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STOCHASTIC AND DETERMINISTIC TREND MODELS

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1. INTRODUCTION

Most information in social sciences, biology, and many other sciences occurs in the form of time series where their main property is that the observations are dependent and the nature of this dependence is of interest in itself. A time series is a finite realization of a stochastic process and is often compiled for consecutive and equal periods, such as weeks, months, quarters, and years. In time series decomposition, four types of movements have been traditionally distinguished, namely, the trend, the cycle, the seasonal variations (for sub annual data), and the irregular fluctuations. As a matter of statistical description, a given series can always be represented by one of these components or a sum of several of them. The four components are usually interrelated and for most series, they influence one another.

The trend corresponds to sustained and systematic variations over a long period of time. It is associated with the structural causes of the phenomenon in question, for example, population growth, technological progress, new ways of organization, capital accumulation. For the majority of socioeconomic time series, the trend is very important because it dominates the total variation of the series. The identification of trend has always posed a serious statistical problem. The problem is not one of mathematical or analytical complexity but of conceptual complexity. This problem exists because the trend as well as the remaining components of a time series are latent (no observables) variables, and therefore, assumptions must be made on their behavioural pattern. The trend is generally thought of as a smooth and slow movement over a long term. The concept of "long" in this connection is relative and what is identified as trend for a given series span might well be part of a long cycle once the series is considerably augmented, such as the Kondratieff economic cycle. Kondratieff (1925) estimated the length of a long cycle to be between 47 and 60 years. Often, a long cycle is treated as a trend because the length of the observed time series is shorter than one complete face of this type of cycle.

To avoid the complexity of the problem posed by a statistically vague definition, statisticians have resorted to two simple solutions: One consists of estimating trend and cyclical fluctuations together calling this combined movement *trend-*

cycle; the other consists of defining the trend in terms of the series length, denoting it as the longest non periodic movement. The estimation of the time series trend can be done via a specified model applied to the whole data called the *global* trend, or by fitting a *local* polynomial function in such a way that, at any time point, its estimates depend on only the observations at that point and some specified neighbouring points. Local polynomial fitting has a long history in the smoothing of noisy data. Henderson (1916), Whittaker and Robinson (1924) and Macaulay (1931) are some of the earliest classical references. These authors were very much concerned with the smoothing properties of linear estimators, being Henderson (1916), the first to show that the smoothing power of a linear filter depends on the shape and values of its weighting system.

On the other hand, more recent contributions (among others, Cleveland and Devlin, 1988; Hardle, 1990; Fan, 1992 and 1993; Fan and Gijbels, 1996; Wand and Jones, 1995; Simonoff, 1995; Green and Silverman, 1994; Eubank, 1999) concentrated on the asymptotic statistical properties of optimally estimated smoothing parameters. Optimality being defined in terms of minimizing a given loss function, usually, the mean square error or the prediction risk.

In this paper we will review some of the stochastic and deterministic trend models formulated for global and local estimation.

2. DETERMINISTIC AND STOCHASTIC GLOBAL TREND MODELS

Deterministic and stochastic global trend models are based on the assumption that the trend or nonstationary mean of a time series can be approximated closely by simple functions of time over the entire span of the series.

The most common representation of a deterministic trend is by means of polynomials and transcendental functions. The time series from which the trend is to be identified is assumed to be generated by a nonstationary process where the nonstationary property results from a deterministic trend . A classical model is the regression or error model (Anderson, 1971) where the observed series is treated as the sum of a systematic part or trend and a random part or irregular. This model can be written as

$$Z_t = Y_t + \mu_t \tag{1}$$

where μ_t is a purely random process, that is, $\mu_t \sim$i.i.d.$(0,\sigma_\mu^2)$ (independent and identically distributed with expected value 0 and variance, σ_μ^2.)

In the case of a polynomial trend,

$$Y_t = a_0 + a_1 t + a_2 t^2 + ... + a_n t^n , \tag{2}$$

where generally $n \leq 3$. The trend is said to be of a deterministic character because it is not affected by random shocks which are assumed to be uncorrelated with the systematic part.

Model (1) can be generalized by assuming that μ_t is a second-order linear stationary stochastic process, that is, its mean and variance are constant and its autocovariance is finite and depends only on the time lag.

Besides polynomials in time, other suitable mathematical functions are used to represent deterministic trends. Three of the most widely applied functions, known as growth curves, are the modified exponential, the Gompertz, and the logistic.

The modified exponential trend can be written as

$$Y_t = a + bc^t, \; a \; real, b \neq 0, c > 0, c \neq 1 \tag{3}$$

For $a=0$, model (3) reduces to the unmodified exponential trend

$$Y_t = bc^t = Y_0 e^{\alpha t}; \, b = Y_0, \alpha = \log c \tag{4}$$

when $b>0$ and $c>1$, and so $\alpha>0$, model (4) represents a trend that increases at a constant relative rate α. For $0<c<1$, the trend decreases at the rate α. Models (3) and (4) are solutions of the differential equation

$$dY_t / dt_t = \alpha(Y - a), \qquad \alpha = \log c, \tag{5}$$

which specifies the simple assumption of no inhibited growth.

Several economic variables during periods of sustained growth or rapid inflation, as well as population growths measured in relative short periods of time, can be well approximated by trend models (3) and (4). But in the long run, socioeconomic and demographic time series are often subject to obstacles that slow their time path, and if there are no structural changes, their growth tend to a stationary state. Quetelet made this observation with respect to population growth and Verhulst (1838) seems to have been the first to formalize it by deducing the logistic model. Adding to eq. (5) an inhibit factor proportional to $-Y^2$, the result is

$$\begin{aligned} &dY_t / dt_t = \alpha Y - \beta Y^2 = \alpha Y(1 - Y / k), \\ &k = \alpha / \beta, \alpha, \beta > 0 \end{aligned} \tag{6}$$

which is a simple null form of the Ricatti differential equation. Solving eq. (6), we obtain the logistic model,

$$Y_t = k(1 + ae^{-\alpha t})^{-1}, \tag{7}$$

where $a>0$ is a constant of integration.

Model (7) belongs to a family of S-shaped curves generated from the differential equation (see Dagum, 1985):

$$dY_t / dt = Y_t \Psi(t) \Phi(Y_t / k), \; \Phi(1) = 0. \tag{8}$$

Solving eq. (8) for Ψ=log c and Φ=log(Y_t/k), we obtain the Gompertz curve used to fit mortality table data; that is,

$$Y_t = kb^{c^t}, \; b > 0, b \neq 1, 0 < c < 1, \tag{9}$$

where b is a constant of integration.

It should be noted that differencing will remove polynomial trends and suitable mathematical transformations plus differencing will remove trends from nonlinear processes; e.g., for (7) using

$$Z_t = \log[Y_t / (k - Y_t)]$$

and then taking differences gives $\Delta Z_t = \alpha$.

The second major class of global trend models is the one that assumes the trend to be a stochastic process, most commonly that the series from which the trend will be identified follows a homogeneous linear nonstationary stochastic process (Yaglom, 1962). Processes of this kind are nonstationary, but applying a homogeneous filter, usually the difference filter, we obtain a stationary process in the differences of a finite order. In empirical applications, the nonstationarity is often present in the level and/or slope of the series; hence, the order of the difference is low. An important class of homogeneous linear nonstationary processes are the ARIMA (autoregressive integrated moving average processes) which can be written as (Box and Jenkins, 1970)

$$\begin{aligned} &\phi_p(B)\Delta^d Y_t = \theta_q(B)a_t, \\ &a_t \sim \text{i.i.d.}(0, \sigma^2{}_a) \end{aligned} \tag{10}$$

where B is the backshift operator such that $B^n Y_t = Y_{t-n}$; $\phi_p(B)$ and $\theta_q(B)$ are polynomials in B of order p and q, respectively, and satisfy the conditions of stationarity and invertibility; $\Delta^d = (1-B)^d$ is the difference operator of order d and a_t is a purely random process. Model (10) is also known as an *ARIMA process* of order (p,d,q). If $p=0$, the process follows an *IMA model.*

Two common stochastic trend models are the *IMA(0,1,1)* and *IMA(0,2,2)* which take the form, respectively,

$$\begin{aligned} &(1-B)Y_t = (1-\theta B)a_t, \quad |\theta| < 1, \\ &a_t \sim \text{i.i.d.}(0, \sigma^2{}_a) \end{aligned} \tag{11}$$

or, equivalently,

$$Y_t = Y_{t-1} + a_t - \theta a_{t-1}, \tag{12}$$

and

$$(1-B)^2 Y_t = (1-\theta_1 B - \theta_2 B^2)a_t,$$
$$\theta_2 + \theta_1 < 1, \theta_2 - \theta_1 < 1, -1 < \theta_2 < 1, \quad (13)$$
$$a_t \sim \text{i.i.d.}(0, \sigma^2{}_a)$$

or equivalently,

$$Y_t = 2Y_{t-1} - Y_{t-2} + a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2}. \quad (14)$$

The *a*'s may be regarded as a series of random shocks that drive the trend and θ can be interpreted as measuring the extent to which the random shocks or "innovations" incorporate themselves into the subsequent history of the trend. For example, in model (11), the smaller the value of θ, the more flexible the trend; the higher the value of θ, the more rigid the trend (less sensitive to new innovations). For $q=1$, model (11) reduces to one type of random walk model which has been used mainly for economic time series such as stock market price data (Granger and Morgenstern,1970). In such models, as time increases the random variables tend to oscillate about their mean value with an ever increasing amplitude. The use of stochastic models in business and economic series has received considerable attention during the last twenty years (see, for example, Nelson and Plosser, 1982, Harvey, 1985, Harvey and Jaegger 1993, Kaiser and Maravall, 1999 and 2001).

3. COMMON LOCAL TREND PREDICTORS

Economists and statisticians are often interested in the "short" term trend of socio-economic time series. The short term trend generally includes cyclical fluctuations, and is referred to as *trend-cycle*. In recent years, there has been an increased interest to use trend-cycle estimates or smoothed seasonally adjusted data to facilitate recession and recovery analysis. Among other reasons, this interest originated from major economic and financial changes of global nature which have introduced more variability in the data, and consequently, in the seasonally adjusted numbers. This makes very difficult to determine the direction of the short-term trend, particularly to assess the presence or the upcoming of a turning point. The local polynomial regression predictors developed by Henderson (1916) and LOESS due to Cleveland (1979) are the most widely applied to estimate the short-term trend of seasonally adjusted economic indicators. Particularly, the former is available in nonparametric seasonal adjustment software such as the U.S. Bureau of the Census X11 method (Shiskin *et al.*, 1967) and its variants the X11ARIMA (Dagum, 1980) and X12ARIMA (Findley *et al.*, 1998), the latter, in STL (Cleveland *et al.*, 1990).

The basic assumption is that the input series $\{y_t, t = 1, 2, ..., N\}$ can be decomposed into the sum of a systematic component called the signal (or nonstationary mean) g_t, plus an erratic component called the noise u_t, such that

$$y_t = g_t + u_t. \tag{15}$$

The noise component u_t is assumed to be either a white noise, $WN(0, \sigma^2_u)$, or, more generally, to follow a stationary and invertible Autoregressive Moving Average (ARMA) process.

Assuming that the input series $\{y_t, t = 1, 2, ...,N\}$ is seasonally adjusted or without seasonality, the signal g_t represents the trend and cyclical components, usually referred to as trend-cycle for they are estimated jointly. The trend-cycle can be represented locally by a polynomial of the time distance j, between y_t and the neighboring observations y_{t+j}.

Hence, given u_t for some time point t, it is possible to find a local polynomial trend estimator

$$g_t(j) = a_0 + a_1 j + ... + a_p j^p + \varepsilon_t(j), \tag{16}$$

where $a_0, a_1, ..., a_p$ are real and ε_t is assumed to be purely random and mutually uncorrelated with u_t. The coefficients $a_0, a_1, ..., a_p$ can be estimated by ordinary or weighted least squares or by summation formulae. The solution for $\hat{a}_0$ provides the trend-cycle estimate $\hat{g}_t(0)$, which equivalently is a weighted average (Kendall, Stuart, and Ord, 1983), applied in a moving average, such that

$$\hat{g}_t(0) = \hat{g}_t = \sum_{j=-m}^{m} w_j y_{t-j} \tag{17}$$

where w_j , $j < N$, denotes the weights to be applied to the observations y_{t+j} to get the estimate $\hat{g}_t$ for each point in time $t = 1, 2, ...,N$.

The weights depend on: (1) the degree of the fitted polynomial, (2) the amplitude of the neighborhood, and (3) the shape of the function used to average the observations in each neighborhood.

Once a (symmetric) span $2m+1$ of the neighborhood has been selected, the w_j's for the observations corresponding to points falling out of the neighborhood of any target point are null or approximately null, such that the estimates of the $N - 2m$ central observations are obtained by applying $2m + 1$ symmetric weights to the observations neighboring the target point. The missing estimates for the first and last m observations can be obtained by applying asymmetric moving averages of variable length to the first and last m observations, respectively. The length of the moving average or time invariant symmetric linear filter is $2m+1$, whereas the asymmetric linear filters length is time varying.

Using the backshift operator B, such that $B^n y_t = y_{t-n}$, equation (17) can be written as

$$\hat{g}_t = \sum_{j=-m}^{m} w_j B^j y_t = W(B) y_t \qquad t = 1,2,...,N \tag{18}$$

where *W(B)* is a linear nonparametric estimator.

The nonparametric estimator *W(B)* is said to be a second order kernel if it satisfies the conditions

$$\sum_{j=-m}^{m} w_j = 1, \quad \sum_{j=-m}^{m} j w_j = 0 \tag{19}$$

hence it preserves a constant and a linear trend. On the other hand, *W(B)* is a higher order kernel if

$$\sum_{j=-m}^{m} w_j = 1, \quad \sum_{j=-m}^{m} j^i w_j = 0 \tag{20}$$

for some $i = 1, 2, ..., p \geq 2$. In other words, it will reproduce a polynomial trend of degree $p - 1$ without distortion.

The nonparametric function estimators are based on different assumptions of smoothing. For example, the locally weighted regression smoother (LOESS) fits local polynomials of a degree *d* where the parameters are estimated either by ordinary or weighted least squares. Thus, it satisfies the property of best fit to the data.

Given a series of equally spaced observations and corresponding target points $\{(y_j, t_j), j = 1, ..., N\}$, $t_1 < ... < t_N$, where t_j denotes the time the observation y_j is taken, Loess produces a smoothed estimate as follows

$$y_j = \mathbf{t}^T \hat{\beta}_j \tag{21}$$

where $\mathbf{t}_j$ is a $(d+1)$-dimensional vector of generic component t_j^p, $p = 0, ..., d$; $d = 0, 1, 2, ...$ denotes the degree of the fitting polynomial, and $\hat{\beta}_j$ is the $(d+1)$-dimensional least squares estimate of a weighted regression computed over a neighborhood of t_j constituting a subset of the full span of the series.

The weights of the regression depend on the distance between the target point t_j^* and any other point belonging to its neighborhood, through a weight function *W(t)*.

The weighting function more often used is the tricube proposed by Cleveland *et al.* (1990), *i.e.*

$$W(t) = (1 - |t|^3)^3 I_{[-1,1]}(t) \tag{22}$$

In particular, at each point in the neighborhood of the target point t_j^*, $N(t_j^*)$, has assigned a weight

$$w(t_k) = W\left(\frac{|t_j^* - t_k|}{\Delta(t_j^*)}\right) \qquad \forall t_k \in N(t_j^*) \tag{23}$$

with $D(t_j^*)$ representing the distance of the furthest near-neighbor from t_j^*.

Each neighborhood is made of the same number of points chosen to be nearest to t_j^*, and the ratio between the amplitude of the neighborhood, k, and the full span of the series, N, defines the bandwidth or smoothing parameter.

Cleveland (1979) derived the filters for the first and last observations by weighting the data belonging to an asymmetric neighborhood which contains the same number of data points of the symmetric one.

On the other hand, the Cubic Smoothing Spline searches for an optimal solution between both fitting and smoothing of the data under the assumption that the signal follows, locally, a second degree polynomial. Hence,

$$\min_{f_\lambda \in C^2} \frac{1}{N}\sum_{i=1}^{N}[y_j - f_\lambda(t_j)]^2 + \lambda \int_a^b [f_\lambda^{''}(u)]^2 du \tag{24}$$

where λ is a smoothing parameter that balances the trade-off between the fit to the data (left hand) and the smoothness of the final estimates (right hand).

In matrix form

$$\hat{y} = S(\lambda)\, y, \tag{25}$$

where $S(\lambda)$ is called the influential matrix.

The well known Hodrick-Prescott (1997) trend filter applied to economic and financial series is a cubic spline where for quarterly series lambda is equal to 1600, indicating that the output data will be very smooth. These authors framework is that a given time series y_t is the sum of a growth component and a cyclical component c_t:

$$y_t = g_t + c_t, \text{ for } t=1,2,...,T. \tag{26}$$

The measure of the smoothness of the $\{g_t\}$ path is the sum of the squares of its second order difference. The c_t are deviations from g_t and the conceptual framework is that over long time periods, their average is near zero. These considerations lead to the following programming problem for determining the growth components

$$\min_{\{g_t\}_{t=1}^{T}} \left\{ \sum_{t=1}^{T} c_t^{\,2} + \lambda \sum_{t=1}^{T} [(g_t - g_{t-1}) - (g_{t-1} - g_{t-2})]^2 \right\} \tag{27}$$

where $c_t = y_t - g_t$. The parameter λ is a positive number which penalizes variability in the growth component series. The larger the value of λ, the smoother is the solution series. For a sufficiently large λ, at the optimum all the $g_{t+1} - g_t$ must be arbitrarily near some constant β and therefore for g_t arbitrarily near $g_0 + \beta t$. This implies that the limit of solution to (27) as λ approaches infinity is the least squares fit of a linear time trend model.

Kaiser and Maravall (1999) showed that under certain restriction the Hodrick-Prescott filter can be well approximated by a IMA model of order 2.

A Kernel is a locally weighted average with a weighting function that follows a probability distribution.

$$\hat{y}_h = \sum_{j=1}^{N} w_{hj} y_j \tag{28}$$

where

$$w_{hj} = \frac{K_b\left(\dfrac{t_h^* - t_j}{b}\right)}{\sum_{i=1}^{N} K_b\left(\dfrac{t_h^* - t_i}{b}\right)}$$

are the weights from a non parametric kernel $K_b(x)=K_b(-x)$, *i.e.* a nonnegative function such that $b>0$ is a smoothing parameter.

An example is provided by the Gaussian kernel given by

$$K_b = (2\pi)^{-\frac{1}{2}b^{-1}} exp\left\{-\frac{1}{2}\left(\frac{t_h{}^* - t_j}{b}\right)^2\right\} \tag{29}$$

The Henderson smoothing filters are derived from the graduation theory, known to minimize smoothing with respect to a third degree polynomial within the span of the filter. It consists of locally fitting a cubic trend by weighted least squares where the weights are chosen to minimize the sum of squares of their third differences (*smoothing criterion*). The objective function to be minimized is

$$\sum_{j=-m}^{m} W_j [y_{t+j} - a_0 - a_1 j - a_2 j^2 - a_3 j^3]^2, \tag{30}$$

where the solution for the constant term $\hat{a}_0$ is the smoothed observation $\hat{g}_t, W_j = W_{-j}$ and the filter length is $2m + 1$.

The solution is a local cubic smoother with weights

$$W_j \propto \{(m+1)^2 - j^2\}\{(m+2)^2 - j^2\}\{(m+3)^2 - j^2\} \tag{31}$$

and the weight diagram known as Henderson's ideal formula is obtained, for a filter length equal to $2m-3$,

$$w_j = \frac{315 \times [(m-1)^2 - j^2](m^2 - j^2)[(m+1)^2 - j^2](3m^2 - 16 - 11j^2)}{8m(m^2-1)(4m^2-1)(4m^2-9)(4m^2-25)} \tag{32}$$

Important studies related to these kind of trend-cycle estimators have been made, among many others, by Pearce (1975), Burman (1980), Cleveland and Tiao (1976), Box et al. (1978), Kenny and Durbin (1982), and Dagum and Luati (2000 and 2001).

Recently, Dagum and Bianconcini (2006 and 2007) have found Reproducing Kernels in Hilbert Spaces (RKHS) of the Henderson and LOESS local polynomial regression predictors with particular emphasis on the asymmetric filters applied to most recent observations. These authors show that the asymmetric filters can be derived coherently with the corresponding symmetric weights or from a lower or higher order kernel within a hierarchy, if preferred. In the particular case of the currently applied asymmetric Henderson and LOESS filters, those obtained by means of the RKHS are shown to have superior properties relative to the classical ones from the view point of signal passing, noise suppression and revisions.

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REFERENCES

T.W. ANDERSON (1971), *The statistical analysis of time series*, Wiley, New York.

G.E.P. BOX, G.M. JENKINS (1970), *Time series analysis: forecasting and control*, Holden Day, San Francisco, CA.

G.E.P. BOX, S.C. HILLMER, G.C. TIAO (1978), *Analysis and modelling of seasonal time series*. In "Seasonal Analysis of Economic Time Series", A. Zellner, ed. U.S. Bureau of Census, Washington, D.C.

J.P. BURMAN (1980), *Seasonal adjustment by signal extraction*, "Journal of the Royal Statistical Society, Series A.", 143, pp. 321-337.

W.S. CLEVELAND (1979), *Robust locally regression and smoothing scatterplots*, "Journal of the American Statistical Association", 74, pp. 829-836.

R. CLEVELAND, W. CLEVELAND, J. MCRAE, I. TERPENNING (1990), *STL: A seasonal trend decomposition procedure based on LOESS*, "Journal of Official Statistics", 6, pp. 3-33.

W.P. CLEVELAND, G. C. TIAO (1976), *Decomposition of seasonal time series: a model for the census X11 program*, "Journal of the American Statistical Association", 71, pp. 581-587.

C. DAGUM (1985), *Analyses of income distribution and inequality by education and sex in Canada*. In "Advances in Econometrics", Vol. IV, R. L. Basmann and G.F. Rhodes, Jr., eds JAI Press, Greenwich, CN, pp. 167-227.

E.B. DAGUM (1980), *The X11ARIMA seasonal adjustment method. Statistics Canada*, Ottawa, Canada, Catalogue No. 12-564.

E.B. DAGUM, S. BIANCONCINI (2006), *Local polynomial trend-cycle predictors in reproducing kernel Hilbert spaces for current economic analysis*, "Anales de Economia Aplicada", pp. 1-22.

E.B. DAGUM, S. BIANCONCINI (2007), *The Henderson smoother in reproducing kernel Hilbert space*, "Journal of Business and Economic Statistics", forthcoming.

E.B. DAGUM, A. LUATI (2000), *Predictive performance of some nonparametric linear and nonlinear smoothers for noisy data*, "Statistica", Anno LX vol. 4, pp. 635-654.

E.B. DAGUM, A. LUATI (2001), *A Study of asymmetric and symmetric weights of Kernel smoothers and* *their spectral properties*, in *Estadistica*, "Journal of the InterAmerican Statistical Institute", special issue on Time Series Analysis, vol. 53, pp. 215-258.

R.L. EUBANK (1988), *Spline smoothing and nonparametric regression*, New York: Marcel Dekker.

J. FAN (1992), *Design-adaptive nonparametric regression*, "Journal of the American Statistical Association", 87, pp. 998-1004.

J. FAN (1993), *Local linear regression smoothers and their minimax efficiencies*, "Annals of Statistics", 21, pp. 196-216.

J. FAN, I. GIJBELS (1997), *Local polynomial modelling and its applications*, Chapman and Hall, New York.

D. FINDLEY, B. MONSELL, W. BELL, M. OTTO, B. CHEN (1998), *New capabilities and methods of the X12arima seasonal adjustment program*, "Journal of business economic statistics", 16, pp. 127-152.

C.W. GRANGER, O. MORGENSTERN (1970), *Predictability of stock market prices*, D.C. Heath, Lexington, MA.

P.J. GREEN, B.W. SILVERMAN (1994), *Nonparametric regression and generalized linear models*, London: Chapman and Hall.

W. HARDLE (1990), *Applied nonparametric regression*, Cambridge: Cambridge University Press.

A.G. HARVEY (1985), *Trends and cycles in macroeconomic time series*, "J. Bus. Econ. Statist.", 3, pp. 216-227.

A.G. HARVEY, A. JAEGGER (1993), *Detrending, stylized facts and the business cycle*, "Journal of Applied Econometrics", vol. 8, pp. 231-247.

R. HENDERSON (1916), *Note on graduation by adjusted average*, "Transaction of Actuarial Society of America", 17, pp. 43-48.

R.J. HODRICK, E. PRESCOTT (1997), *Postwar U.S. business cycles: an empirical investigation*, "Journal of Money, Credit and Banking", vol. 29 No. 1, pp. 1-16.

R. KAISER, A. MARAVALL (1999), *Estimation of the Business-cycle: a modified Hodrick-Prescott filter*, "Spanish Economic Review", vol. 1, pp. 175-206.

R. KAISER, A. MARAVALL (2001), *Measuring cycles in economic statistics*, Lecture Notes in Statistics, 154 New York: Springer-Verlag.

M.G. KENDALL, A. STUART, J. ORD (1983), *The advanced theory of statistics*, Vol. 3, Ed. C. Griffin.

P.B. KENNY, J. DURBIN (1982), *Local trend estimation and seasonal adjustment of economic and social time series*, "Journal of the Royal Statistical Society, Series A", 145, pp. 1-41.

N. KONDRATIEFF (1925), *Long economic cycles. Voprosy Konyuktury*, Vol. 1, No. 1 (English translation: *The long wave cycle*, Richardson and Snyder, New York, 1984).

F. MACAULEY (1931), *The smoothing of time series*, National Bureau of Economic Research, New York.

C.R. NELSON, C.I. PLOSSER (1982), *Trends and random walks in macroeconomic time series: Some evidences and implications*, "Journal of Monetary Economics", 10, pp. 139-162.

D.A. PIERCE (1975), *On trend and autocorrelation*, "Communications in Statistics", 4, pp. 163-175.

J. SHISKIN, A.H. YOUNG, J.C. MUSGRAVE (1967), *The X11 variant of the census method II seasonal adjustment program*, Technical Paper No. 15, U.S. Department of Commerce, U.S. Bureau of Census, Washington, DC.

J.S. SIMONOFF (1995), *Smoothing methods in statistics*, New York: Springer.

P.F. VERHULST (1838), *Notice sur la loi que la population suit dans son accroissement*, Correspondance Mahtematique et Physique, A. Quetelet, ed. Tome X, pp. 113-121.

M. WAND, M. JONES (1995), *Kernel smoothing*, Monographs on statistics and applied probability, 60, Chapman and Hall.

E. WHITTAKER, G. ROBINSON (1924), *Calculus of observations: a treasure on numerical calculations*, Blackie and Son, London.
A.M. YAGLOM (1962), *An introduction to the theory of stationary random functions*, Prenctice-Hall, Englewood Cliffs, NJ.

RIASSUNTO

Modelli stocastici e deterministici per la stima del trend

Nel lavoro proponiamo una rassegna di alcuni modelli per la stima di trend globali e locali. Modelli per la stima globale del trend si basano sull'assunzione che la media non stazionaria della serie storica possa essere ben approssimata da semplici funzioni del tempo sull'intero campo di osservazione della serie. Le rappresentazioni più comuni di trend deterministici e stocastici sono quindi introdotte. In particolare, nel contesto dei trend deterministici vengono analizzate funzioni polinomiali e trascendentali, mentre per la stima di trend stocastici assumiamo che la serie originale segua un processo stocastico non stazionario, ma lineare omogeneo. Recentemente, una maggiore attenzione è stata orientata all'analisi del trend di breve periodo, caratterizzato da fluttuazioni cicliche e generalmente noto come trend-ciclo. A tale riguardo, nel lavoro consideriamo il classico filtro sviluppato da Henderson (1916) e la LOESS ideato da Cleveland (1979), che sono i predittori maggiormente utilizzati per la stima del trend locale di breve periodo di serie storiche destagionalizzate.

SUMMARY

Stochastic and deterministic trend models

In this paper we provide an overview of some trend models formulated for global and local estimation. Global trend models are based on the assumption that the trend or nonstationary mean of a time series can be approximated closely by simple functions of time over the entire span of the series. The most common representation of deterministic and stochastic trend are introduced. In particular, for the former we analyze polynomial and transcendental functions, whereas for the latter we assume that the series from which the trend will be identified follows a homogeneous linear nonstationary stochastic process. Recently more attention has been oriented on the analysis of the short term trend, that includes cyclical fluctuations and is referred to as trend-cycle. At this regard, we analyze the local polynomial regression predictors developed by Henderson (1916) and LOESS due to Cleveland (1979), which are the most widely applied to estimate the short term local trend of seasonally adjusted economic indicators.