of retrial situations. In computer network systems, messages which are to be transmitted could consist of a random number of packets; comparable work on optimal control policies for batch arrival case is seldom found in the literature. This motivates us to develop a realistic model for queueing system with batch arrivals. Chaudhry and Templeton (1983) have provided a comprehensive review on bulk queues and their applications. Takahashi (1987) has discussed the mixed loss and delay queueing system with batch arrival. Langaris and Moutzoukis (1995) have investigated a retrial queue with structured batch arrivals, preemptive resume priorities for a single vacation model. Martin and Artalejo (1995) have dealt with a single server retrial queueing system in which the server must serve two types of customers with control retrial policy by using Markov renewal process technique. For detailed survey of retrial queues with two classes of customers and results on several models, one can refer to Choi and Chang (1999).

In this paper, we consider a retrial queueing system with two types of customers of class-1 and class-2 where class-1 customers arrive singly and class-2 customers arrive in a random batch size. The retrial time, the time interval between two consecutive attempts made by a class-2 customer in the retrial group is exponentially distributed and is independent of all previous retrial times and all other stochastic processes in the system. The organization of the paper is as follows: The model under consideration is described in section 2 along with the necessary and sufficient condition for the system to be stable. The steady state distributions of the server state and the orbit length are discussed in section 3. In section 4, some performance measures are obtained. Finally, in section 5, we establish a general stochastic decomposition law for our retrial queueing system.

2. MODEL DESCRIPTION AND STABILITY CONDITION

We consider a single-server retrial queueing system with two classes of customers, known as class-1 (impatient) customers and class-2 (non-impatient) customers. The class-1 customers arrive singly at the server as independent Poisson stream with rate λ_1 whereas the class-2 customers arrive in groups according to a time homogeneous Poisson process with parameter λ_2. The batch size Y of the class-2 customers is a random variable and $P(Y=k)=c_k$, $k=1, 2, 3, \ldots$ with $\sum_{k=1}^{\infty} c_k = 1$. Denote by $C(z)=\sum_{k=1}^{\infty} c_k z^k$ the generating function of the batch size distribution of the class-2 customers , and $\overline{C}$, the mean batch size and $\overline{C^2}$, second moment. Let class-m customers have independently and identically distributed service times S_m, with distribution functions $B_m(x)$ and probability density functions $b_m(x)$, $m=1, 2.$ Also, Laplace Stieltjes transforms and the first two moments of S_m, are denoted by $\beta_m^*(s) = \int_0^\infty e^{-sx} dB_m(x)$, $\overline{\beta_m}$ and $\overline{\beta_m^{(2)}}$, $m = 1, 2$, respectively.

If a class-1 customer arrives at the service system and finds the server free, then it immediately occupies the server and leaves the system after completion of service; whereas if the server is busy, the arriving class-1 customer leaves the system forever without getting its service. On the other hand, if the server is busy when a batch of class-2 customers arrives at the server, then all these class-2 customers leave the service area and enter the group of blocked customers called "orbit" and wait there to be served later; whereas if the server is free, then one of the arriving class-2 customers begins its service and the other class-2 customers join the "orbit" and wait there to be served later.

The class-2 customers in the "orbit" behave independently of each other and are persistent in the sense that they keep making retrials until they receive their requested service, after which they have no future effects on the system. Successive inter-retrial times of any class-2 customers are independent, exponentially distributed with a common mean $\frac{1}{\nu}$. We assume that the input flow of primary customers of both classes of customers, service times and intervals between repeated attempts are mutually independent. From this description, it is clear that either at any service completion epoch of class-1 customer or class-2 customer, the server becomes free, in such a case, the possible primary arrivals and the one (if any) in the orbit compete for service.

The state of the system at time t can be described by the Markov process $\{N(t);\ t \geq 0\} = \{(C(t), X(t), \xi(t));\ t \geq 0\}$, where $C(t)$ denotes the server state (0, 1 or 2 according as the server being free, busy with a class-1 customer or busy with a class-2 customer, respectively) and X(t) corresponds to the number of customers in orbit at time *t.* If C(t) $= 1$, then $\xi(t)$ represents the extended service time of the class-1 customer being served at time t, if $C(t)=2$, then $\xi(t)$ corresponds the extended service time of the class-2 customer being served at time t and if $C(t)=0$, then $\xi(t)=0$. The functions $\eta_1(x)=\frac{b_1(x)}{1-B_1(x)}$ and $\eta_2(x)=\frac{b_2(x)}{1-B_2(x)}$ are the conditional completion rates (at time x) for services of the class-1 and class-2 customers respectively.

In this section, we derive the necessary and sufficient condition for the system to be stable. To this end, in the following theorem we investigate the ergodicity of the embedded Markov chain at the customer departure epochs. Let $\{t_n;\ n \in N\}$ be the sequence of epochs of the service completion times at which the server is idle, i.e., the sequence of epochs of either service completion times of class-1 (impatient) customers or class-2 (non impatient) customers. The sequence $\{X_n = X(t_n+)\}$ forms a Markov chain which is embedded in our retrial queueing system on the state space N.

Theorem 1 : Let X_n be the orbit length at the time of the n^{th} customer's departure, $n \geq 1$. Then $\{X_n;\ n \geq 1\}$ is ergodic if and only if $\lambda_2 \overline{C}\ \ \overline{\beta_2} < 1$.

Proof: It is not difficult to see that $\{X_n;\ n \geq 1\}$ is irreducible and aperiodic. To prove it is also positive recurrent, we shall use the following Foster's criterion: An irreducible and aperiodic Markov chain is ergodic if there exists a non negative function $f(j),\ j \in N$ and $\varepsilon > 0$ such that the mean drift $\chi_j = E[f(X_{n+1}) - f(X_n) / X_n = j]$ is finite for all $j \in N$ and $\chi_j \leq -\varepsilon$ for all $j \in N$, except perhaps for a finite number of j's. In our case, we consider the function $f(j) = j$. Then, we have for j =0,

$$\chi_0 = \frac{\lambda_1}{\lambda_1 + \lambda_2}[\lambda_2 \overline{C}\ \ \overline{\beta_1}] + \frac{\lambda_2}{\lambda_1 + \lambda_2}[\lambda_2 \overline{C}\ \ \overline{\beta_2}] = \frac{\overline{C}\,\lambda_2\,(\lambda_1 \overline{\beta_1}\ + \lambda_2 \overline{\beta_2})}{\lambda_1 + \lambda_2}$$

and for j=1, 2, 3, ... ,

$$\chi_j = \frac{\nu\ j}{\lambda_1 + \lambda_2 + \nu j}[j - 1 + \lambda_2 \overline{C}\ \ \overline{\beta_2} - j] + \frac{\lambda_1}{\lambda_1 + \lambda_2 + j\ \ \nu}[j + \lambda_2 \overline{C}\ \ \overline{\beta_1} - j]$$

$$+ \frac{\lambda_2}{\lambda_1 + \lambda_2 + j\ \ \nu}[j + \lambda_2 \overline{C}\ \overline{\beta_2} - j]$$

$$= \frac{(\nu\ j + \lambda_2)\lambda_2 \overline{C}\ \overline{\beta_2} + \lambda_1 \overline{\beta_1}\,\lambda_2 \overline{C} - \nu\ j}{\lambda_1 + \lambda_2 + \nu\ j},$$

so that

$$\lim_{j \to \infty} \chi_j\ = \lambda_2 \overline{C}\ \ \overline{\beta_2} - 1.$$

Thus, if condition $\lambda_2\ \overline{C}\ \ \ \overline{\beta_2} < 1$ is fulfilled then Foster's criterion guarantees that the embedded Markov chain $\{X_n;\ n \geq 1\}$ is ergodic. Hence the sufficient condition for ergodicity is proved.

The same inequality is also necessary for ergodicity. As noted in Sennott et al. (1983), we can guarantee nonergodicity, if the Markov chain $\{X_n;\ n \geq 1\}$ satisfies Kaplan's condition, namely $\chi_j < \infty$ for all $j \geq 0$ and there exists $j_0 \in N$ such that $\chi_j \geq 0$ for $j \geq j_0$. It should be noted that in our case, Kaplan's condition is satisfied because there is a k such that $r_{ij}=0$ for $j<i-k$ and $i>0$, where $R=(r_{ij})$ is the one step transition matrix of $\{X_n;\ n\geq 1\}$. Then $\lambda_2 \overline{C}\ \overline{\beta_2}\ \geq 1$ implies the nonergodicity of the Markov chain.

As $\{(X_n,\ t_n);\ n\ \geq\ 0\}$ is an embedded Markov renewal process of the semi-regenerative process $\{(C(t), X(t), \xi(t));\ t\ \geq 0\}$, it can be shown from the results in Cinlar (1975, pp. 343-350), that the limiting probabilities of $\{(C(t), X(t), \xi(t));\ t\ \geq\ 0\}$ exist and are positive if $\lambda_2 \overline{C}\ \overline{\beta_2} < 1$ and $B_1(.)$ and $B_2(.)$ satisfy regular conditions (i.e., the existence of the first two moments or both $1-B_1(x)$ and $1-B_2(x)$ being Riemann integrable over $[0,\infty)$).

3. STATIONARY DISTRIBUTION

In this section, we study the steady state distribution for the system under consideration. For the Markov process $\{N(t);\ t\ \geq\ 0\}$, we define the unconditional probabilities:

$$R_n(t)=P\{C(t)=0,\ X(t)=n\},\qquad n=0,\ 1,\ 2,\ ...$$

and the unconditional probability densities:

$$P_n(x,t)dx=P\{C(t)=1,\ X(t)=n,\ x\leq\xi(t)<x+dx\}$$

$$\text{for } t\geq 0,\ x\geq 0\ \ \&\ \ n\geq 0$$

and

$$Q_n(x,t)dx=P\{C(t)=2,\ X(t)=n,\ x\leq\xi(t)<x+dx\}$$

$$\text{for } t\geq 0,\ x\geq 0\ \&\ n\geq 0.$$

Following routine procedures (see, for instant, Keilson et al.(1968)), we obtain the following system of equations that govern the dynamics of the system behaviour:

$$\frac{dR_n(t)}{dt}=-(\lambda_1+\lambda_2+n\nu)R_n(t)+\int_0^\infty P_n(x,t)\eta_1(x)dx+\int_0^\infty Q_n(x,t)\eta_2(x)dx,$$

$$n=0,\ 1,\ 2,... \tag{3.1}$$

$$\frac{\partial P_0(x,t)}{\partial t}+\frac{\partial P_0(x,t)}{\partial x}=-(\lambda_2+\eta_1(x))P_0(x,t) \tag{3.2}$$

$$\frac{\partial P_n(x,t)}{\partial t}+\frac{\partial P_n(x,t)}{\partial x}=-(\lambda_2+\eta_1(x))P_n(x,t)+\lambda_2\sum_{i=1}^{n}c_iP_{n-i}(x,t),$$

$$n=1, 2, 3,\ldots \tag{3.3}$$

$$\frac{\partial Q_0(x,t)}{\partial t}+\frac{\partial Q_0(x,t)}{\partial x}=-(\lambda_2+\eta_2(x))\ Q_0(x,t) \tag{3.4}$$

$$\frac{\partial Q_n(x,t)}{\partial t}+\frac{\partial Q_n(x,t)}{\partial x}=-(\lambda_2+\eta_2(x))\ Q_n(x,t)+\lambda_2\sum_{i=1}^{n}c_iQ_{n-i}(x,t),$$

$$n=1, 2, 3\ \ldots\ . \tag{3.5}$$

The boundary conditions are

$$P_n(0,t)=\lambda_1\,\mathrm{R}_n(t),\qquad n=0,\ 1,\ 2,\ \ldots \tag{3.6}$$

$$Q_0(0,t)=c_1\lambda_2\mathrm{R}_0(t)+\nu\ \mathrm{R}_1(t) \tag{3.7}$$

and

$$Q_n(0,t)=c_{n+1}\lambda_2\mathrm{R}_0(t)+(n+1)\nu\mathrm{R}_{n+1}(t)+\lambda_2\sum_{i=1}^{n}c_i\mathrm{R}_{n-i+1}(t),$$

$$n=1,\ 2,\ 3,\ \ldots \tag{3.8}$$

We assume that the condition $\lambda_2\overline{C}\ \overline{\beta_2}<1$ is fulfilled, so that the limiting probabilities $R_n=\lim_{t\to\infty}R_n(t)$, for $n=0, 1, 2, \ldots$ and the limiting densities $P_n(x)=\lim_{t\to\infty}P_n(x,t)$, for $x\geq 0$ and $n=0, 1, 2, \ldots$ and $Q_n(x)=\lim_{t\to\infty}Q_n(x,t)$, for $x\geq 0$ and $n=0, 1, 2, \ldots,$ exist. Letting $t\to\infty$ in equations (3.1) - (3.8), we get

$$(\lambda_1+\lambda_2+n\nu)\mathrm{R}_n=\int_0^\infty P_n(x)\eta_1(x)dx+\int_0^\infty Q_n(x)\eta_2(x)dx,$$

$$n=0,\ 1,\ 2,\ \ldots \tag{3.9}$$

$$\frac{dP_0(x)}{dx}=-(\lambda_2+\eta_1(x))P_0(x) \tag{3.10}$$

$$\frac{dP_n(x)}{dx}=-(\lambda_2+\eta_1(x))P_n(x)+\lambda_2\sum_{i=1}^{n}c_iP_{n-i}(x),\quad n=1, 2, 3, \ldots \tag{3.11}$$

$$\frac{dQ_0(x)}{dx} = -(\lambda_2 + \eta_2(x))\ Q_0(x) \tag{3.12}$$

$$\frac{dQ_n(x)}{dx} = -(\lambda_2 + \eta_2(x))\ Q_n(x) + \lambda_2 \sum_{i=1}^{n} c_i Q_{n-i}(x),\ \ n=1, 2, 3, \dots . \tag{3.13}$$

The steady state boundary conditions are

$$P_n(0) = \lambda_1 R_n, \qquad n = 0, 1, 2, \dots \tag{3.14}$$

$$Q_0(0) = c_1\lambda_2 R_0 + \nu\ R_1 \tag{3.15}$$

and

$$Q_n(0) = c_{n+1}\lambda_2 R_0 + (n+1)\nu\ R_{n+1} + \lambda_2 \sum_{i=1}^{n} c_i R_{n-i+1},\quad n = 1, 2, 3, \dots \tag{3.16}$$

and the normalizing condition is

$$\sum_{n=0}^{\infty} R_n + \sum_{n=0}^{\infty} \int_0^{\infty} P_n(x)dx + \sum_{n=0}^{\infty} \int_0^{\infty} Q_n(x)dx = 1. \tag{3.17}$$

In order to solve the equations (3.9) - (3.16), we define the generating functions:

$$R(z) = \sum_{n=0}^{\infty} R_n z^n, \quad P(x,z) = \sum_{n=0}^{\infty} P_n(x) z^n \quad \text{and} \quad Q(x,z) = \sum_{n=0}^{\infty} Q_n(x) z^n.$$

The following theorem discusses the steady state distribution of the system.

Theorem 2: If $\lambda_2 \overline{C}\ \overline{\beta_2} < 1$, then the joint steady state distributions of $\{N(t);\ t \geq 0\}$ under different server states are obtained as

$$R(z) = R_0\ \phi(z) \tag{3.18}$$

$$P(x,z) = \lambda_1 R_0 \phi(z) e^{-\lambda_2(1-C(z))x - \int_0^{\infty} \eta_1(u)du} \tag{3.19}$$

$$Q(x,z) = R_0 \phi(z) \frac{\{\lambda_2(1-C(z)) + \lambda_1[1-\beta_2^*(\lambda_2(1-C(z)))]\}}{[\beta_2^*(\lambda_2(1-C(z))) - z]}\ e^{-\lambda_2(1-C(z))x - \int_0^x \eta_2(u)du} \tag{3.20}$$

where

$$\phi(z) = \exp\left\{-\int_0^z \frac{\{\lambda_1 u[1-\beta_1^*(\lambda_2(1-C(u)))]+\lambda_2[u-C(u)\beta_2^*(\lambda_2(1-C(u)))]\}}{\nu\, u[u-\beta_2^*(\lambda_2(1-C(u)))]}du\right\} \tag{3.21}$$

and the probability R_0 is to be determined from the normalization condition.

Proof: Multiplying equations (3.9) - (3.16) by z^n and summing over all possible values of n, we obtain the following equations:

$$(\lambda_1+\lambda_2)R(z) + \nu z\frac{dR(z)}{dz} = \int_0^\infty P(x,z)\eta_1(x)dx + \int_0^\infty Q(x,z)\eta_2(x)dx \tag{3.22}$$

$$\frac{\partial P(x,z)}{\partial x} = -[\lambda_2(1-C(z))+\eta_1(x)]\, P(x,z) \tag{3.23}$$

$$\frac{\partial Q(x,z)}{\partial x} = -[\lambda_2(1-C(z))+\eta_2(x)]\, Q(x,z) \tag{3.24}$$

$$P(0,z) = \lambda_1 R(z) \tag{3.25}$$

$$Q(0,z) = \frac{1}{z}\left[\nu z\frac{dR(z)}{dz} + \lambda_2 C(z)R(z)\right]. \tag{3.26}$$

Solving equations (3.23) and (3.24), we have

$$P(x,z) = P(0,z)\, e^{-\lambda_2(1-C(z))x-\int_0^x \eta_1(u)du} \tag{3.27}$$

$$Q(x,z) = Q(0,z)\, e^{-\lambda_2(1-C(z))x-\int_0^x \eta_2(u)du}. \tag{3.28}$$

Substituting the right hand side of (3.25) into (3.27), we get

$$P(x,z) = \lambda_1 R(z)\, e^{-\lambda_2(1-C(z))x-\int_0^x \eta_1(u)du}. \tag{3.29}$$

Using (3.26) into (3.28), we obtain

$$Q(x,z)=\frac{1}{z}\left[\nu z\frac{dR(z)}{dz}+\lambda_2 C(z)R(z)\right] e^{-\lambda_2(1-C(z))x-\int_0^x \eta_2(u)du}. \tag{3.30}$$

Now, applying the equations (3.29) and (3.30) to (3.22) and integrating, we have after rearrangement:

$$\frac{dR(z)}{dz}+\frac{\{\lambda_1 z[1-\beta_1^*(\lambda_2(1-C(z)))]+\lambda_2[z-C(z)\beta_2^*(\lambda_2(1-C(z)))]\}}{\nu\, z[z-\beta_2^*(\lambda_2(1-C(z)))]}R(z)=0. \tag{3.31}$$

Solving the above equation gives

$$R(z)=D\phi(z) \tag{3.32}$$

where $\phi(z)$ is defined as in (3.21) and D is a constant. To determine the constant D, we set $z=0$ in equation (3.32) and obtain $D=R_0$, so that

$$R(z)=R_0\,\phi(z)\,. \tag{3.33}$$

Substituting (3.31) and (3.33) back into (3.29) and (3.30), we have after algebraic manipulation:

$$P(x,z)=\lambda_1 R_0\phi(z)\; e^{-\lambda_2(1-C(z))x-\int_0^{\infty}\eta_1(u)du} \tag{3.34}$$

and

$$Q(x,z) = R_0\phi(z)\frac{\{\lambda_2(1-C(z))+\lambda_1[1-\beta_1^*(\lambda_2(1-C(z)))]\}}{[\beta_2^*(\lambda_2(1-C(z)))-z]} \times e^{-\lambda_2(1-C(z))x-\int_0^{x}\eta_2(u)du}. \tag{3.35}$$

Hence the theorem follows from (3.33) - (3.35).

We define the partial generating function $\psi(z)=\int_0^{\infty}\psi(x,z)dx$ for any generating function $\psi(x,z)$. Then, from (3.19) - (3.20), we have the following partial generating functions under steady state condition:

$$P(z) = \frac{R_0\lambda_1\phi(z)[1-\beta_1^*(\lambda_2(1-C(z)))]}{\lambda_2(1-C(z))} \tag{3.36}$$

and

$$Q(z) = \frac{R_0\phi(z)\{\lambda_2(1-C(z))+\lambda_1[1-\beta_1^*(\lambda_2(1-C(z)))]\}\times[1-\beta_2^*(\lambda_2(1-C(z)))]}{\lambda_2(1-C(z))[\beta_2^*(\lambda_2(1-C(z)))-z]} \tag{3.37}$$

where

$$R_0 = \frac{1-\lambda_2\overline{\beta_2}\ \overline{C}}{(1+\lambda_1\overline{\overline{\beta_1}})}\times\exp\left\{\int_0^1\frac{\{\lambda_1 u[1-\beta_1^*(\lambda_2(1-C(u)))]+\lambda_2[u-C(u)\beta_2^*(\lambda_2(1-C(u)))]du\}}{\nu\ u[u-\beta_2^*(\lambda_2(1-C(u)))]}\right\}$$
(3.38)

which is determined using the normalizing condition $R(1)+P(1)+Q(1)=1$. The probability generating function for the number of customers in the system denoted by $K(z)$ is given by $K(z)=R(z)+zP(z)+zQ(z)$. Substituting for $R(z)$, $P(z)$ and $Q(z)$, we have

$$K(z)=\frac{\{R_0(1-z)\phi(z)\{\lambda_1 z[1-\beta_1^*(\lambda_2(1-C(z)))]+\lambda_2(1-C(z))\beta_2^*(\lambda_2(1-C(z)))\}}{\lambda_2(1-C(z))\{\beta_2^*(\lambda_2(1-C(z)))-z\}}.$$
(3 39)

Note that $R(z)$ is the probability generating function of the orbit size when the server is free, $P(z)$ is the probability generating function of the orbit size when the server is busy with class-1 (impatient) customer, $Q(z)$ is the probability generating function of the orbit size when the server is busy with class-2 (non-impatient) customer and R_0 is the probability that the server is free in the system, i.e., no customer in the system. These expressions are used in the next section for obtaining performance measures.

4. PERFORMANCE MEASURES

In this section, we derive some performance measures for the system under steady state condition. Let U be the steady state probability that the server is busy for providing the service for either a class-1 (impatient) customer or a class-2 (non-impatient) customer, I be the steady state probability that the server is idle during the retrial time or no customer in the system, V be the steady state probability that the server is idle during the retrial time and D be the steady state loss probability for the class-1 customers. From the results of the previous section, we obtain:

$$U = P(1)+Q(1) = \frac{\lambda_1\overline{\beta_1}+\lambda_2\overline{\beta_2}\ \overline{C}}{1+\lambda_1\overline{\overline{\beta_1}}} \tag{4.1}$$

$$I = R(1) = \frac{1-\lambda_2\overline{\beta_2}\ \overline{C}}{1+\lambda_1\overline{\overline{\beta_1}}} \tag{4.2}$$

$$V = R(1)-R_0 = \frac{1-\lambda_2\overline{\beta_2}\ \overline{C}}{1+\lambda_1\overline{\overline{\beta_1}}}\ \{1-[\phi(1)]^{-1}\} \tag{4.3}$$

and

$$D = 1 - R(1) = \frac{\lambda_1 \overline{\beta_1} + \lambda_2 \overline{\beta_2}\ \overline{C}}{1 + \lambda_1 \overline{\beta_1}} \tag{4.4}$$

which is same as (4.1).

The mean number of customers in the system L_s under steady state condition is obtained by differentiating (3.39) with respect to z and evaluating at z=1 as

$$\begin{aligned} L_s = K'(1) &= \frac{\{2\lambda_1 \overline{\beta_1} + \lambda_2 \overline{C}(\lambda_1 \overline{\beta_1^{(2)}} + 2\overline{\beta_2})\}}{2(1+\lambda_1 \overline{\beta_1})} \\ &+ \frac{\lambda_2 \{\overline{C}(\lambda_1 \overline{\beta_1} + \lambda_2 \overline{\beta_2} + 1) - 1\}}{\nu[1 - \lambda_2 \overline{\beta_2}\ \overline{C}]} + \frac{\lambda_2 \{\overline{\beta_2}(\overline{C^2} - \overline{C}) + \lambda_2 \overline{\beta_2^{(2)}}(\overline{C})^2\}}{2[1 - \lambda_2 \overline{\beta_2} \overline{C}]}. \end{aligned} \tag{4.5}$$

Let W be the average time a class-2 customer spends in the system under steady state. Due to Little's formula, we have

$$W = \frac{L_s}{\lambda \overline{C}}.$$

Define $H(z) = R(z) + P(z) + Q(z)$. Then $H(z)$ represents the probability generating function for the number of customers in the orbit. Using (3.18) - (3.21) and (3.38) and simplifying, we get

$$H(z) = \frac{\{R_0(1-z)\phi(z)\{\lambda_1[1 - \beta_1^*(\lambda_2(1-C(z)))] + \lambda_2(1-C(z))\}\}}{\lambda_2(1-C(z))\{\beta_2^*(\lambda_2(1-C(z))) - z\}}. \tag{4.6}$$

Hence, the mean number of customers in the orbit is given by

$$\begin{aligned} L_q = H'(1) = &\frac{\lambda_1 \overline{\beta_1^{(2)}} \lambda_2 \overline{C}}{2(1+\lambda_1 \overline{\beta_1})} + \frac{\lambda_2 \{\overline{C}(\lambda_1 \overline{\beta_1} + \lambda_2 \overline{\beta_2} + 1) - 1\}}{\nu\,(1 - \lambda_2 \overline{\beta_2}\ \overline{C})} \\ &+ \frac{\lambda_2 \{\overline{\beta_2}(\overline{C^2} - \overline{C}) + \lambda_2 \overline{\beta_2^{(2)}}(\overline{C})^2\}}{2(1 - \lambda_2 \overline{\beta_2}\ \overline{C})}, \end{aligned} \tag{4.7}$$

so that the mean rate of flow of repeated class-2 customers for our system is given by

$$\nu\, L_q \;=\; \frac{\nu\lambda_1 \overline{\beta_1^{(2)}}\lambda_2 \overline{C}}{2(1+\lambda_1\overline{\beta_1})} + \frac{\lambda_2\{\overline{C}(\lambda_1\overline{\beta_1}+\lambda_2\overline{\beta_2}+1)-1\}}{[1-\lambda_2\overline{\beta_2}\;\overline{C}]} + \frac{\nu\lambda_2\{\overline{\beta_2}(\overline{C^2}-\overline{C})+\lambda_2\overline{\beta_2^{(2)}}(\overline{C})^2\}}{2[1-\lambda_2\overline{\beta_2}\;\overline{C}]}. \tag{4.8}$$

Thus $\nu\, L_q \;\to\; \dfrac{\lambda_2\{\overline{C}\,(\lambda_1\overline{\beta_1}+\lambda_2\overline{\beta_2}+1)-1\}}{(1-\lambda_2\overline{\beta_2}\;\overline{C})}$, as $\nu\to 0$.

Furthermore, let M_0^1 and M_1^1 denote the partial moments defined by $M_0^1=\sum_{n=0}^{\infty} nR_n$ and $M_1^1=\sum_{n=0}^{\infty}\int_0^{\infty}[P_n(x)+Q_n(x)]dx$. If $\lambda_2\overline{C}\;\overline{\beta_2} < 1$, then routine differentiation of the partial probability generating functions $R(z)$, $P(z)$ and $Q(z)$ yield

$$M_0^1 = R_0\phi'(1) \tag{4.9}$$

and

$$\begin{aligned} M_1^1 \;=\; & \frac{\lambda_1 R_0}{2}\{2\overline{\beta_1}\phi'(1)+\overline{\beta_2^{(2)}}\lambda_2\overline{C}\phi(1)\}+\frac{R_0}{2(1-\lambda_2\overline{C}\overline{\beta_2})^2} \\ & \{(1-\lambda_2\overline{C}\;\overline{\beta_2})[(1+\lambda_1\overline{\beta_1})\;(2\lambda_2\overline{C}\overline{\beta_2}\phi'(1)+\lambda_2^2(\overline{C})^2\overline{\beta_2^{(2)}}\phi(1))+ \\ & (\overline{C^2}-\overline{C})(1+\lambda_1\overline{\beta_1})+\lambda_1\overline{\beta_1^{(2)}}\lambda_2^2(\overline{C})^2\overline{\beta_2}\phi(1)]+ \\ & \overline{\beta_2}(1+\lambda_1\overline{\beta_1})\phi(1)(\overline{\beta_2^{(2)}}\lambda_2(\overline{C})^2+\overline{\beta_2}(\overline{C^2}-\overline{C})\lambda_2^2\overline{C}\}. \end{aligned} \tag{4.10}$$

It is observed that the mean number of customers in the system in steady state is

$$L_S = M_0^1 + M_1^1 + \lim_{t\to\infty} P(C(t)=1) + \lim_{t\to\infty} P(C(t)=2).$$

Remark. 1: If $\lambda_1 = 0$, i.e., no class-1 customers arrive at the system, then the model reduces to the classical M/G/1 batch arrival retrial queue (Falin and Templeton (1997)). In this case, the probability generating function of the number of customers present in the system K(z) is given as

$$K(z)=\frac{(1-\lambda_2\overline{\beta_2}\;\overline{C})(1-z)\beta_2^*(\lambda_2(1-C(z)))}{\{\beta_2^*(\lambda_2(1-C(z)))-z\}}\,\frac{\phi(z)}{\phi(1)} \tag{4.11}$$

where

$$\phi(z) = \exp\left\{-\frac{\lambda_2}{\nu}\int_0^z \frac{C(u)\beta_2^*(\lambda_2(1-C(u)))-u]}{u[\beta_2^*(\lambda_2(1-C(u)))-u]}du\right\}$$

and the mean number of customers in the system L_S under steady state conditions is obtained from (4.5) as

$$L_S = \lambda_2\overline{C}\,\overline{\beta_2} + \frac{\lambda_2\{\overline{C}(1+\lambda_2\overline{\beta_2})-1\}}{\nu[1-\lambda_2\overline{\beta_2}\,\overline{C}]} + \frac{\lambda_2\{\overline{\beta_2}(\overline{C^2}-\overline{C})+\lambda_2\overline{\beta_2^{(2)}}(\overline{C})^2\}}{2[1-\lambda_2\overline{\beta_2}\,\overline{C}]}. \qquad (4.12)$$

Remark 2: If $\lambda_2 = 0$, i.e., no class-2 customers arrive at the system, then our model becomes an M/G/1 queueing system with no waiting line. Thus, the probability generating function of the number of customers in the system $K(z)$ is reduces to

$$K(z) = \frac{1+\lambda_1\overline{\beta_1}z}{1+\lambda_1\overline{\beta_1}} \qquad (4.13)$$

and the corresponding mean number of customers in the system L_S under steady state condition is obtained as

$$L_S = \frac{\overline{\beta_1}}{\frac{1}{\lambda_1}+\overline{\beta_1}}. \qquad (4.14)$$

We now consider a busy period of the system for the model under consideration. The mean of the system busy period is an interesting and important performance measure in the retrial context. The system busy period L is defined as the period that starts at an epoch when an arriving customer finds an empty system and ends at the next departure epoch at which the system is empty. The mean length of the system busy period of our model is obtained in a direct way by the theory of regenerative processes which leads to the limiting probabilities

$$R_0 = \lim_{t\to\infty} P\{(C(t), X(t)) = (0,0)\}$$

as follows:

$$R_0 = \frac{E(T_{00})}{\frac{1}{\lambda_1+\lambda_2}+E(L)}$$

where T_{00} is the amount of time in a regenerative cycle during which the system is in the state (0, 0). It is clear that

$$E(T_{00}) = \frac{1}{\lambda_1 + \lambda_2}$$

so that

$$E(L) = \frac{1}{\lambda_1 + \lambda_2}(R_0^{-1} - 1).$$

Using equation (3.38), we get

$$E(L) = \frac{\{(\lambda_2 \overline{C}\ \overline{\beta_2} - 1) + (1 + \lambda_1 \overline{\beta_1})\phi(1)\}}{(\lambda_1 + \lambda_2)(1 - \lambda_2 \overline{\overline{C}}\ \overline{\overline{\beta_2}})}. \tag{4.15}$$

Remark 3: If $\lambda_1 = 0$ and $\overline{C} = 1$, then the mean of the system busy period (4.15) reduces to

$$E(L) = \frac{1}{\lambda_2}\left\{\frac{\phi(1)}{1 - \lambda_2 \overline{\overline{\beta_2}}} - 1\right\} \tag{4.16}$$

which agrees with Artalejo and Lopez-Herrero (2000).

5. STOCHASTIC DECOMPOSITION

In this section, we analyse the stochastic decomposition property of the system size distribution. The literature on vacation models recognizes this problem as one of the most interesting features (see, for example, Cooper (1970), Doshi (1986) and Fuhrmann and Cooper (1985)). A key result in these analyses shows that the number of customers present in the system in steady state at an arbitrary point in time is distributed as the sum of two independent random variables, one of which is the number of customers present at an arbitrary point in the corresponding queueing model without server vacations and the other random variable may have different probabilistic interpretations in specific cases depending on how the vacations are scheduled (see Doshi (1986) and Takagi (1991) for more details).

Stochastic decomposition has also been observed to hold for some M/G/1 retrial queues (Artalejo (1997), Artalejo and Gomez-Corral (1997), Yang and Templeton (1987) and Neuts and Ramalhoto (1984)). The retrial queue, under consideration, can be thought of as a queueing system with generalized vacations (Fuhrmann and Cooper (1985)) in which the vaction begins at the end of either class-1 customer service or class-2 customer service time. Let $\Pi(z)$ be the probability generating function of the number of customers in the $M_1, M_2^X / G_1, G_2 / 1/1, \infty$ mixed loss and delay queueing system (Takahashi

(1987)) in the steady state at a random point in time, $\chi(z)$ be the probability generating function of the number of customers in the generalized vacation system at a random point in time given that the server is idle due to retrials and $K(z)$ be the probability generating function of the random variable being decomposed. Then the mathematical version of the stochastic decomposition is

$$K(z) = \Pi(z)\chi(z). \tag{5.1}$$

We now verify that the decomposition law is applicable to our retrial queue with two classes of customers analyzed in this paper. For the $M_1, M_2^X / G_1, G_2 / 1/1, \infty$ queue with mixed loss and delay system, we have

$$\Pi(z) = \frac{\{(1-z)(1-\lambda_2 \overline{C}\,\overline{\beta_2})\{\lambda_1 z[1-\beta_1^*(\lambda_2(1-C(z)))] + \lambda_2(1-C(z))\beta_2^*(\lambda_2(1-C(z)))\}}{(1+\lambda_1\overline{\beta_1})\lambda_2(1-C(z))\{\beta_2^*(\lambda_2(1-C(z))) - z\}}. \tag{5.2}$$

To obtain an expression for $\chi(z)$, we first define vacation in our retrial context. We say that the server is on vacation if the server is idle (there may be customers in the system even when the server is idle in the retrial queue context). Under this definition, we have

$$\chi(z) = \frac{R(z)}{R(1)} = \frac{\phi(z)}{\phi(1)}. \tag{5.3}$$

From (3.38) and (3.39), we can see that $K(z) = \Pi(z)\chi(z)$, which confirms that the stochastic decomposition law of Fuhrmann and Cooper (1985) is also valid for this special vacation system. However, we must point out that if the idle periods were not considered as vacations, the decomposition law would not apply here due to interference between customer retrials and server vacation.

For the appropriate choice of parametric values and distributions, the decomposition property for the models considered in Keilson *et al.* (1968) and Falin and Templeton (1997) can be deduced as special cases.

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RIASSUNTO

Un sistema di code "retrial" con perdite e attese con due tipologie di utenti

Il presente studio è dedicato all'analisi di una coda "retrial" con arrivi raggruppati in un singolo punto di servizio, in presenza di due tipologie di utenti. In caso di intasamento della coda, gli utenti del primo tipo escono dal sistema senza più ritornarvi, mentre gli utenti del secondo tipo attendono di venire serviti più tardi. Viene definita una condizione necessaria e sufficiente per la stabilità del sistema e vengono studiati la distribuzione della lunghezza della coda e la *performance* del sistema sotto particolari condizioni. Infine, viene definito un criterio di scomposizione per un sistema di code di questo tipo.

SUMMARY

Mixed loss and delay retrial queueing system with two classes of customers

This paper is concerned with the analysis of a single-server batch arrival retrial queue with two classes of customers. In the case of blocking, the class-1 customers leave the system forever whereas the class-2 customers leave the service area and enter the orbit and wait to be served later. The necessary and sufficient condition for the system to be stable is derived and analytic results for the queue length distribution as well as some performance measures of the system under steady state condition are obtained. A general decomposition law for the retrial queueing system is established.