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MATRIX POLYNOMIALS AND THEIR INVERSION: THE ALGEBRAIC FRAMEWORK OF UNIT-ROOT ECONOMETRICS REPRESENTATION THEOREMS (*)

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1. WHAT IS THIS PAPER FOR?

Classical proofs of unit-root econometrics representation theorems – in the wake of Granger's seminal paper and Johansen's key contributions – are somewhat cumbersome. Thus an attempt to develop a more convenient analytical toolkit to tackle the matter looks attractive and worth exploring. This is what this paper aims to achieve.

2. MATRIX POLYNOMIALS: PRELIMINARIES

Definition 2.1. A matrix polynomial of degree K in the scalar argument z is an expression of the form

$$A(z) = \sum_{k=0}^{K} A_k z^k \tag{2.1}$$

where the $A_k' s$ are (square) coefficient matrices and $A_K \neq \mathbf{0}$.

A Taylor series expansion of the matrix polynomial (1) about $z = 1$ leads to the representation

$$A(z) = A(1) + \sum_{k=0}^{K} (1-z)^k \cdot (-1)^k \cdot \frac{1}{k!} A^{(k)}(1) \tag{2.2}$$

where

$$A^{(k)}(1) = \left.\frac{\partial^k A(z)}{\partial z^k}\right]_{z=1} = k! \sum_{j=k}^{K} \binom{j}{k} \cdot A_j \tag{2.3}$$

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The dot-matrix notation $\dot{A}\ (1)$, $\ddot{A}(1)$ will be henceforth used for k = 1, 2.
The following reparametrizations

$$A\ (z) = \boldsymbol{Q}\ (z) \cdot (1 - z) + A\ (1) \tag{2.4}$$

$$A\ (z) = \boldsymbol{\Psi}\ (z) \cdot (1 - z)^2 - \dot{A}\ (1) \cdot (1 - z) + A\ (1) \tag{2.5}$$

where

$$\boldsymbol{Q}\ (z) = \sum_{k=1}^{K} (1-z)^{k-1} \cdot (-1)^k \cdot \frac{1}{k!} A^{(k)}\ (1), \quad \boldsymbol{Q}\ (1) \equiv - \dot{A}\ (1) \tag{2.6}$$

$$\boldsymbol{\psi}\ (z) = \sum_{k=2}^{K} (1-z)^{k-2} \cdot (-1)^k \cdot \frac{1}{k!} A^{(k)}\ (1), \quad \boldsymbol{\Psi}\ (1) \equiv \frac{1}{2} \ddot{A}\ (1) \tag{2.7}$$

are of special interest for the subsequent analysis.

Assumption 2.1. We will henceforth assume that the roots of the characteristic polynomial

$$\pi\ (z) = \det \{A\ (z)\} \tag{2.8}$$

lie either outside or on the unit circle and, in the latter case, are equal to one.

Theorem 2.1. Under the assumption above, the inverse of the matrix polynomial $A\ (z)$ can be given the following Laurent form

$$A^{-1}\ (z) = \underbrace{\sum_{j=1}^{k \le K} \frac{1}{(1-z)^j} \cdot \boldsymbol{N}_j}_{principal\ \ part} + \underbrace{\sum_{i=0}^{\infty} z^i \cdot \boldsymbol{M}_i}_{regular\ \ part}, \tag{2.9}$$

in a *deleted* neighborhood of z = 1, where the coefficient matrices $\boldsymbol{M}_i$ of the regular part have exponentially decreasing entries and the coefficient matrices $\boldsymbol{N}_j$ of the principal part disappear if $A\ (1)$ is of full rank.

Proof. The statements of the theorem are matrix extensions of classical results of Laurent series theory (see, *e.g.*, Jeffrey, 1992).

The following generalized-inverse-like property

$$\lim_{z \to 1} \left[A(z) \cdot \left\{ \sum_{j=1}^{K} \frac{1}{(1-z)^j} \cdot \boldsymbol{N}_j + \sum_{i=0}^{\infty} z^i \cdot \boldsymbol{M}_i \right\} \cdot A(z) \right] = \lim_{z \to 1} A\ (z) \tag{2.10}$$

of the Laurent expansion, in the right-hand side of (2.9), at the isolated singularity $z = 1$, is noteworthy.

For notational convenience the regular part of the Laurent expansion will be henceforth written $\boldsymbol{M}(z)$.

Special forms of the Laurent expansion (2.9) are:

$$(i) \quad \boldsymbol{A}^{-1}(z) = \frac{1}{(1-z)} \cdot \boldsymbol{N}_1 + \boldsymbol{M}(z) \tag{2.11}$$

which corresponds to the case of a simple pole at $z = 1$, with

$$\boldsymbol{N}_1 = \lim_{z \to 1} \{(1-z) \cdot \boldsymbol{A}^{-1}(z)\} \tag{2.12}$$

playing the role of the (matrix) residue.

$$(ii) \quad \boldsymbol{A}^{-1}(z) = \sum_{j=1}^{2} \frac{1}{(1-z)^j} \cdot \boldsymbol{N}_j + \boldsymbol{M}(z) \tag{2.13}$$

which corresponds to the case of a second-order pole at $z = 1$, with

$$\boldsymbol{N}_2 = \lim_{z \to 1} \{(1-z)^2 \cdot \boldsymbol{A}^{-1}(z)\} \tag{2.14}$$

$$\boldsymbol{N}_1 = -\lim_{z \to 1} \frac{d\{(1-z)^2 \cdot \boldsymbol{A}^{-1}(z)\}}{dz} \tag{2.15}$$

as coefficient matrices of the principal part.

3. CHARACTERIZATION OF MATRIX-POLYNOMIAL INVERSES ABOUT A POLE

Lemma 3.1. Let $\boldsymbol{A}$ be a square matrix of order n and rank $\rho < n$. Then there exist pairs of full column-rank $n \times \rho$ matrices $\boldsymbol{B}$ and $\boldsymbol{C}$ such that

$$\boldsymbol{A} = \boldsymbol{B}\boldsymbol{C}' \tag{3.1}$$

The representation (3.1) is called rank factorization of $\boldsymbol{A}$.

A specular result holds for the Moore-Penrose generalized inverse $\boldsymbol{A}^g$ of $\boldsymbol{A}$, namely

$$\boldsymbol{A}^g = (\boldsymbol{B}\boldsymbol{C}')^g = (\boldsymbol{C}')^g \cdot \boldsymbol{B}^g = \boldsymbol{C} \cdot (\boldsymbol{C}'\boldsymbol{C})^{-1} \cdot (\boldsymbol{B}'\boldsymbol{B})^{-1} \cdot \boldsymbol{B}' \tag{3.2}$$

Proof. Result (3.1) is well-known (see, *e.g.*, Rao and Mitra, 1971, p. 5). Factorization (3.2) is due to Greville (1960).

Corollary 3.1.1. With A as in Lemma 3.1 the following rank factorizations hold

$$(i) \quad AA^g = BB^g = B \cdot (B'B)^{-1} \cdot B' \tag{3.3}$$
$$(ii) \quad A^gA = (C')^g C' = C \cdot (C'C)^{-1} \cdot C' \tag{3.4}$$
$$(iii) \quad I - AA^g = I - BB^g = B_\perp B_\perp^g \tag{3.5}$$
$$(iv) \quad I - A^gA = I - (C')^g C' = (C_\perp')^g \cdot C_\perp' \tag{3.6}$$

where $B_\perp$ and $C_\perp$ are the orthogonal complements of B and C, respectively.
Proof. Proofs of (i) and (ii) are simple and are omitted. To prove (iii) observe that (see Pringle and Rayner, 1971, corollary 4, p. 44)

$$[B, B_\perp]^g = [B, B_\perp]^{-1} = \begin{bmatrix} B^g \\ B_\perp^g \end{bmatrix} \tag{3.7}$$

Proof of (iv) is similar.

Theorem 3.2. Consider the matrix polynomial

$$A(z) = Q(z) \cdot (1 - z) + BC' \tag{3.8}$$

where BC' is a rank factorization of $A(1)$. Let the inverse of $A(z)$ have a simple pole at the point $z = 1$, which entails the Laurent series expansion about $z = 1$

$$A^{-1}(z) = \frac{1}{1-z} N_1 + M(z), \; N_1 \neq \mathbf{0} \tag{3.9}$$

Then the following hold:

(*a*) The matrix-residue N_1 satisfies

$$A(1) N_1 = \mathbf{0} \tag{3.10}$$
$$N_1 A(1) = \mathbf{0} \tag{3.11}$$

Hence, the following representation

$$N_1 = C_\perp V B_\perp' \tag{3.12}$$

holds for a suitable choice of V.

(*b*) The coefficient matrix $M(1) = \sum_{i=0}^{\infty} M_i$ satisfies

$$C'M(1) B = I \tag{3.13}$$

Proof. Since the equalities

$$A(z) \cdot [\frac{1}{1-z}N_1 + M(z)] = I \tag{3.14}$$

$$[\frac{1}{1-z}N_1 + M(z)] \cdot A(z) = I \tag{3.15}$$

hold true in a deleted neighborhood of $z = 1$, the term containing the negative power of $(1 - z)$ in the left-hand sides of (3.14) and (3.15) must vanish, which occurs if N_1 satisfies (3.10) and (3.11). This proves the first part of (*a*).

According to lemma 2.3.1 in Rao and Mitra, 1971, the equations (3.10) and (3.11) have the common solution

$$N_1 = [I - A^g(1)\,A(1)] \cdot \boldsymbol{\Lambda} \cdot [I - A(1)\,A^g(1)] \tag{3.16}$$

where $\boldsymbol{\Lambda}$ is arbitrary. In view of corollary 3.1.1 the solution can be given the form (3.12) where $V = C_{\perp}^{g}\boldsymbol{\Lambda}\,(B'_{\perp})^{g}$. This proves the second part of (*a*).

Finally, by applying (2.10) simple computations give

$$A(1)\,M(1)\,A(1) = A(1) \tag{3.17}$$

which leads to (3.13) by pre and postmultiplication by B^g and $(C')^g$, respectively. This proves (*b*)

Corollary 3.2.1. Either pre or postmultiplying $A^{-1}(z)$ by $A(1)$ the isolated singularity at $z = 1$ disappears leading to the Taylor expansions

$$A^{-1}(z)\,B = M(z)\,B \tag{3.18}$$
$$C'A^{-1}(z) = C'M(z) \tag{3.19}$$
$$C'A^{-1}(z)\,B = C'M(z)\,B \tag{3.20}$$

Proof. Proof is straightforward in view of (3.10) and (3.11).

Theorem 3.3. Let the inverse of the matrix polynomial

$$A(z) = \boldsymbol{\Psi}(z) \cdot (1-z)^2 - \dot{A}(1) \cdot (1-z) + BC' \tag{3.21}$$

have a second-order pole at the point $z = 1$, which entails the Laurent series expansion about $z = 1$

$$A^{-1}(z) = \frac{1}{(1-z)^2}N_2 + \frac{1}{(1-z)}N_1 + M(z) \tag{3.22}$$

Then the following hold:

(*a*) The principal-part matrices N_2 and N_1 satisfy

$$A(1)\, N_2 = \mathbf{0} \tag{3.23}$$
$$N_2 A(1) = \mathbf{0} \tag{3.24}$$
$$\dot{A}(1)\, N_2 = A(1)\, N_1 \tag{3.25}$$
$$N_2\, \dot{A}(1) = N_1 A(1) \tag{3.26}$$

(*b*) $N_2 \neq \mathbf{0} \Rightarrow \det\, [C'_\perp\, \dot{A}(1)\, B_\perp] = 0$ (3.27)

(*c*) N_2 has the representation

$$N_2 = C_\perp S_\perp W R'_\perp B'_\perp \tag{3.28}$$

for a suitable choice of W, given the rank factorization

$$C'_\perp\, \dot{A}(1)\, B_\perp = RS' \tag{3.29}$$

(*d*) N_1 has the representation

$$N_1 = A^g(1)\, \dot{A}(1)\, N_2 + N_2\, \dot{A}(1)\, A^g(1) + C_\perp U B'_\perp \tag{3.30}$$

for a suitable choice of U.

(*e*) The coefficient matrix $M(1) = \sum_{i=0}^{\infty} M_i$ satisfies

$$C' M(1)\, B - B^g\, \dot{A}(1)\, N_2\, \dot{A}(1)\, (C^g)' = I \tag{3.31}$$

Proof. Since the equalities

$$A(z) \cdot \left[\frac{1}{(1-z)^2} N_2 + \frac{1}{1-z} N_1 + M(z)\right] = I \tag{3.32}$$

$$\left[\frac{1}{(1-z)^2} N_2 + \frac{1}{1-z} N_1 + M(z)\right] \cdot A(z) = I \tag{3.33}$$

hold true in a deleted neighborhood of $z = 1$, the terms containing the negative powers of $(1 - z)$ in the left-hand sides of (3.32) and (3.33) must vanish, which occurs if N_2 and N_1 satisfy (3.23), (3.24), (3.25) and (3.26). This proves (*a*).

Using the same argument as in the proof of (3.12) N_2 can be written as

$$N_2 = C_\perp \boldsymbol{\Omega} B'_\perp \tag{3.34}$$

where $\boldsymbol{\Omega}$ is arbitrary.

Substituting (3.34) into (3.25) and pre and B' postmultiplying by $B'_\perp$ and $(B'_\perp)^g$, respectively, we get

$$B'_\perp\, \dot{A}(1)\, C_\perp \boldsymbol{\Omega} = \mathbf{0} \tag{3.35}$$

By inspection of (3.34) and (3.35) the conclusion

$$N_2 \neq \mathbf{0} \Rightarrow \mathbf{\Omega} \neq \mathbf{0} \Rightarrow \det(\boldsymbol{B}'_{\perp}\,\dot{\boldsymbol{A}}\,(1)\,\boldsymbol{C}_{\perp}) = 0 \tag{3.36}$$

is easily drawn. This proves (*b*).

The result

$$\mathbf{\Omega}\,\boldsymbol{B}'_{\perp}\,\dot{\boldsymbol{A}}\,(1)\,\boldsymbol{C}_{\perp} = \mathbf{0} \tag{3.37}$$

which is specular to (3.35), can be derived similarly.

Let

$$\boldsymbol{B}'_{\perp}\,\dot{\boldsymbol{A}}\,(1)\,\boldsymbol{C}_{\perp} = \boldsymbol{R}\boldsymbol{S}' \tag{3.38}$$

be a rank factorization of $\boldsymbol{B}'_{\perp}\,\dot{\boldsymbol{A}}\,(1)\,\boldsymbol{C}_{\perp}$. Hence (3.35) and (3.37) can be more conveniently rewritten

$$\begin{cases} S'\,\Omega = 0 \\ \Omega R = 0 \end{cases} \tag{3.39}$$

and, in view of the usual arguments, $\mathbf{\Omega}$ can be written as

$$\mathbf{\Omega} = \boldsymbol{S}_{\perp}\boldsymbol{W}\boldsymbol{R}'_{\perp} \tag{3.40}$$

where $\boldsymbol{W}$ is arbitrary.

Substituting (3.40) into (3.34) eventually leads to (3.28). This proves (*c*).

According to Theorem 2.3.3 in Rao and Mitra, 1971, the equations (3.25) and (3.26) have the common solution

$$\begin{aligned} N_1 = {} & A^g(1)\ \dot{A}(1)\,N_2 + N_2\,\dot{A}(1)\,A^g(1) - A^g(1)\,A\,(1)\,N_2\,\dot{A}(1)\,A^g(1) + \\ & + [\boldsymbol{I} - A^g(1)\,A(1)] \cdot \mathbf{\Phi} \cdot [\boldsymbol{I} - A\,(1)\,A^g(1)] \end{aligned} \tag{3.41}$$

where $\mathbf{\Phi}$ is arbitrary. In view of (3.23) and of Corollary 3.1.1 the solution can be given the form (3.30) where $\boldsymbol{U} = \boldsymbol{C}_{\perp}^{g}\mathbf{\Phi}\,(\boldsymbol{B}'_{\perp})^{g}$. This proves (*d*).

Finally by applying (2.10) and reminding (3.23) and (3.26), simple computations give

$$-\dot{\boldsymbol{A}}\,(1)\,N_2\,\dot{\boldsymbol{A}}\,(1) + \boldsymbol{A}\,(1)\,\boldsymbol{M}\,(1)\,\boldsymbol{A}\,(1) = \boldsymbol{A}\,(1) \tag{3.42}$$

which leads to (3.31) by pre and postmultiplying by $\boldsymbol{B}^g$ and $(\boldsymbol{C}')^g$, respectively. This proves (*e*).

Corollary 3.3.1 – The following statements are true:

(*i*) Should $A^{-1}(z)$ be either premultiplied by $(C_\perp S_\perp)'_\perp$ or postmultiplied by $(B_\perp R_\perp)_\perp$ then the isolated singularity at $z = 1$ reduces to a simple pole, in view of (3.28). The following Laurent expansions

$$(C_\perp S_\perp)'_\perp \cdot A^{-1}(z) = \frac{1}{1-z}(C_\perp S_\perp)'_\perp \cdot [A^g(1)\ \dot{A}\ (1)N_2 + C_\perp U B'_\perp] + \\ + (C_\perp S_\perp)'_\perp \cdot M(z) \tag{3.43}$$

$$A^{-1}(z)\ (B_\perp R_\perp)_\perp = \frac{1}{1-z} \cdot [N_2\ \dot{A}\ (1) A^g(1) + C_\perp U B'_\perp]\ (B_\perp R_\perp)_\perp + \\ + M(z) \cdot (B_\perp R_\perp)_\perp \tag{3.44}$$

hold accordingly.

(*ii*) Similar conclusions are drawn, in view of (3.34), by pre/postmultiplying by C' and B (which incidentally are submatrices of $(C_\perp S_\perp)'_\perp$ and $(B_\perp R_\perp)_\perp$, respectively).

This eventually leads to the noteworthy expansions:

$$C' A^{-1}(z) = \frac{1}{1-z} B^g\ \dot{A}\ (1) N_2 + C' M(z) \tag{3.45}$$

$$A^{-1}(z)\ B = \frac{1}{1-z} N_2\ \dot{A}\ (1)(C')^g + M(z)\ B \tag{3.46}$$

$$C' A^{-1}(z)\ B = C' M(z)\ B \tag{3.47}$$

(*iii*) Premultiplying $A^{-1}(z)$ by $(1-z)\ \dot{A}\ (1) - A\ (1)$ the isolated singularity at $z = 1$ disappears. The following Taylor expansion

$$[(1-z)\ B^g\ \dot{A}\ (1) - C'] \cdot A^{-1}(z) = B^g\ \dot{A}\ (1) N_1 - C' M(z) + (1-z)\ B^g\ \dot{A}\ (1)\ M(z) \tag{3.48}$$

holds accordingly.

Proof. Proofs of (i) and (ii) are simple and are omitted. For what concerns proposition (iii) simple computations show that

$$[(1-z)\ \dot{A}(1) - A\ (1)] \cdot A^{-1}(z) = \frac{1}{1-z}[\dot{A}(1) N_2 - A\ (1) N_1] + \dot{A}(1) N_1 + \\ + [(1-z)\ \dot{A}\ (1) - A\ (1)] \cdot M(z) \tag{3.49}$$

where the terms in $(1-z)^{-1}$ cancel out each other in view of (3.25). Hence (3.48) follows by premultiplying both sides by B^g.

4. THE COEFFICIENT-MATRICES ASSOCIATED TO FIRST AND SECOND ORDER POLES IN MATRIX-POLYNOMIAL INVERSION

Lemma 4.1. Let $\boldsymbol{A}$ and $\boldsymbol{D}$ be square matrices of order n and m respectively, and $\boldsymbol{B}$ and $\boldsymbol{C}$ two matrices of order $n \times m$. Consider the partitioned matrix

$$P = \begin{bmatrix} A & B \\ C' & D \end{bmatrix} \tag{4.1}$$

Then any one of the following sets of conditions is sufficient for the existence of P^{-1}:

(*a*) Both A and its Schur complement $E = D - C'A^{-1}B$ are non singular matrices.

(*b*) Both D and its Schur complement $F = A - B\,D^{-1}C'$ are non singular matrices.

(*c*) D is a null matrix, B and C are full column-rank matrices and $H = B'_{\perp}AC_{\perp}$ is non singular.

Further the following results hold:

(*i*) Under (*a*) the partitioned inverse of P is expressible as follows

$$P^{-1} = \begin{bmatrix} A^{-1} + A^{-1}BE^{-1}C'A^{-1} & -A^{-1}BE^{-1} \\ -E^{-1}C'A^{-1} & E^{-1} \end{bmatrix} \tag{4.2}$$

(*ii*) Under (*b*) the partitioned inverse of P is expressible as follows

$$P^{-1} = \begin{bmatrix} F^{-1} & -F^{-1}BD^{-1} \\ -D^{-1}C'F^{-1} & D^{-1} + D^{-1}C'F^{-1}BD^{-1} \end{bmatrix} \tag{4.3}$$

(*iii*) Under (*c*) the partitioned inverse of P is expressible as follows

$$P^{-1} = \begin{bmatrix} C_{\perp}H^{-1}B'_{\perp} & (I - C_{\perp}H^{-1}B'_{\perp}A)\cdot(C')^{g} \\ B^{g}(I - AC_{\perp}H^{-1}B'_{\perp}) & B^{g}(AC_{\perp}H^{-1}B'_{\perp}A - A)\cdot(C')^{g} \end{bmatrix} \tag{4.4}$$

(*iv*) If both (*a*) and (*b*) hold then

$$(A - BD^{-1}C')^{-1} = A^{-1} + A^{-1}B\cdot(D - C'A^{-1}B)^{-1}\cdot C'A^{-1} \tag{4.5}$$

(*v*) If both (*c*) and (*a*) hold then

$$C_{\perp}\cdot(B'_{\perp}AC_{\perp})^{-1}\cdot B'_{\perp} = A^{-1} - A^{-1}B\cdot(C'A^{-1}B)^{-1}\cdot C'A^{-1} \tag{4.6}$$

(*vi*) With $\boldsymbol{D} = -\lambda\boldsymbol{I}$, let (*b*) hold in a deleted neighborhood of $\lambda = 0$ and (*c*) hold in $\lambda = 0$, then

$$\boldsymbol{C}_\perp \cdot (\boldsymbol{B}'_\perp \boldsymbol{A}\boldsymbol{C}_\perp)^{-1} \cdot \boldsymbol{B}'_\perp = \lim_{\lambda \to 0} \{\lambda\,(\lambda\boldsymbol{A} + \boldsymbol{B}\boldsymbol{C}')^{-1}\} \tag{4.7}$$

Proof. The partitioned inversion formulas (4.2) and (4.3) under the assumption (*a*) and (*b*) respectively, are standard results of the algebraic toolkit of econometricians (see, *e.g.*, Faliva, 1987).

For the proof of the inversion formula (4.4) under the assumptions (*c*) see Faliva and Zoia (2002, Theorem 1). Result (4.5) arises from equating the upper diagonal blocs of the right-hand sides of (4.3) and (4.2). Result (4.6) arises from equating the upper diagonal blocs of the right-hand sides of (4.4) and of (4.2) for $\boldsymbol{D} = 0$.

For the proof of (4.7) see Faliva and Zoia (2002, section 2).

Theorem 4.2. Let

$$\underset{(n,n)}{\boldsymbol{A}(z)} = (1-z) \cdot \boldsymbol{Q}(z) + \boldsymbol{A}(1) \tag{4.8}$$

If

$$\mathrm{r}(\boldsymbol{A}(1)) = \rho < n \Rightarrow \boldsymbol{A}(1) = \underset{(n,\rho)}{\boldsymbol{B}}\ \underset{(\rho,n)}{\boldsymbol{C}'},\ \mathrm{r}(\boldsymbol{B}) = \mathrm{r}(\boldsymbol{C}) = \rho \tag{4.9}$$

$$\mathrm{r}(\boldsymbol{B}'_\perp \dot{\boldsymbol{A}}(1)\,\boldsymbol{C}_\perp) = n - \rho \tag{4.10}$$

then the inverse of the matrix polynomial (4.8) has the Laurent expansion

$$\boldsymbol{A}^{-1}(z) = \frac{1}{1-z}\boldsymbol{N}_1 + \boldsymbol{M}(z) \tag{4.11}$$

about the simple pole $z = 1$ with matrix-residue $\boldsymbol{N}_1$ given by

$$\boldsymbol{N}_1 = -\boldsymbol{C}_\perp \cdot [\boldsymbol{B}'_\perp \dot{\boldsymbol{A}}(1)\,\boldsymbol{C}_\perp]^{-1} \cdot \boldsymbol{B}'_\perp \tag{4.12}$$

Proof. Simple computations show that

$$\boldsymbol{B}'_\perp \boldsymbol{A}(z)\,\boldsymbol{C}_\perp = (1-z) \cdot \boldsymbol{B}'_\perp \boldsymbol{Q}(z)\,\boldsymbol{C}_\perp \tag{4.13}$$

where the matrix $\boldsymbol{B}'_\perp \boldsymbol{Q}(z)\,\boldsymbol{C}_\perp$ turns out to be non singular for $z = 1$ under the rank condition (4.10), since

$$\boldsymbol{B}'_\perp \boldsymbol{Q}(1)\,\boldsymbol{C}_\perp = -\boldsymbol{B}'_\perp \dot{\boldsymbol{A}}(1)\,\boldsymbol{C}_\perp \tag{4.14}$$

According to (2.12) the matrix-residue for a simple pole at $z = 1$ is given by:

$$N_1 = \lim_{z\to 1}\{(1-z)\,A^{-1}(z)\} = \lim_{z\to 1}\{(1-z)\,[(1-z)\,Q\,(z) + BC']^{-1}\} \tag{4.15}$$

and straightforward application of result (vi) of lemma 4.1 leads to (4.12) in view of (4.14).

For an alternative proof check that, by (4.6) of lemma 4.1 and in view of corollary 3.2.1, the following holds

$$N_1 = \lim_{z\to 1}\{(1-z)\,A^{-1}(z)\} = \lim_{z\to 1}\{(1-z)\,C_\perp\cdot[B'_\perp\,A\,(z)\,C_\perp]^{-1}\cdot\,B'_\perp +$$

$$+ (1-z)\,A^{-1}(z)\,B\cdot\,[C'A^{-1}(z)\,B]^{-1}\cdot\,C'A^{-1}(z)\} =$$

$$= \lim_{z\to 1}\{C_\perp[B'_\perp\,Q\,(z)\,C_\perp]^{-1}\cdot\,B'_\perp + (1-z)\,M\,(z)\,B\cdot\,[C'M\,(z)\,B]^{-1}\cdot\,C'M\,(z)\} =$$

$$= -\,C_\perp\cdot\,[B'_\perp\,\dot{A}\,(1)\,C_\perp]^{-1}\cdot\,B'_\perp \tag{4.16}$$

Theorem 4.3. Let

$$\underset{(n,\,n)}{A(z)} = \boldsymbol{\Psi}\,(z)\cdot(1-z)^2 - \dot{A}\,(1)\cdot(1-z) + A\,(1) \tag{4.17}$$

If

$$\mathrm{r}\,(A\,(1)) = \rho < n \Rightarrow A\,(1) = \underset{(n,\,\rho)}{B}\;\underset{(\rho,\,n)}{C'}\,,\ \mathrm{r}\,(B) = \mathrm{r}\,(C) = \rho \tag{4.18}$$

$$\mathrm{r}\,(B'_\perp\,\dot{A}\,(1)\,C_\perp) = \vartheta < n-\rho \Rightarrow B'_\perp\,\dot{A}\,(1)\,C_\perp =$$

$$= \underset{(n-\rho,\,\vartheta)}{R}\;\underset{(\vartheta,\,n-\rho)}{S'}\,,\ \mathrm{r}\,(R) = \mathrm{r}\,(S) = \vartheta \tag{4.19}$$

$$\mathrm{r}\,(R'_\perp\,B'_\perp\cdot[\frac{1}{2}\,\ddot{A}\,(1) - \dot{A}\,(1)A^g(1)\,\dot{A}\,(1)]\cdot\,C_\perp S_\perp) = n-\rho-\vartheta \tag{4.20}$$

then the inverse of the matrix polynomial (4.17) has the Laurent expansion

$$A^{-1}\,(z) = \frac{1}{(1-z)^2}N_2 + \frac{1}{1-z}N_1 + M\,(z) \tag{4.21}$$

about the second order pole $z = 1$ with principal-part matrices N_2 and N_1 given by

$$N_2 = C_\perp S_\perp\cdot\{R'_\perp\,B'_\perp\cdot[\frac{1}{2}\,\ddot{A}\,(1) - \dot{A}\,(1)\,A^g\,(1)\,\dot{A}\,(1)]\cdot\,C_\perp S_\perp\}^{-1}\cdot\,R'_\perp\,B'_\perp \tag{4.22}$$

$$N_1 = A^g\,(1)\,\dot{A}\,(1)\,N_2 + N_2\,\dot{A}\,(1)\,A^g\,(1) + C_\perp U B'_\perp \tag{4.23}$$

for a suitable choice of U.

Proof. Let us remember the statements of corollary 3.3.1 together with (3.31), namely

$$A^{-1}(z)\, B = \frac{1}{1-z} N_2\, \dot{A}\,(1)\,(C^g)' + M\,(z)\, B \tag{4.24}$$

$$C'A^{-1}(z) = \frac{1}{1-z} B^g\, \dot{A}\,(1)\, N_2 + C'M\,(z) \tag{4.25}$$

$$C'A^{-1}(z)\, B = C'M\,(z)\, B \tag{4.26}$$

$$C'M\,(1)\, B - B^g\, \dot{A}\,(1)\, N_2\, \dot{A}\,(1)\,(C^g)' = I \tag{4.27}$$

Then by applying result (v) of lemma 4.1 to the matrix polynomial (4.17) and multiplying by $(1-z)^2$, we find after proper substitutions:

$$\begin{aligned}&(1-z)^2\, C_\perp \cdot [\,B'_\perp A\,(z)\, C_\perp]^{-1} \cdot B'_\perp = \\ &= (1-z)^2 A^{-1}(z) - [N_2\, \dot{A}(1)\,(C^g)' + (1-z)\, M\,(z)\, B] \cdot \\ &[C'M\,(z)\, B]^{-1} \cdot [B^g\, \dot{A}\,(1)\, N_2 + (1-z)\, C'M\,(z)]\end{aligned} \tag{4.28}$$

Taking the limit as $z \to 1$ we get

$$\begin{aligned}&\lim_{z\to 1}\{(1-z)^2 C_\perp \cdot [\,B'_\perp A\,(z)\, C_\perp]^{-1} \cdot B'_\perp\} = \\ &= N_2 - N_2\, \dot{A}\,(1)\,(C^g)' \cdot [B^g\, \dot{A}\,(1)\, N_2\, \dot{A}\,(1)\,(C^g)' + I]^{-1} \cdot B^g\, \dot{A}\,(1)\, N_2\end{aligned} \tag{4.29}$$

But

$$\begin{aligned}&N_2 - N_2\, \dot{A}(1)\,(C^g)' \cdot [B^g\, \dot{A}(1)\, N_2\, \dot{A}(1)\,(C^g)' + I]^{-1} \cdot B^g\, \dot{A}(1)\, N_2 = \\ &= \lim_{z\to 1}\Big((1-z)^2 \cdot \{A^{-1}(z) - A^{-1}(z)\, \dot{A}(1)\,(C^g)' \cdot \\ &\cdot [B^g\, \dot{A}(1)\, A^{-1}(z)\, \dot{A}(1)\,(C^g)' + \frac{1}{(1-z)^2} I]^{-1} \cdot B^g\, \dot{A}(1)\, A^{-1}(z)\}\Big) = \\ &= \lim_{z\to 1}\{(1-z)^2 \cdot [A\,(z) + (1-z)^2 \cdot \dot{A}(1)\, A^g(1)\, \dot{A}(1)\,]^{-1}\}\end{aligned} \tag{4.30}$$

where the latter result is obtained by applying a by-product of result (iv) of lemma 4.1, recalling the rank factorization (3.2).

Combining (4.29) and (4.30) gives:

$$\begin{aligned}&\lim_{z\to 1}\{(1-z)^2 C_\perp \cdot [\,B'_\perp A\,(z)\, C_\perp]^{-1} \cdot B'_\perp\} = \\ &= \lim_{z\to 1}\{(1-z)^2 \cdot [A\,(z) + (1-z)^2 \cdot \dot{A}(1)\, A^g(1)\, \dot{A}(1)\,]^{-1}\}\end{aligned} \tag{4.31}$$

Now, replace the matrix polynomial $A\,(z)$ in (4.31) by the matrix polynomial

$$\tilde{A}(z) = A\,(z) - (1-z)^2 \cdot \dot{A}(1)\, A^g\,(1)\, \dot{A}(1) \tag{4.32}$$

Then the major result

$$\lim_{z \to 1} \{(1-z)^2 \cdot C_\perp \cdot [B'_\perp \tilde{A}(z)\ C'_\perp]^{-1} \cdot B'_\perp\} = \lim_{z \to 1} [(1-z)^2 A^{-1}(z)] = N_2 \qquad (4.33)$$

ensues.

Consider now the matrix $B'_\perp \tilde{A}(z)\, C_\perp$ appearing in the left-hand side of (4.33). Simple computations show that:

$$B'_\perp \tilde{A}(z)\, C_\perp = (1-\mathrm{z}) \cdot [(1-\mathrm{z})\ B'_\perp \tilde{\Psi}(z)\, C_\perp + RS'] \qquad (4.34)$$

where

$$\tilde{\Psi}(z) = \Psi(z) + \dot{A}\ (1)\, A^g\,(1)\ \dot{A}\ (1) \qquad (4{,}35)$$

From (4.34) it follows that

$$(1-z)^2 \cdot C_\perp \cdot [B'_\perp \tilde{A}(z)\ C_\perp]^{-1} \cdot B'_\perp =$$
$$= C_\perp \cdot \{(1-z) \cdot [(1-z)\ B'_\perp \tilde{\Psi}(z)\ C_\perp + RS']^{-1}\} \cdot B_\perp \qquad (4.36)$$

which, taking the limit as z → 1 and applying – under the rank condition (4.20) – result (vi*)* of lemma 4.1 to the right-hand side, eventually gives:

$$N_2 = C_\perp S_\perp \cdot \{R'_\perp B'_\perp \cdot [\frac{1}{2}\ddot{A}(1) - \dot{A}(1)\, A^g\,(1)\ \dot{A}(1)] \cdot C_\perp S_\perp\}^{-1} \cdot R'_\perp B'_\perp \qquad (4.37)$$

as

$$\tilde{\Psi}(1) = \Psi(1) - \dot{A}(1)\, A^g\,(1)\ \dot{A}(1) \qquad (4.38)$$

and

$$\Psi(1) = \frac{1}{2}\ddot{A}(1) \qquad (4.39)$$

This proves (4.22).

As regards the expression of N_1, as quoted in (4.23), see proposition (*d*) of theorem 3.3.

5. REPRESENTATION THEOREMS OF UNIT-ROOT ECONOMETRICS REVISITED

Application of the results obtained so far to modelling integrated processes leads to elegant closed-form solutions, a glimpse of which is given here below.

Theorem 5.1. Any process $y_t \sim I(d)$, with $0 \le d \le 2$, specified as a VAR model, namely

$$A(L)\, y_t = \boldsymbol{\varepsilon}_t \tag{5.1}$$

where

$$\underset{(n,\,n)}{A(L)} = \boldsymbol{\Psi}(L)\nabla^2 - \dot{A}(1)\nabla + A(1),\ \nabla = \mathrm{I} - L,\ \boldsymbol{\varepsilon}_t \sim WN \tag{5.2}$$

has a closed-form representation such as:

$$y_t = k_1 + k_2 t + N_2 \cdot \sum_{\tau \le t}(t+1-\tau)\boldsymbol{\varepsilon}_t + N_1 \sum_{\tau \le t} \varepsilon_\tau + \sum_{i=0}^{\infty} M_i \boldsymbol{\varepsilon}_{t-i} \tag{5.3}$$

where $N_j = \mathbf{0}$ and $k_j = \mathbf{0}$ if $j > d$, and $k_1 + k_2 t$ is the autonomous component associated with the unit roots.

The following major results hold true:

$$(i)\quad y_t \sim I(0) \tag{5.4}$$

under the rank condition

$$\mathrm{r}\,[A(1)] = n \tag{5.5}$$

$$(ii)\quad \left.\begin{matrix} y_t \sim I(1) \\ C' y_t \sim I(0) \end{matrix}\right\} \Rightarrow y_t \sim CI(1,1) \tag{5.6}$$

under the rank conditions

$$\mathrm{r}\,[A(1)] = \rho < n \tag{5.7}$$

$$\mathrm{r}\,[B'_\perp \dot{A}(1)\, C_\perp] = n - \rho \tag{5.8}$$

which entail

$$N_1 = -C_\perp \cdot [B'_\perp \dot{A}(1)\, C_\perp]^{-1} \cdot B'_\perp \tag{5.9}$$

$$k_1 = N_1 v \tag{5.10}$$

where the symbols have the same meaning as in sections 3 and 4 and v is a vector depending on the initial conditions.

$$(iii)\quad \left.\begin{matrix} y_t \sim I_2 \\ (C_\perp S_\perp)'_\perp y \sim I_1 \\ B^g \dot{A}(1)\nabla y_t - C' y_t \sim I_0 \end{matrix}\right\} \Rightarrow \begin{cases} y_t \sim CI(2,1) \\ y_t \sim PCI(2,0) \end{cases} \tag{5.11}$$

under the rank conditions

$$\mathrm{r}\,[\boldsymbol{A}\,(1)]\ \rho < n \tag{5.12}$$

$$\mathrm{r}\,[\,\boldsymbol{B}'_{\perp}\ \dot{\boldsymbol{A}}\ (1)\ \boldsymbol{C}_{\perp}] = \vartheta < n - \rho \tag{5.13}$$

$$\mathrm{r}\,\{\,\boldsymbol{R}'_{\perp}\ \boldsymbol{B}'_{\perp} \cdot [\frac{1}{2}\ \ddot{\boldsymbol{A}}\ (1) - \dot{\boldsymbol{A}}(1)\ \boldsymbol{A}^{g}\,(1)\ \ \dot{\boldsymbol{A}}\ (1)] \cdot \boldsymbol{C}_{\perp}\boldsymbol{S}_{\perp}\} = n - \rho - \vartheta \tag{5.14}$$

which entail

$$\boldsymbol{N}_2 = \boldsymbol{C}_{\perp}\boldsymbol{S}_{\perp}\ \{\boldsymbol{R}'_{\perp}\ \boldsymbol{B}'_{\perp} \cdot [\frac{1}{2}\ \ddot{\boldsymbol{A}}(1) - \dot{\boldsymbol{A}}\ (1)\ \boldsymbol{A}^{g}\,(1)\ \ \dot{\boldsymbol{A}}\ (1)] \cdot \boldsymbol{C}_{\perp}\boldsymbol{S}_{\perp}\}^{-1}\ \boldsymbol{R}'_{\perp}\ \boldsymbol{B}'_{\perp} \tag{5.15}$$

$$\boldsymbol{N}_1 = \boldsymbol{A}^{g}\,(1)\ \ \dot{\boldsymbol{A}}\ (1)\ \boldsymbol{N}_2 + \boldsymbol{N}_2\,\dot{\boldsymbol{A}}\ (1)\ \boldsymbol{A}^{g}(1) + \boldsymbol{C}_{\perp}\boldsymbol{U}\boldsymbol{B}'_{\perp}\,,\ \textit{for a suitable choice of}\ \boldsymbol{U} \tag{5.16}$$

$$\boldsymbol{k}_2 = \boldsymbol{N}_2\boldsymbol{v} \tag{5.17}$$

$$\boldsymbol{k}_1 = \boldsymbol{A}^{g}\,(1)\ \ \dot{\boldsymbol{A}}\ (1)\ \boldsymbol{k}_2 + \boldsymbol{C}_{\perp}\boldsymbol{w} \tag{5.18}$$

where *PCI* is an acronym for Polynomially CoIntegrated process and $\boldsymbol{v}$ and $\boldsymbol{w}$ are vectors depending on the initial conditions.

Proof (hint). Basically the propositions quoted in the theorem can be established by means of the algebraic toolkit developed in sections 3 and 4. As regards the autonomous component of the solution (5.3), the classical theory of difference equations applies with proper matrix technicalities.

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RIASSUNTO

Polinomi matriciali e loro inversione: i fondamenti analitici dei teoremi di rappresentazione dell'econometria delle serie storiche

L'articolo affronta il problema dell'inversione con lo sviluppo di Laurent di un polinomio matriciale in un intorno di una radice unitaria e perviene ad una caratterizzazione delle matrici dei coefficienti della parte principale dello sviluppo suddetto in corrispondenza di un polo di primo e di secondo ordine.

Le espressioni in forma chiusa delle matrici all'oggetto vengono quindi derivate grazie ad un recente risultato sull'inversione per parti (Faliva e Zoia, 2002), creando così i presupposti per un'elegante formalizzazione di un teorema generale di rappresentazione per processi (co)integrati fino al secondo ordine.

SUMMARY

Matrix polynomials and their inversion: the algebraic framework of unit-root econometrics representation theorems

In this paper the issue of the inversion of a matrix polynomial about a unit root is tackled by resorting to Laurent expansion.

The principal-part matrix coefficients associated with a simple and a second order pole are properly characterized and closed-form expressions are derived by virtue of a recent result on partitioned inversion (Faliva and Zoia, 2002).

This eventually sheds more light on the analytical foundations of unit-root econometrics which in turn paves the way to an elegant unified representation theorem for (co)integrated processes up to the second order.