STATISTICA, anno LXXXI, n. 1, 2021

ROBUST ESTIMATIONS OF SURVIVAL FUNCTION FOR WEIBULL DISTRIBUTION

Derya Karagöz [1]
Department of Statistics, Hacettepe University, Beytepe, Ankara, Turkey
Nihal Ata Tutkun
Department of Statistics, Hacettepe University, Beytepe, Ankara, Turkey

1. Introduction

Survival analysis plays an important role in many fields such as medicine, epidemiology, biology, demography, economics, engineering and so forth. In many survival studies the outcome of interest is the time to an event. Such events may be adverse, such as death, recurrence of an illness, failure of an equipment, births, divorces, promotions and so forth. In analyzing time to event (survival) data, however, because of possible censoring rate (CR), the summary statistics may not have the desired statistical properties, such as unbiasedness. So other methods to present the data have to be used. One way is to estimate the underlying true distribution of survival time (T) (Borovkova, 2002).

Let T be the survival time having a continuous distribution with finite expectation and represents the time being in a given state or the time between two events. The distribution of the random variable T can be described in a number of equivalent ways. Probability density function $f(t)$, survival function $S(t)$ or hazard function $h(t)$ characterize the distribution of T. In survival analysis, it is often common to use survival function,

$$S(t) = P(T > t) = \int_t^\infty f(x)dx, \qquad 0 < t < \infty \tag{1}$$

which gives the probability that failure will occur after time t. Many parametric and non-parametric approximations are used to estimate survival function. The estimation of the survival function turns into estimating the unknown parameters of the distributions for parametric distributed survival data.In generally, if the parameters of distributions are not known, they are usually estimated. The well known classical estimation method is the maximum likelihood (ML). So the parameters of the distribution are usually estimated by the ML method.

[1] Corresponding Author. E-mail: deryacal@hacettepe.edu.tr

First of all, ML relies on numerical methods since there is no explicit solution to the log-likelihood maximization problem. Second, the ML estimator is very sensitive to outliers in the data. Empirical and simulation evidence for this is given in He and Fung (1999), Shier and Lawrence (1984) and among others. Boudt *et al.* (2011) complements this by deriving formal robustness measures. They consider the breakdown point, which is defined as the smallest proportion of observations that needs to be replaced with arbitrary values in order for the estimation of λ or β to be arbitrarily close to zero (implosion) or infinity (ex-losing). It is easy to see that the breakdown point of the ML estimator is $1/n \to 0$. The influence function, which quantifies the effect of small contaminations on the estimator, is considered as a second robustness measure .

The remainder of our paper is organized as follows. In the next section, we first represent the methods used to estimate the parameters of Weibull distribution. In Section 3 the estimation of survival function is given. We present the ML, quantile (Q) and quantile least squares (QLS) estimators of scale and shape parameters for the survival function in Section 3.1. We illustrate the proposed estimators with Hodgkin's data in Section 4. In Section 5, Monte Carlo studies that compare the survival function of the Q and QLS estimators with the ML estimator of Weibull distribution are presented. Finally, the results sum up in the last Section 6.

2. METHODOLOGY

Any distribution of non-negative random variables could be used to describe survival time. If the assumption of a particular probability distribution for the data is valid, inferences based on such assumption will be more precise (Collett, 2003). Throughout the literature on survival analysis, certain parametric distributions such as exponential, Gamma and Weibull have been widely used.

The simplest possible censored data distribution is exponential distribution. However, its constant hazard rate is improper and unrealistic in many cases. Gamma distribution is another candidate distribution for censored data. Nevertheless, distribution function or survival function of gamma distribution cannot be expressed in a closed form if the shape parameter is not an integer. Since it is in terms of an incomplete gamma function, one needs to obtain the distribution function, survival function or the hazard rate by numerical integration. This makes gamma distribution little bit unpopular compared to the Weibull distribution, which has a nice distribution function, survival function and hazard function (Gupta and Kundu, 2001).

Weibull distribution, also known as Extreme Value Type III minimum distribution, is widely used distribution in reliability and survival analysis. The Weibull distribution was introduced by the Swedish physicist Weibull (1951). He claimed that his distribution applied to a wide range of problems and illustrated this point with seven examples ranging from the strength of steel to the height of adult males in the British Isles (Abernethy *et al.*, 1983). It has been used in many different fields like material science, engineering, physics, chemistry, meteorology,medicine, pharmacy, economics and business, quality control, biology, geology and geography (Almalki and Nadarajah, 2014).

The primary advantage of Weibull distribution is the ability to provide reasonably accurate survival analysis and survival forecasts with extremely small samples. Solutions are possible at the earliest indications of a problem without having to "crash a few more". Small samples also allow cost effective component testing. For example, "sudden death" Weibull tests are completed when the first failure occurs in each group of components, (say,groups of four bearings). If all the bearings are tested to failure, the cost and time required is much greater. Second advantage of Weibull is that it provides a simple and useful graphical plot of the failure data. The data plot is extremely important and particularly informative. The horizontal scale is a measure of life or ageing. The vertical scale is the cumulative percentage failed. The slope of the line, β, is particularly significant and may provide a clue to the physics of the failure (Abernethy *et al.*, 1983). Another advantage is that the Weibull probability distribution of censored data is defined by two parameters; λ is called the scale parameter and β is called the shape parameter. These two parameters provide additional flexibility that potentially increases the accuracy of the description of collected censored data (Bain, 1976).

The probability density function of Weibull distribution has the form

$$f(t) = \frac{\beta}{\lambda}\left(\frac{t}{\lambda}\right)^{\beta-1} \exp\left(-(t/\lambda)^{\beta}\right), \tag{2}$$

where $t, \lambda, \beta > 0$. Weibull cumulative distribution function is given by

$$F(t) = 1 - \exp\left(-(t/\lambda)^{\beta}\right) \tag{3}$$

and survival function is then

$$S(t) = \exp\left(-(t/\lambda)^{\beta}\right). \tag{4}$$

The mean and variance of a Weibull distribution can be expressed as

$$E(t) = \lambda\Gamma(1 + \frac{1}{\beta}) \tag{5}$$

$$V(t) = \lambda\left[\Gamma(1 + \frac{2}{\beta}) - (\Gamma(1 + \frac{1}{\beta}))^2\right], \tag{6}$$

where Γ is the gamma function. Weibull distribution has broader application since it does not assume a constant hazard rate, unlike exponential distribution. So that, the hazard function and corresponding cumulative hazard and corresponding cumulative hazard functions are,

$$h(t) = \frac{\beta}{\lambda}\left(\frac{t}{\lambda}\right)^{\beta-1}, \tag{7}$$

$$H(t) = \frac{1}{\lambda^{\beta}} t^{\beta}. \tag{8}$$

When $\beta = 1$, the hazard function remains constant as time increases and Weibull and the exponential survival probabilities are identical. Weibull distribution allows the hazard function to increase when $\beta > 1$ with increasing time. Human mortality and disease patterns typically have increasing hazard rates with age. When $\beta < 1$, the hazard function decreases also in a non-linear pattern with increasing time. For example, the risk of a recurrence of a tumour after surgery might decrease as time passes. Decreasing hazard rates are less commonly encountered in epidemiology and medical survival data.

3. Estimations of survival function for Weibull distribution

The estimator of the survival function for Weibull distributed survival data is obtained as follows by using Eq. (4):

$$\hat{S}(t) = \exp\left(-\left(t/\hat{\lambda}\right)^{\hat{\beta}}\right), \tag{9}$$

where $\hat{\lambda}$ and $\hat{\beta}$ is the parameter estimators of Weibull distribution which are given in Section 3.1. Survival function estimation based on ML estimators is obtained by using Eq. (12) and Eq. (13). Survival function estimation based on Q estimators is obtained by setting the parameter estimators given by Eq. (14) and Eq. (15). Finally, survival function estimation based on QLS estimators is obtained by setting the parameter estimators given by Eq. (20) and Eq. (19).

In order to obtain the variance of survival function, we use of a general result known as Taylor series approximation to the variance of a function of a random variable. According to this result, the variance of a function $g(X)$ of the random variable X is given by $V\{g(x)\} \approx \{dg(X)/dX\}^2 V(X)$ (Collett, 2003). In our study, the variance of the survival function is obtained as follows:

$$V\left\{\hat{S}(t)\right\} \approx \left\{-\exp\left(-t/\hat{\lambda}\right)^{\hat{\beta}} \frac{\hat{\beta}}{\hat{\lambda}^{\hat{\beta}}} t^{\hat{\beta}-1}\right\}^2 \hat{V(t)}, \tag{10}$$

where $\hat{V(t)}$ is the estimation for variance of survival time which has the Weibull distribution.

As seen in Eq. (10) the variance of survival function is the function of the estimated distribution parameters, $\hat{\lambda}$ and $\hat{\beta}$, and variance of the distribution $V(t)$. To obtain the variance of survival function, firstly the parameter estimators of Weibull must be obtained by using ML, Q and QLS methods.

3.1. The parameter estimators of Weibull distribution for the censored data

ML estimators of the Weibull parameters are the most efficient estimates (lowest variances) but are numerically complex. The ML process takes into account censored observations making the estimates unbiased (Selvin, 2005). ML estimator of the Weibull parameters λ and β involves equations to be solved simultaneously. Numerical methods such as the Newton-Raphson iterative procedure can be applied. When data are progressively censored, we have $t_1 \leq t_2 \leq ... \leq t_r, t_{r+1}^+, ..., t_n^+$. The log-likelihood function is given by,

$$\begin{aligned} I(\lambda,\beta) &= r\log(\beta) + r\log(1/\hat{\lambda}) + \sum_{i=1}^{r}(\beta-1)\log t_i - (1/\hat{\lambda})^{\hat{\beta}}\sum_{i=1}^{r} t_i^{\hat{\beta}} + \log t_i \\ &\quad -(1/\hat{\lambda})^{\hat{\beta}}\sum_{i=r+1}^{n} t_i^{+\hat{\beta}} = 0. \end{aligned} \tag{11}$$

The ML estimators of the λ and β may be obtained by solving the following two equations simultaneously (Lee and Wang, 2003):

$$r - (1/\hat{\lambda})^{\hat{\beta}}\left(\sum_{i=1}^{r} t_i^{\hat{\beta}} + \sum_{i=r+1}^{n} t_i^{+\hat{\beta}}\right) = 0, \tag{12}$$

$$\begin{aligned} \frac{r}{\hat{\beta}} &+ r\log(1/\hat{\lambda}) + \sum_{i=1}^{r}\log t_i - (1/\hat{\lambda})^{\hat{\beta}}\sum_{i=1}^{r} t_i^{\hat{\beta}}(\log(1/\hat{\lambda}) + \log t_i) \\ &- (1/\hat{\lambda})^{\hat{\beta}}\sum_{i=r+1}^{n} t_i^{+\hat{\beta}}(\log(1/\hat{\lambda}) + \log t_i^+) = 0. \end{aligned} \tag{13}$$

As an alternative to the ML estimators of the Weibull parameters we consider two robust and explicit Weibull parameter estimators proposed by Boudt *et al.* (2011): the Q and QLS which are all robust to left, right and interval censored data. These estimators are identical to the estimator based on the data without censoring if the amount of right censoring is moderate. If the robust estimators are used, the proportion of right, left and interval censored data is 33% which is bigger than the method of median proposed by He and Fung (1999).

In the rest of the paper, the robust shape and scale parameter estimators are based on $\alpha_1 = \text{CR}$ and $\alpha_2 = 1 - \text{CR}$. Boudt *et al.* (2011) illustrated not only that these are the estimators for which the shape parameter has the highest breakdown point, but also that the influence function of ML estimator is unbounded, whereas the influence functions of the Q and QLS estimators are all bounded.

Denote $\hat{q}_\alpha$, quantile of the observations $t_1, t_2, ..., t_n$. The difference between the logs of any two high and low Weibull quantiles $q_{\alpha 2} = \lambda[-\log(1-\alpha_2)]^{1/\beta}$ and $q_{\alpha 1} = \lambda[-\log(1-\alpha_1)]^{1/\beta}$, $(0 < \alpha_1 < \alpha_2 < 1)$ depends only on the shape parameter. Replacing the theoretical quantiles with the corresponding empirical quantiles $\log(\hat{q}_{\alpha 1})$ and $\log(\hat{q}_{\alpha 2})$, yields the following quantile-estimator of shape parameter (Boudt *et al.*, 2011)

$$\hat{\beta}_Q = \frac{1}{\log \hat{q}_{\alpha 2}/\hat{q}_{\alpha 1}} \log\frac{\log(1-\alpha_2)}{\log(1-\alpha_1)}. \tag{14}$$

The corresponding scale estimator is obtained by plugging the quantile-estimator of shape in the Weibull quantile function. After some algebra, this yields the following estimate for the scale parameter (Boudt *et al.*, 2011)

$$\hat{\lambda}_Q = \frac{F(\alpha)^{-1}}{(-\log(1-\alpha))^{1/\hat{\beta}_Q}}. \tag{15}$$

The Weibull distribution function in $q_\alpha = \lambda[-\log(1-\alpha)]^{1/\beta}$ can be transformed into a straight line by a double logarithmic transformation, with parameters that are non-linear functions of λ and β. The quantiles of the general log-Weibull distribution is

$$\log q_\alpha = \log\lambda + \frac{1}{\beta}\log(-\log(1-\alpha)) \tag{16}$$

a linear with intercept $b_0 = \log\lambda$ and slope $b_1 = 1/\beta$. The QLS estimators are:

$$\hat{b}_1 = \frac{r\sum_{i=\lfloor\frac{n-r}{2}\rfloor+1}^{n-\lfloor\frac{n-r}{2}\rfloor} z_i y_i - \sum_{i=\lfloor\frac{n-r}{2}\rfloor+1}^{n-\lfloor\frac{n-r}{2}\rfloor} z_i \sum_{i=\lfloor\frac{n-r}{2}\rfloor+1}^{n-\lfloor\frac{n-r}{2}\rfloor} y_i}{r\sum_{i=\lfloor\frac{n-r}{2}\rfloor+1}^{n-\lfloor\frac{n-r}{2}\rfloor} z_i^2 - \left(\sum_{i=\lfloor\frac{n-r}{2}\rfloor+1}^{n-\lfloor\frac{n-r}{2}\rfloor} z_i\right)^2}, \tag{17}$$

$$\hat{b}_0 = \frac{1}{r}\sum_{i=\lfloor\frac{n-r}{2}\rfloor+1}^{n-\lfloor\frac{n-r}{2}\rfloor} y_i - \frac{1}{r}\hat{b}_{1QLS}\sum_{i=\lfloor\frac{n-r}{2}\rfloor+1}^{n-\lfloor\frac{n-r}{2}\rfloor} z_i, \tag{18}$$

where $y_i = \log\hat{q}_{i/(n+1)}$, $z_i = \log(-\log(1-\alpha_i))$, $0 < \frac{n-r}{n} < 0.5$ (Boudt *et al.*, 2011). Considering these two estimators of the regression parameters, we can easily obtain the estimators of the Weibull distribution parameters

$$\hat{\lambda}_{QLS} = \exp\left(\hat{b}_{0QLS}\right), \tag{19}$$

$$\hat{\beta}_{QLS} = 1/\hat{b}_{1QLS}. \tag{20}$$

4. NUMERICAL EXAMPLE

Our example consists of survival times in months for a group of patients with advanced Hodgkin's disease who received little or no previous therapy: 1.25, 1.41, 4.98, 5.25, 5.38, 6.92, 8.89, 10.98, 11.18, 13.11, 13.21, 16.33, 19.77, 21.08, 22.07, 42.92, 21.84+, 31.38+, 32.62+, 37.18+ (He and Fung, 1999). The last four observations were right censored, then the censoring rate is 20%. With this Weibull distributed real data, we obtain survival function estimates and also their variances of ML, Q and QLS estimators. The results are given by Table 1. For Hodgkin's disease data, the variance of Q estimator is the smallest variances whereas the variance of ML estimator is the largest within the three estimators for each survival time.

TABLE 1
Survival estimation and its variance using ML, Q and QLS with respect to Hodgkin's data.

	ML		Q		QLS	
t	$\hat{S}(t)$	$V(\hat{S}(t))$	$\hat{S}(t)$	$V(\hat{S}(t))$	$\hat{S}(t)$	$V(\hat{S}(t))$
1.25	0.96	0.45	0.96	0.25	0.96	0.30
1.41	0.95	0.45	0.96	0.26	0.95	0.31
4.98	0.81	0.44	0.81	0.33	0.81	0.36
5.25	0.80	0.44	0.80	0.33	0.80	0.36
5.38	0.80	0.43	0.79	0.32	0.79	0.36
6.92	0.74	0.40	0.73	0.31	0.73	0.33
8.89	0.67	0.35	0.65	0.28	0.66	0.29
10.98	0.61	0.29	0.57	0.24	0.58	0.25
11.18	0.60	0.29	0.57	0.23	0.57	0.25
13.11	0.55	0.25	0.50	0.19	0.51	0.27
13.21	0.54	0.25	0.50	0.19	0.51	0.21
16.33	0.46	0.19	0.40	0.14	0.42	0.15
19.77	0.38	0.14	0.32	0.09	0.34	0.10
21.08	0.36	0.12	0.29	0.08	0.31	0.09
22.07	0.34	0.11	0.28	0.07	0.29	0.08
42.92	0.34	0.11	0.27	0.07	0.29	0.08

5. Simulation study

In this section, Monte Carlo simulation is performed to compare the behaviours of the proposed robust estimators.

TABLE 2
Robust quantile estimator for interval, right and left censored Weibull data.

$[\beta, \lambda]$	Censoring	$\hat{\beta}$	SD($\hat{\beta}$)	MSE($\hat{\beta}$)	$\hat{\lambda}$	SD($\hat{\lambda}$)	MSE($\hat{\lambda}$)
	Right	2.216	0.357	0.132	1.000	0.066	0.004
[2.50,1]	Left	2.219	0.358	0.133	1.000	0.066	0.004
	Interval	2.227	0.364	0.138	0.998	0.067	0.005
	Right	2.216	0.357	0.132	1.000	0.066	0.004
[2.15,1]	Left	2.219	0.358	0.133	1.001	0.066	0.004
	Interval	2.227	0.364	0.138	0.998	0.067	0.005
	Right	1.617	0.264	0.072	1.002	0.091	0.008
[1.57,1]	Left	1.619	0.264	0.072	1.002	0.091	0.008
	Interval	1.620	0.259	0.070	1.002	0.092	0.009
	Right	1.2410	0.2004	0.042	1.002	0.122	0.015
[1.20,1]	Left	1.243	0.201	0.042	1.002	0.121	0.015
	Interval	1.242	0.198	0.041	1.001	0.120	0.014
	Right	1.031	0.165	0.028	1.004	0.146	0.021
[1.00,1]	Left	1.032	0.165	0.028	1.004	0.145	0.021
	Interval	1.037	0.166	0.029	1.004	0.143	0.020
	Right	0.887	0.142	0.021	1.007	0.170	0.029
[0.86,1]	Left	0.888	0.143	0.021	1.006	0.169	0.029
	Interval	0.892	0.144	0.022	1.006	0.170	0.029
	Right	0.793	0.127	0.017	1.011	0.189	0.036
[0.77,1]	Left	0.794	0.127	0.017	1.010	0.188	0.036
	Interval	0.801	0.129	0.018	1.009	0.189	0.036

In the first part of simulation study, by considering increasing, constant, and decreasing failure rates, the Weibull distribution is used with the shape parameter (β) which is equal to 2.5, 1.57, 1.20, 1.00, 0.86, 0.77 , whereas the scale parameter is set to 1. For Weibull distribution it is mentioned that the skewness increases when the value of shape parameter decreases. Weibull (λ, β) observations are generated among 10,000 samples of size n=100. Then for the left censored data we replaced $Y_{68} \dots Y_{100}$ by Y_{67} and for the right censored data we replaced $Y_1 \dots Y_{32}$by Y_{33}.

We use 10,000 repetitions to obtain the estimations of parameters achieved via Eq. (14) and Eq. (15) for Q. The estimations of parameters with their standard deviations are given in Table 2. The mean square error of an estimator shows how much an estimated value of parameter differs from the true value. For a Monte-Carlo simulation using $M = 10000$ repetitions, the mean-square error MSE_β for the shape parameter β is given by

$$\mathrm{MSE}_\beta = \frac{1}{M}\sum_{i=1}^{M}(\hat{\beta}_i - \beta)^2 \tag{21}$$

and the mean-square error MSE_λ for the shape parameter λ is given by

$$\mathrm{MSE}_\lambda = \frac{1}{M}\sum_{i=1}^{M}(\hat{\lambda}_i - \lambda)^2. \tag{22}$$

The estimations of distribution parameters, standard deviation of estimation parameters and MSE of estimation parameters are obtained for right, left and interval censoring data under Weibull distribution. Then $\hat{\lambda}$, $\hat{\beta}$, standard deviations and MSEs of these parameters were estimated by using Q method and the results obtained are given in Table 2.

In the presence of 33% right, left and interval censored observations, Monte Carlo simulation results shows that these estimators are applicable. And also that the bias and variability of shape parameter estimations $\hat{\beta}$ decreases as skewness increases and the bias and variability of scale parameter estimations $\hat{\lambda}$ increases as skewness increases.

In the second part of simulation study, the reference distribution is the Weibull distribution with the shape parameter (β) is equal to 0.5, 1, 2 corresponding to decreasing, constant, and increasing failure rates, respectively whereas the scale parameter is set to 1. We generate $M = 10,000$ samples of sizes $n = 20, n = 80$ to represent small and large sample sizes. Simulating censoring time C_i's, $i = 1, 2, ..., n$ are drawn from the uniform distribution $U(0, a)$ where a is chosen to ensure a desired censoring rate (CR). Therefore, CR is set equal to 10% and 30% in each sample, respectively. For each sample, we obtained the empirical values of upper percentiles(Up)at arbitrarily selected points, 0.1, 0.3, 0.5, 0.7, and 0.9, based on the Q, QLS and ML estimators. Suppose the Q and QLS estimators are computed using $\alpha_1 = 1/3$ and $\alpha_2 = 2/3$. We compute for each sample the scale estimate $\hat{\lambda}_j$ and shape estimate $\hat{\beta}_j$, for $j = 1, ..., M$ according to different simulation schemes. For each censoring rate and each type of estimator, we obtain the survival function estimators to compare the ML and robust estimators. For a Monte-Carlo simulation using $M = 10000$ repetitions, the root-mean-square error RMSE for the variance of survival function $V(\hat{S}(t))$ is given by

$$\mathrm{RMSE}_{V(\hat{S}(t))} = \sqrt{\frac{1}{M}\sum_{i=1}^{M}\Big[V(\hat{S}(t))_i - V(S(t))\Big]^2}. \tag{23}$$

We give the RMSE of survival function estimations for right, left and interval censored data respectively in Tables 3-5.

Estimates of the median value or other percentiles of a Weibull distribution follow directly from the estimated survival function. For the probability p, the corresponding p-level percentile is denoted tp. The percentile estimate is found by solving the relationship $S(tp) = 1 - p$ for the value tp. The expression for tp produces estimates of any number of percentiles characterizing the entire survival distribution (Selvin, 2005). For example, the estimated Weibull parameters yield the estimated pth-percentile of A months. Thus, this means that p% are expected to failed within A months based on the estimated Weibull distribution. One possible approach for the analysis of time-to-event data is the evaluation of survival percentiles, defined as the time by which a certain fraction of the population has experienced the event of interest (Orsini *et al.*, 2012). When focusing on survival percentiles, a specific probability of the event is fixed and is the time point to be estimated (Bellavia *et al.*, 2016).

Survival percentiles can be defined as the time points by which specific proportions of the study population have experienced the event D. For example, the time by which the first 50% of the individuals have experienced the event is defined as 50th survival percentile or median survival. The survival curve depicts a complete summary of the entire range of observed survival percentiles, presenting the proportion of events during the follow-up time (Bellavia *et al.*, 2016). A common approach to evaluate the survival function is to fix a specific time t usually the end of follow up and to estimate survival probabilities or rates of the event D in the time interval $[0, t]$, possibly according to levels of specific exposures or risk factors. In a percentile approach, on the other hand, the incidence proportion p is fixed to a specific level and the outcome to be evaluated is the corresponding survival percentile, the time t by which the study population reaches the specific fraction of events p (Bellavia *et al.*, 2016). In this simulation study we consider survival percentiles as 0.10, 0.30, 0.5, 0.7, 0.9.

The flow diagram for the simulation study 2 is given in Figure 1. The results of RMSE for the simulation study 2 are given in Table 3-5.

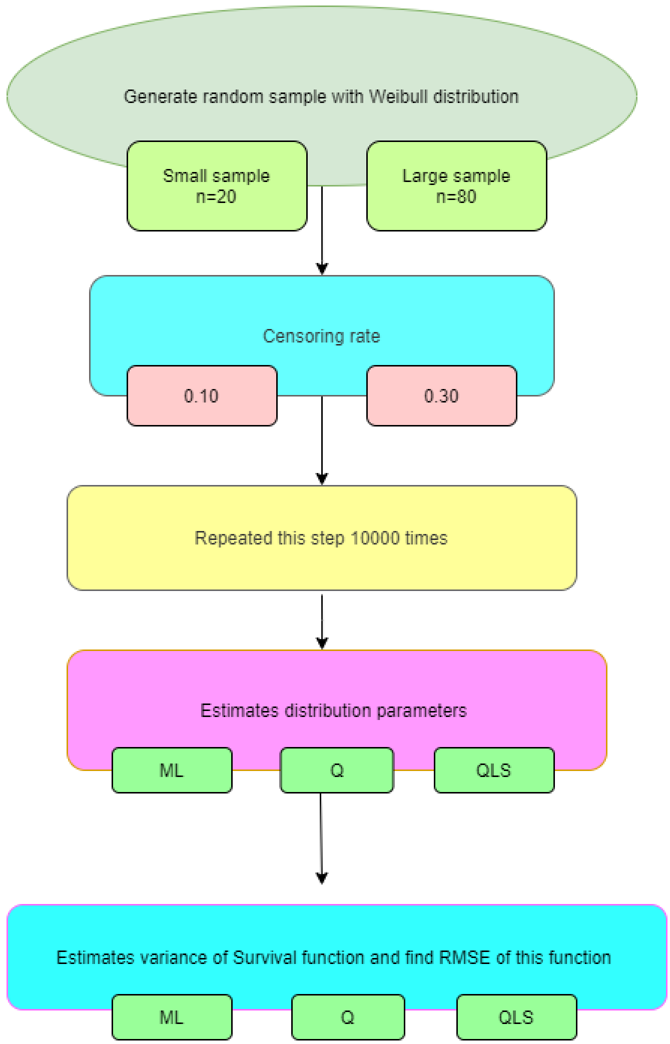

Figure 1 – Flow diagram for the simulation study 2.

TABLE 3
RMSE($V(\hat{S}(t))$) by using ML, Q and QLS estimators for right censored data.

		n=20					
Censoring rate		10%			30%		
β	t	ML	Q	QLS	ML	Q	QLS
0.5	18	0.158	0.152	0.176	0.294	0.266	0.243
	14	0.279	0.263	0.283	0.371	0.389	0.335
	10	0.263	0.273	0.257	0.370	0.399	0.338
	6	0.155	0.158	0.158	0.249	0.338	0.287
	2	0.064	0.061	0.077	0.069	0.082	0.083
1	18	0.075	0.090	0.091	0.142	0.152	0.130
	14	0.103	0.114	0.107	0.174	0.194	0.145
	12	0.111	0.120	0.106	0.176	0.203	0.149
	6	0.099	0.105	0.093	0.170	0.264	0.204
	2	0.059	0.064	0.058	0.100	0.133	0.116
2	18	0.313	0.340	0.302	0.217	0.301	0.308
	14	0.235	0.268	0.230	0.138	0.132	0.173
	10	0.122	0.128	0.131	0.113	0.088	0.146
	6	0.047	0.063	0.054	0.063	0.143	0.087
	2	0.075	0.081	0.063	0.108	0.144	0.124
		n=80					
Censoring rate		10%			30%		
β	t	ML	Q	QLS	ML	Q	QLS
0.5	72	0.165	0.139	0.162	0.280	0.243	0.186
	56	0.287	0.259	0.281	0.389	0.423	0.342
	40	0.248	0.243	0.241	0.389	0.427	0.345
	24	0.083	0.085	0.087	0.200	0.300	0.254
	8	0.070	0.059	0.074	0.041	0.041	0.040
1	72	0.066	0.054	0.065	0.130	0.113	0.078
	56	0.084	0.071	0.079	0.160	0.194	0.117
	40	0.081	0.079	0.075	0.160	0.198	0.120
	24	0.062	0.069	0.058	0.144	0.247	0.198
	8	0.031	0.039	0.031	0.071	0.126	0.109
2	72	0.305	0.340	0.310	0.224	0.289	0.342
	56	0.240	0.274	0.246	0.127	0.095	0.172
	40	0.125	0.135	0.132	0.119	0.082	0.163
	24	0.029	0.036	0.034	0.034	0.128	0.082
	8	0.061	0.072	0.058	0.0970	0.149	0.137

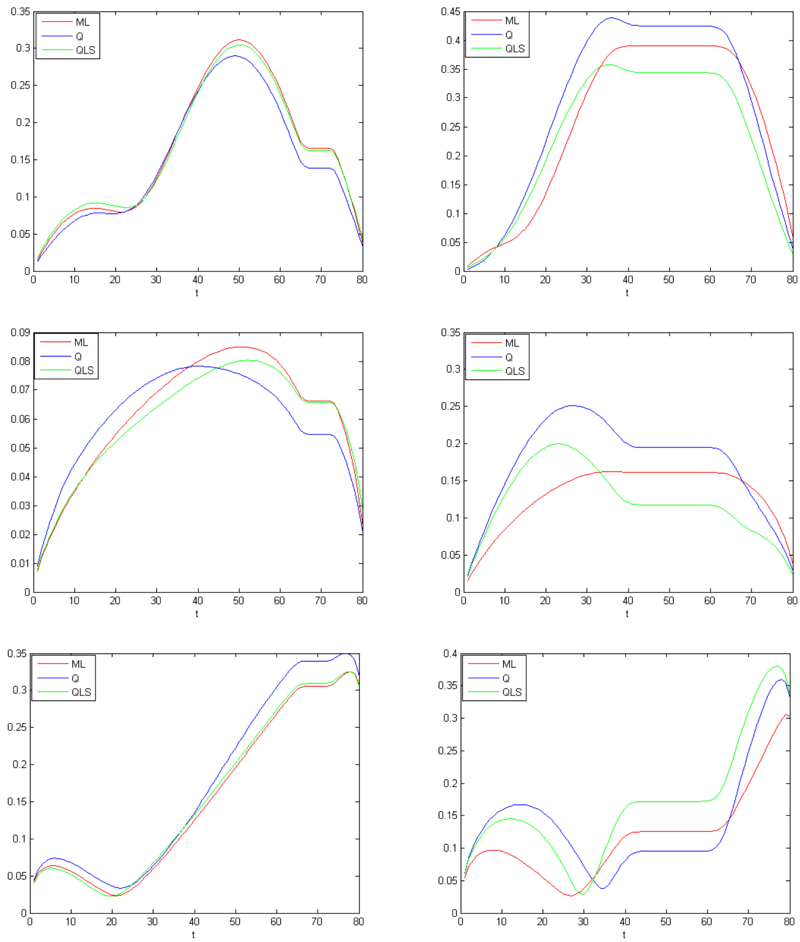

Figure 2 – RMSE($V(\hat{S}(t))$ versus t by using ML, Q and QLS estimators for right censored data n=80, on the left side figures for 10% CR, $\beta = 0.5, 1, 2$, on the right side figures for 30% CR, $\beta = 0.5, 1, 2$, respectively.

TABLE 4
RMSE$(V(\hat{S}(t))$ *by using ML, Q and QLS estimators for left censored data.*

n=20							
Censoring rate		10%			30%		
β	t	ML	Q	QLS	ML	Q	QLS
0.5	18	0.135	0.126	0.137	0.193	0.211	0.207
	14	0.235	0.207	0.206	0.267	0.268	0.288
	10	0.200	0.202	0.184	0.163	0.262	0.193
	6	0.112	0.145	0.165	0.153	0.258	0.188
	2	0.061	0.086	0.105	0.069	0.137	0.096
1	18	0.072	0.103	0.086	0.087	0.209	0.204
	14	0.088	0.134	0.105	0.121	0.316	0.314
	10	0.093	0.117	0.109	0.127	0.188	0.115
	6	0.080	0.086	0.094	0.125	0.184	0.111
	2	0.056	0.056	0.064	0.091	0.102	0.095
2	18	0.363	0.392	0.367	0.320	0.508	0.500
	14	0.244	0.312	0.286	0.131	0.473	0.462
	10	0.111	0.169	0.180	0.063	0.218	0.126
	6	0.054	0.067	0.072	0.068	0.208	0.111
	2	0.075	0.055	0.042	0.098	0.098	0.104
n=80							
Censoring rate		10%			30%		
β	t	ML	Q	QLS	ML	Q	QLS
0.5	72	0.121	0.089	0.118	0.180	0.012	0.026
	56	0.241	0.181	0.198	0.259	0.196	0.161
	40	0.119	0.116	0.104	0.079	0.264	0.171
	24	0.096	0.136	0.169	0.075	0.264	0.172
	8	0.069	0.090	0.123	0.051	0.140	0.095
1	72	0.031	0.061	0.039	0.045	0.161	0.152
	56	0.045	0.090	0.063	0.085	0.330	0.267
	40	0.049	0.079	0.080	0.086	0.194	0.100
	24	0.042	0.054	0.077	0.085	0.192	0.098
	8	0.029	0.032	0.055	0.059	0.074	0.060
2	72	0.364	0.403	0.369	0.325	0.502	0.492
	56	0.251	0.321	0.296	0.133	0.521	0.453
	40	0.116	0.178	0.189	0.037	0.236	0.136
	24	0.027	0.045	0.070	0.040	0.231	0.131
	8	0.060	0.042	0.021	0.085	0.067	0.074

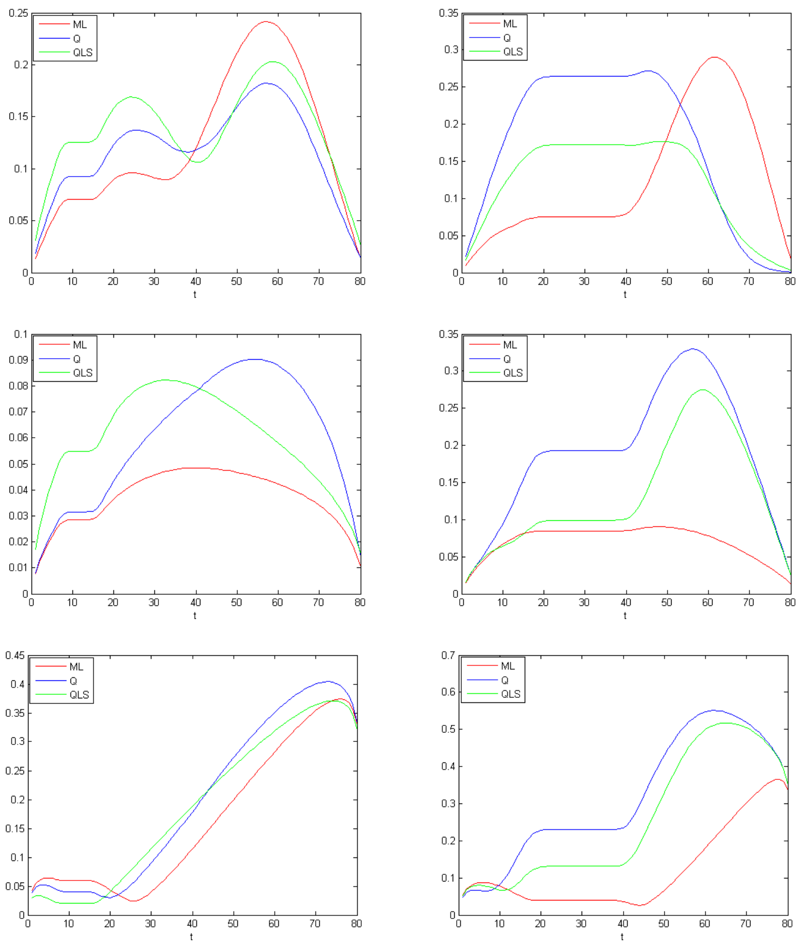

Figure 3 – RMSE($V(\hat{S}(t))$) versus t by using ML, Q and QLS estimators for left censored data n=80, on the left side figures for 10% CR, $\beta = 0.5, 1, 2$, on the right side figures for 30% CR, $\beta = 0.5, 1, 2$, respectively.

TABLE 5
RMSE($V(\hat{S}(t))$) *by using ML, Q and QLS estimators for interval censored data.*

				n=20			
Censoring rate		10%			30%		
β	t	ML	Q	QLS	ML	Q	QLS
0.5	18	0.132	0.135	0.150	0.245	0.184	0.207
	14	0.260	0.239	0.248	0.326	0.245	0.257
	10	0.220	0.223	0.202	0.273	0.236	0.218
	6	0.130	0.143	0.155	0.160	0.172	0.171
	2	0.062	0.075	0.092	0.069	0.100	0.115
1	18	0.063	0.088	0.078	0.103	0.121	0.078
	14	0.090	0.113	0.091	0.135	0.128	0.099
	10	0.101	0.108	0.093	0.151	0.119	0.1018
	6	0.091	0.089	0.087	0.138	0.113	0.114
	2	0.056	0.055	0.057	0.095	0.081	0.115
2	18	0.344	0.360	0.334	0.265	0.358	0.324
	14	0.240	0.283	0.254	0.177	0.293	0.262
	10	0.118	0.150	0.156	0.062	0.146	0.145
	6	0.048	0.067	0.066	0.078	0.087	0.078
	2	0.075	0.064	0.050	0.104	0.078	0.068
				n=80			
Censoring rate		10%			30%		
β	t	ML	Q	QLS	ML	Q	QLS
0.5	72	0.134	0.112	0.138	0.233	0.160	0.209
	56	0.275	0.236	0.256	0.344	0.241	0.273
	40	0.178	0.160	0.149	0.245	0.169	0.158
	24	0.086	0.105	0.124	0.079	0.108	0.128
	8	0.068	0.077	0.100	0.045	0.095	0.133
1	72	0.040	0.045	0.046	0.0850	0.063	0.074
	56	0.058	0.060	0.049	0.117	0.070	0.065
	40	0.0601	0.058	0.048	0.124	0.066	0.055
	24	0.049	0.048	0.050	0.103	0.058	0.060
	8	0.028	0.029	0.036	0.066	0.043	0.057
2	72	0.337	0.367	0.335	0.273	0.355	0.300
	56	0.244	0.290	0.264	0.172	0.284	0.244
	40	0.121	0.157	0.161	0.050	0.156	0.158
	24	0.024	0.044	0.057	0.056	0.059	0.079
	8	0.062	0.054	0.033	0.092	0.052	0.036

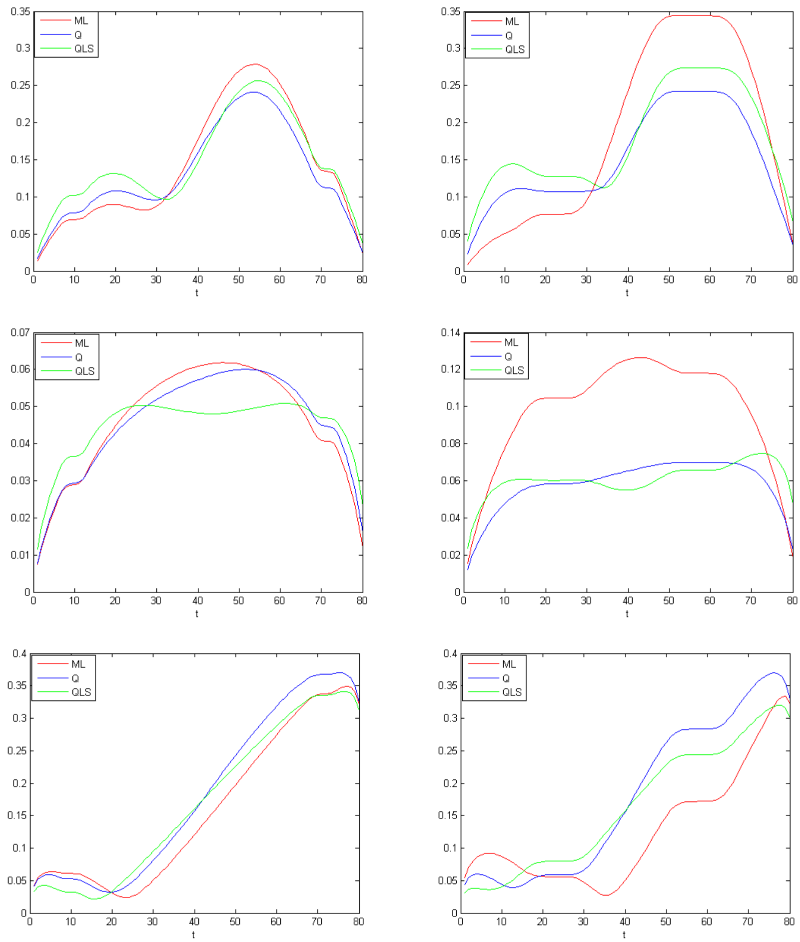

Figure 4 – RMSE($V(\hat{S}(t))$ versus t by using ML, Q and QLS estimators for interval censored data n=80, on the left side figures for 10% CR, $\beta = 0.5, 1, 2$, on the right side figures for 30% CR, $\beta = 0.5, 1, 2$, respectively.

The main findings from these tables for n=20 are summarized below.

- For right censored data and 10% CR:
  - ML is best estimator at the first points of time, then Q is the best estimator for decreasing failure rates,
  - QLS is best estimator at the first points of time, then ML is the best estimator for constant failure rates,
  - ML is best estimator at the first points of time, then QLS is the best estimator for increasing failure rates.
- For right censored data and 30% CR:
  - ML is best estimator at the first points of time, then QLS is the best estimator for decreasing failure rates,
  - ML is best estimator at the first points of time, then QLS is the best estimator for constant failure rates,
  - ML is best estimator at the first points of time, then Q is the best estimator for increasing failure rates.
- For left censored data and 10% CR:
  - ML is best estimator at the first points of time, then Q is the best estimator for decreasing failure rates,
  - ML is best estimator for constant and increasing failure rates.
- For left censored data and 30% CR:
  - ML is best estimator for increasing failure rates.
- For interval censored data and 10% CR:
  - ML is best estimator at the first points of time, then Q is the best estimator for decreasing failure rates,
  - ML is best estimator for constant and increasing failure rates.
- For interval censored data and 10% CR:
  - Q is best estimator at the first points of time, then ML is the best estimator for constant failure rates,
  - ML is best estimator for constant and increasing failure rates.
- For interval censored data and 30% CR:
  - Q is best estimator at the first points of time, then QLS is the best estimator for constant failure rates,

  - QLS is best estimator at the first points of time, then ML is the best estimator for increasing failure rates.

- Considering upper percentile cases, we can see that the RMSE increases while the upper percentile increases, except the case of $\beta < 1$, $Up = 0.9$.

- When the shape parameter of Weibull distribution increases, the Weibull distribution converges to normal distribution. As seen from the Table 3 to 5, the results of RMSE for $\beta = 2$ is obviously larger than the others. Therefore, as the skewness decreases the RMSE of survival function increases.

Tables 3-5 give the results for n=80 for different scenarios. Figures 2-4 allow us to see the results of the estimations of survival function at different time points visually for right, left and interval censored data with n=80, respectively.

- For 10% CR, Q is better for decreasing and constant failure rates and ML is similar with QLS for increasing failure rates. For 30% CR, ML is best estimator at the first points of time, then QLS is the best estimator for decreasing and constant failure rates and Q is the best estimator for increasing failure rates.

- For 10% CR, ML is best estimator at the first points of time, then Q is the best for decreasing failure rates and ML is the best estimator for constant failure rates. QLS is best estimator at the first points of time, then ML is the best for increasing failure rates. For 10% CR, ML is best estimator at the first points of time, then Q is the best for decreasing failure rates and ML is the best estimator for constant and increasing failure rates.

- The results are similar with left censored data except the case of 30% CR for increasing failure rates. For this case, Q is best estimator at the first points of time, then ML is the best estimator.

6. Conclusions

In this paper, we propose to use of new Weibull parameter estimators which are explicit and robust for censored data. We obtain survival function estimators and their variances based on Q, QLS and ML methods. We consider Monte Carlo simulation study to obtain the RMSE of these estimators and a numerical example for comparing the variances of survival function estimators.

The real dataset example suggests that Q estimator has the smallest variance whereas ML has the largest variance. Then, to see the differences over sample sizes, censoring types and censoring rates, two simulation studies are conducted. In simulation studies, RMSE is used to compare between the methods of estimator over 10000 replications. Different censoring types right, left, interval are considered. The shape parameters are selected differently to show increasing, constant, and decreasing failure rates where the scale parameter is set to 1.

In simulation 1, sample size is chosen as 100 and censoring rate is chosen as 33%. The bias and the variability of the shape parameter estimations decreases whereas the bias and the variability of the scale parameter increases as skewness increases.

In simulation 2, we choose the samples with size n = 20 (small) and 80 (large) with varieties of shape parameter values namely 0.5, 1, 2 corresponding to decreasing, constant, and increasing failure rates, respectively.

We can conclude that for right data set with n=20, generally ML is prepared to robust methods at the first time points whereas robust methods are preferred to ML later in time points. For left censored data with n=20, ML is the best estimator. For interval censored data with n=20, ML is preferred to robust estimators for decreasing hazard rates for 10%. For right censored data with n=80, ML is best estimator at the first points of time, then robust estimators. Q is the best estimator for decreasing and constant failure rates for 10% CR. For left censored data with n=80, ML is generally the best estimator. For interval censored data with n=80, ML is the best estimator for constant failure rates. For decreasing failure rates, ML is prepared to robust methods at the first time points whereas robust methods are preferred to ML later in time points and just the opposite for increasing failure rates. Consequently, for the estimation of survival function based on Weibull distribution, the robust estimators can be alternative to ML for right censored data, The most common estimator, ML, maintains its superiority for left censored data set. For interval censored data set, no generalization can be derived. For future work, extensive simulations can be done for large samples and censoring rates.

Acknowledgements

We are grateful for the contribution, comments and suggestions made by the anonymous referees, which improved the presentation and content of this paper. We also thank the editor of the journal for his time and attention.

References

R. Abernethy, J. Breneman, C. Medlin, G. Reinman (1983). Tech. rep., Air Force Wright Aeronautical Laboratories, Washington, D.C.

S. Almalki, S. Nadarajah (2014). *Modifications of the Weibull distribution: A review*. Reliability Engineering and System Safety, 124, pp. 32–55.

L. J. Bain (1976). *Statistical Analysis of Reliability and Life Testing Model*. Marceland Dekker Inc., New York.

A. Bellavia, M. Bottai, N. Orsini (2016). *Evaluating additive interaction using survival percentiles*. Epidemiology, 27, no. 3, p. 360–364.

S. Borovkova (2002). *Analysis of survival data*. Nieuw Archief voor Wiskunde, 3, pp. 302–307.

K. Boudt, D. Caliskan, C. Croux (2011). *Robust explicit estimators of Weibull parameters*. Metrika, 73, pp. 187–209.

D. Collett (2003). *Modelling Survival Data in Medical Research*. Chapman&Hall, New York.

R. Gupta, D. Kundu (2001). *Exponentiated exponential family: An alternative to Gamma and Weibull distributions*. Biometrical Journal, 43, no. 1, pp. 117–130.

X. He, W. Fung (1999). *Method of medians for lifetime data with Weibull models*. Statistics in Medicine, 18, pp. 1993–2009.

E. Lee, J. Wang (2003). *Statistical Methods for Survival Data Analysis*. John Wiley&Sons, New York.

N. Orsini, A. Wolk, M. Bottai (2012). *Evaluating percentiles of survival*. Epidemiology, 23, p. 770–771.

S. Selvin (2005). *Survival Analysis for Epidemiologic and Medical Research Analysis of Epidemiologic Data*. Cambridge University Press, New York.

D. Shier, K. Lawrence (1984). *A comparison of robust regression techniques for the estimation of Weibull parameters*. Communications in Statistics - Simulation and Computation, 13, pp. 743–750.

Summary

The aim of this study is to estimate the robust survival function for the Weibull distribution. Since the survival function of Weibull distribution is based on the parameters, we consider two robust and explicit Weibull parameter estimators proposed by Boudt *et al.* (2011). The quantile and the quantile least squares which are all robust to censored data is used as an alternative to the maximum likelihood estimation of the Weibull parameters. The proposed estimators are applied to Hodgin's disease data which produces smaller variances for the robust survival function. The advantage of new methods is that they are numerically explicit in applications. Monte Carlo simulation is performed to compare the behaviours of the proposed robust estimators in the presence of right, left and interval censored observations considering different censoring rates. The simulation results show that the proposed robust estimators are better than the maximum likelihood estimator.

Keywords: Quantile estimators; Quantile least squares estimators; Survival function; Weibull distribution.