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ESTIMATION OF THE PARAMETERS OF POWER FUNCTION DISTRIBUTION BASED ON PROGRESSIVE TYPE-II RIGHT CENSORING WITH BINOMIAL REMOVAL

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SUMMARY

In this article, we propose estimates of the unknown parameters of the power function distribution in the context of progressive type-II censoring with binomial removals, where the number of units removed at each failure time follows a binomial distribution. The maximum likelihood estimators (MLEs) for the power function parameters are derived using the expectation-maximization (EM) algorithm. The EM-algorithm is also used to obtain the asymptotic variance-covariance matrix. By using the variance-covariance matrix of the MLEs, the asymptotic 95% confidence intervals for the parameters are constructed. Bayes estimators under different loss functions are obtained using the Lindley approximation method and the importance sampling procedure. We also introduce one and two sample prediction estimates and corresponding confidence intervals by using Bayesian techniques. To compare performance of the proposed estimators, we introduce simulation and real-life data studies.

Keywords: Power function distribution; Maximum likelihood estimation; Lindley approximation; Importance sampling procedure; Prediction.

1. INTRODUCTION

The Power Function distribution (PFD) is also known as a flexible lifetime distribution, and it provides a good fit to some sets of failure data. It has applications in several scientific areas, including finance, economics, reliability, etc. In addition to this, the PFD has received considerable attention in the literature as it can also be applied to model the reliability growth of complex systems and repairable systems. Meniconi and Barry (1996) have observed that reliability and hazard function plots of some data set suggest

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that the PFD model is the most suitable model compared to Exponential, Lognormal and Weibull models. This paper aims at developing Bayes estimators for the parameters of the PFD when the sample data available is a progressive type-II right censored sample. We will create point and interval estimators for the parameters of PFD. The probability density function (pdf) of the PFD is defined as

$$f(y,\beta,\alpha)=\frac{\alpha}{\beta}\left(\frac{\beta}{y}\right)^{-(\alpha-1)}, \quad 0<y<\beta, \quad \alpha>0,\ \beta>0. \tag{1}$$

The cumulative distribution function (cdf) of the PFD is defined as

$$F(y,\beta,\alpha)=\left(\frac{y}{\beta}\right)^{\alpha}, \quad \alpha>0,\ \beta>0, \tag{2}$$

where α and β are the shape and the scale parameter, respectively.

Censoring is an important concept used in reliability and life-testing experiments. When we face experimental difficulties and other restrictions during data collection in life testing experiments, we cannot precisely identify the survival period of all the experimental units. Researchers in medicine and industry use censored data frequently because they rarely have the time to follow all of the patients in the study throughout their lives. Furthermore, subjects/items may fail for reasons unrelated to the ones being studied. Among different censoring schemes, progressive type-II censoring is generally used in reliability and survival analysis. One can refer to Balakrishnan and Aggarwala (2000) and Balakrishnan (2007) for a detailed discussion of progressive censoring and its applications.

Progressive censoring can be formulated as follows. Assume that $Y_1,\ldots,Y_n$ are n random lifetimes that are independent and identically distributed (iid). We put n items on test, and the test is terminated at the time of the m^{th} failure. At the time of first failure, r_1 surviving units are randomly selected and removed from the experiment. At the second failure, r_2 of the remaining $n-r_1-1$ units are randomly selected and removed from the experiment. Finally, the remaining units $r_m=n-m-r_1-r_2-\ldots-r_{m-1}$ are removed at the m^{th} failure. In a clinical trial, Yuen and Tse (1996) pointed out that the number of patients that drop out at each stage is random and cannot be prefixed. In such cases, the pattern of removal at each failure is random. For studies related to estimation using progressive censoring with binomial removals or random removals, one may refer to Yan *et al.* (2011), Tse *et al.* (2000), Wu and Chang (2002), Wu and Chang (2003), Sarhan and Abuammoh (2008) and Hashemi and Amiri (2011). In the present work, we focus on estimating the parameters of the PFD when both the parameters are unknown using progressive censored samples with binomial removals.

The rest of the paper is arranged as follows. In Sections 2 and 3, we introduce the problem and derive the MLE of the unknown parameters. In Section 4, the Bayes estimators of the parameters are obtained. The problem of one and two sample predictions is discussed in Section 5. A simulation study is carried out in Section 6, and a real-life data study is carried out in Section 7. A brief conclusion is given in Section 8.

2. Estimation under progressive censoring with binomial removal

Let $Y = (y_{1:m:n}, y_{2:m:n}, \ldots, y_{m:m:n})$ denote a progressive type-II censored sample of size m drawn from a parent sample of size n. With censoring scheme $R_1 = r_1$, $R_2 = r_2$, $\ldots$, $R_m = r_m$, the conditional likelihood takes the following form:

$$L_1(\alpha, \beta, y|R = r) = B \prod_{i=1}^{m} g(y_i)[1 - G(y_i)]^{r_i}, \tag{3}$$

where $B = n(n-1-r_1)(n-2-r_1-r_2)\ldots(n-m+1-r_1-\ldots-r_m)$, $g(y_i)$ and $G(y_i)$ represent the pdf and the cdf of the population from which the sample is drawn, respectively. Also, y_i is used instead of $y_{i:m:n}$ to simplify notation. Using Eq. (1) and (2) in Eq. (3) the conditional likelihood simplifies to

$$L_1(\alpha, \beta, y|r) \propto \left(\frac{\alpha}{\beta^\alpha}\right)^m \exp\left[(\alpha - 1)\sum_{i=1}^{m} \log(y_i) + \sum_{i=1}^{m} r_i \log\left[1 - \left(\frac{y_i}{\beta}\right)^\alpha\right]\right]. \tag{4}$$

The scheme $R_1, R_2, \ldots, R_m$ is pre-fixed in typical progressive type-II censoring. However, in some practical situations, these numbers may occur at random. For example, in some reliability trials, an experimenter may decide that continuing to test any of the tested units, even though they have not failed, is unsuitable or too unsafe. In such cases, the pattern of removal at each failure is random. This leads to progressive censoring with random removals. In this paper, we assume that the random removal R_i follows a binomial distribution with parameter p. It means that each unit leaves with equal probability p, and the probability of R_i units leaving after the i^{th} failure is

$$P(r_1) = \binom{n-m}{r_1} p^{r_1}(1-p)^{n-m-r_1} \quad 0 \le r_1 \le n, \tag{5}$$

and

$$P(r_i|r_{i-1}, \ldots, r_1) = \binom{n - m - \sum_{k=1}^{i-1} r_k}{r_i} p^{r_i}(1-p)^{n-m-\sum_{k=1}^{i-1} r_k}, \tag{6}$$

where $0 \le r_i \le n - m - \sum_{k=1}^{i-1} r_k$; $i = 2, 3, \ldots, m-1$. Furthermore, we assume that R_i is independent of Y_i for all i. A schematic illustration of progressive type-II censoring with binomial removals is given in Table 1.

TABLE 1

Schematic illustration of progressive type-II censoring with binomial removals.

Process	The number in life testing	Failures	Binomial removals	Remains
1	n	1	$R_1 \sim B(n-m, p)$	$n-1-R_1$
2	$n-1-R_1$	1	$R_2 \sim B(n-m-R_1, p)$	$n-2-R_1-R_2$
…	…	…	…	…
$m-1$	$n-(m-2)-\sum_{k=1}^{m-2} R_k$	1	$R_{m-1} \sim B(n-(m-2)-\sum_{k=1}^{m-2} R_k, p)$	$n-(m-1)-\sum_{k=1}^{m-1} R_k$
m	$n-(m-1)-\sum_{k=1}^{m-1} R_k$	1	$R_m = n-m-\sum_{k=1}^{m-1} R_k$	0

The joint likelihood function of $Y=(Y_1,Y_2,\dots,Y_m)$ and $R=(R_1,R_2,\dots,R_m)$ can be expressed as

$$L(\alpha,\beta,p;y,r)=L(\alpha,\beta,y|R=r)P(R=r), \tag{7}$$

where

$$\begin{aligned} P(R=r) &= P(R_{m-1}=r_{m-1}|R_{m-2}=r_{m-2},\dots,R_1=r_1) \\ &\dots\ P(R_2=r_2|R_1=r_1)P(R_1=r_1). \end{aligned} \tag{8}$$

Substituting Eq. (5) and (6) in Eq. (8), we get

$$P(R=r)=\frac{(n-m)!}{\prod_{i=1}^{m-1} r_i!(n-m-\sum_{i=1}^{m-1} r_i)!}\ p^{\sum_{i=1}^{m-1} r_i}\ (1-p)^{(m-1)(n-m)-\sum_{i=1}^{m-1}(m-i)r_i}. \tag{9}$$

Now using Eq. (4), (7) and (9), the full likelihood function can be written as

$$L(\alpha,\beta,p)=WL_1(\alpha,\beta)L_2(p),$$

where $W=\frac{W^*\,(n-m)!}{\prod_{i=1}^{m-1} r_i!(n-m-\sum_{i=1}^{m-1} r_i)!}$, and $W^*=2^m B$, which is independent of the parameters α, β, and p. Further, we have

$$L_1(\alpha,\beta)\propto\left(\frac{\alpha}{\beta^\alpha}\right)^m\times\exp\left[(\alpha-1)\sum_{i=1}^m\log(y_i)+\sum_{i=1}^m r_i\log\left[1-\left(\frac{y_i}{\beta}\right)^\alpha\right]\right] \tag{10}$$

and

$$L_2(p)\propto p^{\sum_{i=1}^{m-1} r_i}\ (1-p)^{(m-1)(n-m)-\sum_{i=1}^{m-1}(m-i)r_i}. \tag{11}$$

In the next section, we use the MLE method to estimate the unknown parameters.

3. Maximum likelihood estimation

In this Section, we discuss the process of obtaining the MLEs of the parameters α, β, and p based on progressive type-II censored data with binomial removals. Both point and interval estimators are derived. From Eq. (10), we can observe that L_1 does not involve the unknown parameter p. Therefore, the MLE of α and β can be derived by maximizing Eq. (10) directly, and the log of L_1 can be written as

$$\log L_1(\alpha,\beta)=m\log\left(\frac{\alpha}{\beta^\alpha}\right)+\left[(\alpha-1)\sum_{i=1}^m\log(y_i)+\sum_{i=1}^m r_i\log\left[1-\left(\frac{y_i}{\beta}\right)^\alpha\right]\right]. \tag{12}$$

To obtain the normal equation for the unknown parameter α, we differentiate Eq. (12) partially with respect to the parameter α and then equate to zero. The resulting normal equation of α will be of the following form:

$$\frac{\partial\log[L_1(\alpha,\beta)]}{\partial\alpha}=\frac{m}{\alpha}-m\log[\beta]+\sum_{i=1}^m\log[y_i]+\sum_{i=1}^m r_i\frac{\log[\frac{y_i}{\beta}]\left(\frac{y_i}{\beta}\right)^\alpha}{\left[1-\left(\frac{y_i}{\beta}\right)^\alpha\right]}=0. \tag{13}$$

The MLE of β can be obtained as

$$\hat{\beta}_{\mathrm{mle}} = Y_{(m)}.$$

Similarly, since L_2 does not involve α and β, the MLE of p can be derived by maximizing Eq. (11). The MLE of binomial parameter p is of the following form:

$$\hat{p}_{mle} = \frac{\sum_{i=1}^{m-1} r_i}{(m-1)(n-m) - \sum_{i=1}^{m-1}(m-i-1)r_i}.$$

Since Eq. (13) has no analytical solution, we have to use the expectation-maximization (EM) algorithm to derive the MLE of α. This is discussed in the next Section.

3.1. *EM algorithm*

The EM algorithm is a beneficial iterative method for computing MLEs in estimation problems involving missing information such as censored samples. It was first introduced by Dempster *et al.* (1977) and further explored by Ng *et al.* (2002), who obtained several applications of this algorithm in life testing experiments. In this section, we derive the MLE of α on the basis of progressive censored samples. Now, suppose that $Y = (y_{1:m:n}, y_{2:m:n}, \ldots, y_{m:m:n})$ denotes the observed data, and that $Q = (q_1, q_2, \ldots, q_m)$ denotes the censored data. Here, Q_j represents $1 \times r_j$ vector with $Q_j = (q_{j1}, q_{j2}, \ldots, q_{jm})$, $j = 1, 2, \ldots, m$. The complete data set is of the form $(Y, Q) = X$. The corresponding log of $L_1(\alpha, \beta)$ is given by

$$\log[L_1(\alpha,\beta)] = n\log[\alpha] - \alpha n \log[\beta] + (\alpha-1)\left(\sum_{i=1}^{m}\log[y_i] + \sum_{i=1}^{m}\sum_{j=1}^{r_i}\log[q_{ij}]\right).$$

In the EM algorithm, the E-step involves the computation of conditional expectations of unobserved data given the observed data. This gives the following expression for the log-likelihood function:

$$\log L_1^*(Y,\alpha,\beta) \propto n\log[\alpha] - \alpha n\log[\beta] + (\alpha-1)\left(\sum_{i=1}^{m}\log[y_i] + \sum_{i=1}^{m} r_i A(y_i, \alpha_{(j)}, \beta_{(j)})\right), \tag{14}$$

where $A(y_i, \alpha_{(j)}, \beta_{(j)}) = E[\log[q_{ij}] | q_{ij} > y_i] = \left(\frac{-q_i}{\beta}\right)^{\alpha} \frac{\log[q_{ij}]}{\left[1-\left(\frac{q_i}{\beta}\right)^{\alpha}\right]}$. The second step of the EM algorithm is the M-step, which involves maximization of Eq. (14) with respect to α, and hence the derived estimate of α is given by

$$\hat{\alpha}_{mle} = \frac{n}{n\log[\beta] + D}, \tag{15}$$

where

$$D=\sum_{i=1}^{m}[y_i+r_iA(y_i,\alpha_{(j)},\beta_{(j)})].$$

Using the updated estimate of α from Eq. (15), we can find the initial estimate of α.

3.2. Fisher information matrix

Based on the asymptotic distribution of the MLE of the parameters α, β, and p, we obtain appropriate confidence intervals for the parameters in this section. The Fisher information matrix elements for the parameters based on progressive censored samples are formally derived. The Fisher information matrix can be defined as

$$I=-E\begin{bmatrix} \frac{\partial^2 lnL}{\partial\alpha^2} & \frac{\partial^2 lnL}{\partial\alpha\partial\beta} & \frac{\partial^2 lnL}{\partial\alpha\partial p} \\ -\frac{\partial^2 lnL}{\partial\beta\partial\alpha} & \frac{\partial^2 lnL}{\partial\beta^2} & \frac{\partial^2 lnL}{\partial\beta\partial p} \\ -\frac{\partial^2 lnL}{\partial p\partial\alpha} & \frac{\partial^2 lnL}{\partial p\partial\beta} & \frac{\partial^2 lnL}{\partial p^2} \end{bmatrix}.$$

Unfortunately, exact mathematical expressions for the elements of the above matrix are difficult to obtain. As a result, we give the approximate asymptotic distribution of the MLE of the parameters α, β, and p, which is obtained by dropping the expectation operator E, and hence I can be written as

$$\begin{aligned} I &= -\begin{bmatrix} \frac{\partial^2 lnL}{\partial\alpha^2} & \frac{\partial^2 lnL}{\partial\alpha\partial\beta} & \frac{\partial^2 lnL}{\partial\alpha\partial p} \\ \frac{\partial^2 lnL}{\partial\beta\partial\alpha} & \frac{\partial^2 lnL}{\partial\beta^2} & \frac{\partial^2 lnL}{\partial\beta\partial p} \\ \frac{\partial^2 lnL}{\partial p\partial\alpha} & \frac{\partial^2 lnL}{\partial p\partial\beta} & \frac{\partial^2 lnL}{\partial p^2} \end{bmatrix} \\ &= \begin{bmatrix} L_{\alpha\alpha} & L_{\alpha\beta} & L_{\alpha p} \\ L_{\beta\alpha} & L_{\beta\beta} & L_{\beta p} \\ L_{p\alpha} & L_{p\beta} & L_{pp} \end{bmatrix}, \end{aligned} \tag{16}$$

where

$$L_{\alpha\alpha}=-\frac{n}{\alpha^2},\quad L_{\alpha\beta}=L_{\beta\alpha}=-\frac{n}{\beta},\quad L_{\beta\beta}=\frac{n\alpha}{\beta^2},$$

$$L_{pp}=\frac{\sum_{i=1}^{m-1}r_i}{p^2}+\frac{(m-1)(n-m)\sum_{i=1}^{m-1}(m-i)r_i}{(1-p)^2},$$

and

$$L_{\alpha p}=L_{\beta p}=L_{p\alpha}=L_{p\beta}=0.$$

The variance-covariance matrix can be approximated as

$$I=\begin{pmatrix} T_{\alpha\alpha} & T_{\alpha\beta} & 0 \\ T_{\beta\alpha} & T_{\beta\beta} & 0 \\ 0 & 0 & T_{pp} \end{pmatrix}=\begin{pmatrix} L_{\alpha\alpha} & L_{\alpha\beta} & 0 \\ L_{\beta\alpha} & L_{\beta\beta} & 0 \\ 0 & 0 & L_{pp} \end{pmatrix}^{-1}.$$

Thus, the asymptotic distribution of the MLE of $(\hat{\alpha}, \hat{\beta}, \hat{p})$ is given by

$$I = \begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \\ \hat{p} \end{pmatrix} \approx N \left[\begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \\ \hat{p} \end{pmatrix}, \begin{pmatrix} T_{\alpha\alpha} & T_{\alpha\beta} & 0 \\ T_{\beta\alpha} & T_{\beta\beta} & 0 \\ 0 & 0 & T_{pp} \end{pmatrix} \right].$$

Hence, the asymptotic distribution of the MLE can be written as

$$\left[(\hat{\alpha} - \alpha), (\hat{\beta} - \beta), (\hat{p} - p) \right] \sim N_3(0, T), \tag{17}$$

where T is the variance-covariance matrix. Using the MLE estimators of α, β, and p, we can derive an estimate of T, which is denoted by $\hat{T}$, because T involves the parameters α, β, and p. Utilizing Eq. (17), approximate $100(1-\zeta)\%$ confidence intervals for α, β, and p are obtained, respectively, as

$$\hat{\alpha} \pm z_{\zeta/2} \sqrt{\hat{T}_{\alpha\alpha}}, \; \hat{\beta} \pm z_{\zeta/2} \sqrt{\hat{T}_{\beta\beta}} \; \textit{and} \; \hat{p} \pm z_{\zeta/2} \sqrt{\hat{T}_{pp}},$$

where $z_{\zeta/2}$ is the upper censored $(\zeta/2)^{\text{th}}$ percentile of the standard normal distribution.

4. Bayesian estimation

In this section, we provide the Bayes estimates of the unknown parameters of the PFD using different loss functions. We consider symmetric as well as asymmetric loss functions for our estimation purposes. One of the symmetric loss functions we consider here is the squared error loss function (SELF), which is defined as $L(\eta, \delta(Y)) = [\delta(Y) - \eta]^2$. Varian (1975) proposed an asymmetric linear-exponential loss function known as Linex loss function (LLF), which is defined as $L(\eta, \delta(Y)) = \exp[h(\delta - \eta)] - h(\delta - \eta) - 1$, $h \neq 0$, where p is the shape parameter known as the degree of asymmetry. Also, another asymmetric loss function is the Entropy loss function (ELF) proposed by Calabria and Pulcini (1996). It is defined as $L[\delta, \eta] \propto \left[\frac{\eta}{\delta}\right]^p - p\left[\log\left(\frac{\eta}{\delta}\right)\right] - 1$, $p \neq 0$. Under the SELF, LLF, and ELF, Bayes estimators of η are defined, respectively, as

$$\delta(Y)_{\text{self}} = E(\eta | Y),$$

$$\delta(Y)_{\text{llf}} = -\frac{1}{h} \log(E(e^{(-h\eta)} \mid Y))$$

and

$$\delta(Y)_{\text{elf}} = [E(\eta^{-c}) \mid Y]^{-1/c}.$$

In the Bayesian approach, unknown parameters are considered as random variables that follow some specified distribution, and this distribution is known as the prior distribution. It may be noted that if all the parameters α, β, and p are unknown, joint conjugate priors do not exist. In such cases, there are several ways to choose the priors. One way is to consider the piecewise independent priors. In this article, we assume that α and β have independent gamma priors and p has a beta prior. The priors are formulated as

$$F_1(\alpha) \propto \alpha^{a_1-1} e^{-b_1\alpha}, \quad \alpha > 0, \quad a_1, b_1 > 0,$$
$$F_2(\beta) \propto \beta^{a_2-1} e^{-b_2\beta}, \quad \beta > 0, \quad a_2, b_2 > 0,$$

and

$$F_3(p) \propto p^{a_3-1}(1-p)^{b_3-1}, \quad 0 < p < 1, \quad a_3, b_3 > 0.$$

Then the joint prior distribution of α, β and p is given by

$$F^*(\alpha, \beta, p) \propto \alpha^{a_1-1} \beta^{a_2-1} e^{-(b_1\alpha + b_2\beta)} p^{a_3-1}(1-p)^{b_3-1}.$$

Hence, the joint posterior distribution of α, β, and p is obtained as

$$\Pi(\alpha, \beta, p) \propto \alpha^{\mu_1-1} e^{-\mu_4\alpha} \beta^{\mu_2-1} e^{-\mu_5\beta} \left[1 - \left(\frac{y_i}{\beta}\right)^{\alpha}\right]^{r_i} e^{-S} p^{\mu_3-1}(1-p)^{\mu_6-1}, \tag{18}$$

where $\mu_1 = m + a_1$, $\mu_2 = a_2 - m\alpha$, $\mu_3 = a_3 + \sum_{i=1}^{m-1} r_i$, $\mu_4 = b_1 - S$, $\mu_5 = b_2$, $\mu_6 = (m-1)(n-m) + b_3 - \sum_{i=1}^{m-1}(m-i)r_i$ and $S = \sum_{i=1}^{m} \log[y_i]$.

The Bayes estimator of $\eta = (\alpha, \beta, p)$ under the SELF, LLF, and ELF are the posterior expectations of η. They are defined, respectively, as

$$\hat{\eta}_{\text{self}} = \left[\frac{\int_0^1\int_0^\infty\int_0^\infty \eta\, \Pi(\alpha, \beta, p) \mathrm{d}\alpha\, d\beta\, \mathrm{d}p}{\int_0^1\int_0^\infty\int_0^\infty \Pi(\alpha, \beta, p) \mathrm{d}\alpha\, \mathrm{d}\beta\, \mathrm{d}p}\right], \tag{19}$$

$$\hat{\eta}_{\text{elf}} = \left[\frac{\int_0^1\int_0^\infty\int_0^\infty \eta^{-c}\, \Pi(\alpha, \beta, p) \mathrm{d}\alpha\, d\beta\, \mathrm{d}p}{\int_0^1\int_0^\infty\int_0^\infty \Pi(\alpha, \beta, p) \mathrm{d}\alpha\, \mathrm{d}\beta\, \mathrm{d}p}\right]^{-\frac{1}{c}}, \tag{20}$$

and

$$\hat{\eta}_{\text{llf}} = -\frac{1}{h} log \left[\frac{\int_0^1\int_0^\infty\int_0^\infty e^{-h\eta}\, \Pi(\alpha, \beta, p) \mathrm{d}\alpha\, \mathrm{d}\beta\, \mathrm{d}p}{\int_0^1\int_0^\infty\int_0^\infty \Pi(\alpha, \beta, p) \mathrm{d}\alpha\, \mathrm{d}\beta\, \mathrm{d}p}\right]. \tag{21}$$

Equations (19), (20), and (21) cannot be computed analytically. From the various existing methods to approximate the ratio of integrals of the above form, here we use two approximation methods, namely Lindley approximation and importance sampling method, for obtaining Bayes estimates of α, β, and p. This is discussed in the following sections.

4.1. Lindley approximation

In this section, we discussed the approximation method proposed by Lindley (1980). First we consider the function $I(y)$, which is defined as

$$\begin{aligned}
I(y) &= E[\Psi(\alpha,\beta,p)] = \frac{\int \Psi(\alpha,\beta,p)e^{L(\alpha,\beta,p)+F^*(\alpha,\beta,p)}\mathrm{d}(\alpha,\beta,p)}{\int e^{L(\alpha,\beta,p)+F^*(\alpha,\beta,p)}\mathrm{d}(\alpha,\beta,p)} \\
&\approx \Psi(\alpha,\beta,p)+(U_1\nu_1+U_2\nu_2+U_3\nu_3+\nu_4+\nu_5)+\frac{1}{2}[\Delta_1(U_1\delta_{11} \\
&+ U_2\delta_{12}U_3\delta_{13})+\Delta_2(U_1\delta_{21}+U_2\delta_{22}+U_3\delta_{23}) \\
&+ \Delta_3(U_1\delta_{31}U_2\delta_{32}+U_3\delta_{33})], \qquad (22)
\end{aligned}$$

where $\Psi(\alpha,\beta,p)$ is a function of α, β, and p, $L(\alpha,\beta,p)$ is the log likelihood, and $F^*(\alpha,\beta,p)$ is the log joint prior. Here we denote

$$\Delta_1 = \delta_{11}L^*_{\alpha\alpha\alpha}+2\delta_{12}L^*_{\alpha\beta\alpha}+2\delta_{13}L^*_{\alpha p\alpha}+2\delta_{23}L^*_{\beta p\alpha}+\delta_{22}L^*_{\beta\beta\alpha}+\delta_{33}L^*_{pp\alpha},$$

$$\Delta_2 = \delta_{11}L^*_{\alpha\alpha\beta}+2\delta_{12}L^*_{\alpha\beta\beta}+2\delta_{13}L^*_{\alpha p\beta}+2\delta_{23}L^*_{\beta p\beta}+\delta_{22}L^*_{\beta\beta\beta}+\delta_{33}L^*_{pp\beta},$$

$$\Delta_3 = \delta_{11}L^*_{\alpha\alpha p}+2\delta_{12}L^*_{\alpha\beta p}+2\delta_{13}L^*_{\alpha pp}+2\delta_{23}L^*_{\beta pp}+\delta_{22}L^*_{\beta\beta p}+\delta_{33}L^*_{ppp},$$

and

$$\nu_i = \varepsilon_1\delta_{i1}+\varepsilon_2\delta_{i2}+\varepsilon_3\delta_{i3}, \quad i=1,2,3,$$

$$\nu_4 = u_{12}\delta_{11}+u_{13}\delta_{13}+u_{23}\delta_{23} \quad \text{and} \quad \nu_5=\frac{1}{2}(u_{11}\delta_{11}+u_{22}\delta_{22}+u_{33}\delta_{33}).$$

Let $\phi_1=\alpha$, $\phi_2=\beta$, and $\phi_3=p$, $\varepsilon_i=\left[\frac{\partial\varepsilon}{\partial\phi_i}\right]$, $i=1,2,3$, $U_i=\left[\frac{\partial\Psi(\phi_1,\phi_2,\phi_3)}{\partial\phi_i}\right]$, $i=1,2,3$, $U_{ij}=\left[\frac{\partial\Psi^2(\phi_1,\phi_2,\phi_3)}{\partial\phi_i\partial\phi_j}\right]$, $i,j=1,2,3$, $L^*_{ij}=\left[\frac{\partial^2L^*(\phi_1,\phi_2,\phi_3)}{\partial\phi_i\partial\phi_j}\right]$, $i,j=1,2,3$, $L^*_{ijk}=\left[\frac{\partial^3L^*(\phi_1,\phi_2,\phi_3)}{\partial\phi_i\partial\phi_j\partial\phi_k}\right]$, $i,j,k=1,2,3$, where $\alpha=1$, $\beta=2$ and $p=3$. Also, δ_{ij} is the $(i,j)^{\text{th}}$ element of the inverse of the matrix L^*_{ij}. Moreover, ε_i denotes the derivatives of the log of the prior with respect to ϕ_1, ϕ_2, ϕ_3. The values of L^*_{ijk} are derived as $L_{\alpha\alpha\alpha}=\left[\frac{\partial^3L}{\partial\alpha^3}\right]$, $L_{\beta\beta\beta}=\left[\frac{\partial^3L}{\partial\beta^3}\right]$, $L_{ppp}=\left[\frac{\partial^3L}{\partial p^3}\right]$, $L_{\beta pp}=\left[\frac{\partial^3L}{\partial\beta\partial p^2}\right]$, $L_{\beta\beta p}=\left[\frac{\partial^3L}{\partial\beta^2\partial p}\right]$, $L_{\alpha\alpha\beta}=\left[\frac{\partial^3L}{\partial\alpha^2\partial\beta}\right]$, $L_{\alpha p\beta}=\left[\frac{\partial^3L}{\partial\beta\partial p\partial\beta}\right]$, $L_{\alpha\alpha p}=\left[\frac{\partial^3L}{\partial\alpha^2\partial p}\right]$, $L_{\alpha\beta\beta}=\left[\frac{\partial^3L}{\partial\alpha\partial\beta^2}\right]$, $L_{\alpha pp}=\left[\frac{\partial^3L}{\partial\alpha\partial p^2}\right]$.

Hence, the Bayes estimators of α under the SELF, ELF, and LLF using the Lindley approximation can be approximated as

$$\begin{aligned}
\hat{\alpha}_{\text{self}} &\approx [\alpha+(u_1\nu_1+u_2\nu_2+u_3\nu_3+\nu_4+\nu_5)+\frac{1}{2}[\Delta_1(u_1\delta_{11}+u_2\delta_{12}) \\
&+ \Delta_2(u_1\delta_{21}+u_2\delta_{22})+\Delta_3(u_3\delta_{33})]], \qquad (23)
\end{aligned}$$

$$\begin{aligned}\hat{\alpha}_{\text{elf}} &\approx [\alpha^{-c} + (u_1 v_1 + u_2 v_2 + u_3 v_3 + v_4 + v_5) + \frac{1}{2}[\Delta_1(u_1\delta_{11} + u_2\delta_{12}) \\ &+ \Delta_2(u_1\delta_{21} + u_2\delta_{22}) + \Delta_3(u_3\delta_{33})]]^{-\frac{1}{c}}, \end{aligned} \tag{24}$$

and

$$\begin{aligned}\hat{\alpha}_{\text{llf}} &\approx -\frac{1}{h}\log[e^{-h\alpha} + (u_1 v_1 + u_2 v_2 + u_3 v_3 + v_4 + v_5) + \frac{1}{2}[\Delta_1(u_1\delta_{11} + u_2\delta_{12}) \\ &+ \Delta_2(u_1\delta_{21} + u_2\delta_{22}) + \Delta_3(u_3\delta_{33})]]. \end{aligned} \tag{25}$$

Similarly, retracing the same steps we can also derive the Bayes estimates of β and p.

4.2. *Importance sampling procedure*

The Bayes estimators of α, β and p are derived using the joint posterior distribution in Eq. (18). It can be rearranged as

$$\Pi(\alpha, \beta, p) \propto f(\alpha; \mu_1, \mu_4) f(\beta|\alpha; \mu_2, \mu_5) f(p; \mu_3, \mu_6)\, h(\alpha, \beta, p),$$

where

$$h(\alpha, \beta, p) = \frac{\Gamma(a_2 - m\alpha)\, e^{-\left[S - \sum_{i=1}^{m} r_i \log\left[1 - \left(\frac{y_i}{\beta}\right)^{\alpha}\right] + (\mu_3 - 1)\log(p) + (\mu_6 - 1)\log(1-p)\right]}}{\exp[-(a_2 - m\alpha)\log[b_2]]}. \tag{26}$$

The prior distributions for the parameters are defined as

$$f_1(\alpha) \propto \alpha^{a_1 + m - 1} e^{-\alpha(b_1 - S)}, \tag{27}$$

$$f_2(\beta) \propto \beta^{a_2 - m\alpha - 1} e^{-b_2\beta}, \tag{28}$$

and

$$f_3(p) \propto p^{a_3 + \sum_{i=1}^{m-1} r_i - 1}(1-p)^{b_3 - (m-1)(n-m) - \sum_{i=1}^{m-1}(m-i)r_i - 1}. \tag{29}$$

The following are the steps used in the importance sampling procedure:

1. Generate α_1 from $f(\beta; a_1, b_1)$.
2. For the generated value of α_1, generate β_1 from $f(\beta|\alpha; a_2, b_2)$.
3. Generate p_1 from $f(p; a_3, b_3)$.
4. Repeat steps 1 to 3 K times to obtain the importance sample $(\alpha_1, \beta_1, p_1), (\alpha_2, \beta_2, p_2), \ldots, (\alpha_K, \beta_K, p_K)$.

Hence, using the SELF, ELF, and LLF, the Bayes estimate of β using the importance sampling procedure is defined, respectively, as

$$\hat{\beta}_{\text{self}} = \left[\frac{\sum_{j=1}^{K} \beta\, h\left(\alpha_j, \beta_j, p_j\right)}{\sum_{j=1}^{K} h\left(\alpha_j, \beta_j, p_j\right)}\right],$$

$$\hat{\beta}_{\text{elf}} = \left[\frac{\sum_{j=1}^{K} \beta^{-c}\, h\left(\alpha_j, \beta_j, p_j\right)}{\sum_{j=1}^{K} h\left(\alpha_j, \beta_j, p_j\right)}\right]^{-\frac{1}{c}},$$

and

$$\hat{\beta}_{\text{llf}} = -\frac{1}{h}\log\left[\frac{\sum_{j=1}^{K} e^{-h\beta}\, h\left(\alpha_j, \beta_j, p_j\right)}{\sum_{j=1}^{K} h\left(\alpha_j, \beta_j, p_j\right)}\right],$$

where $h\left(\alpha_j, \beta_j, p_j\right)$, $j = 1, 2, 3$, are given by Eq. (26). Similarly the Bayes estimates of α and p can be derived.

4.3. HPD credible interval

In this Section, we discuss the HPD credible intervals for η as described by Chen and Shao (1999). Define

$$\eta_1 = (\alpha^{\varphi}, \beta^{\varphi}, p^{\varphi}); \varphi = 1, 2, \ldots, \rho,$$

where α^{φ}, β^{φ}, and p^{φ} are given, respectively, by Equations (27), (28), and (29). Let $\alpha^{(\varphi)}$, $\beta^{(\varphi)}$, and $p^{(\varphi)}$ be the ordered values of α^{φ}, β^{φ}, and p^{φ}, respectively.

Also, define

$$\omega_i = \frac{h(\alpha^{\varphi}, \beta^{\varphi}, p^{\varphi})}{\sum_{i=1}^{\rho} h(\alpha^{\varphi}, \beta^{\varphi}, p^{\varphi})}.$$

Then α^{v}, the v^{th} quantile of α, can be obtained as

$$\hat{\alpha}^{v} = \begin{cases} \alpha_{(1)} & \text{if } v = 0 \\ \alpha_{(i)} & \text{if } \sum_{j=1}^{i-1} \omega_j < v < \sum_{j=1}^{i} \omega_j. \end{cases}$$

Hence, the $100(1-\zeta)\%$, where $0 < \zeta < 1$, confidence interval for α is given by

$$\left(\alpha^{j/\rho}, \alpha^{(j+(1-\zeta)\rho)/\rho}\right), \quad j = 1, 2, \ldots, \rho.$$

Similarly, we can obtain the HPD credible interval for β and p.

4.4. Metropolis-Hastings algorithm

In this Section, we used the MCMC approach to generate samples from the posterior distribution Eq. (18) and then compute Bayes estimates of the three parameters using various loss functions. We use a Metropolis-Hastings (M-H) algorithm proposed by Metropolis *et al.* (1953) and Hastings (1970) to generate a sample from the posterior density. The conditional posterior distributions of the parameters are given by

$$\Pi_1(\alpha|\beta,p,Y) \propto \alpha^{m+a_1-1} e^{-\alpha\left(b_1-\sum_{i=1}^{m}\log[y_i]\right)} \left[1-\left(\frac{y_i}{\beta}\right)^{\alpha}\right]^{r_i}, \tag{30}$$

$$\Pi_2(\beta|\alpha,p,Y) \propto \beta^{a_2-m\alpha-1} e^{-b_2\beta} \left[1-\left(\frac{y_i}{\beta}\right)^{\alpha}\right]^{r_i}, \tag{31}$$

and

$$\Pi_3(p|\alpha,\beta,Y) \propto p^{a_3+\sum_{i=1}^{m-1} r_i-1}(1-p)^{b_3+(m-1)(n-m)-\sum_{i=1}^{m-1}(m-i)r_i-1}. \tag{32}$$

The following steps are used to create samples from the posterior distribution.

1. Start with initial guess ($\alpha^{(0)}=\hat{\alpha}, \beta^{(0)}=\hat{\beta}$ and $p^{(0)}=\hat{p}$).
2. Generate α^I using Eq. (30) by assuming the proposal distribution as $N(\alpha_{(n-1)},\sigma_1)$.
3. Generate β^I using Eq. (31) by assuming the proposal distribution as $N(\beta_{(n-1)},\sigma_2)$.
4. Generate p^I using Eq. (32) where
$$p \sim \text{Beta}(a_3+\sum_{i=1}^{m-1} r_i, b_3+(m-1)(n-m)-\sum_{i=1}^{m-1}(m-i)r_i).$$
5. Set $I=I+1$.
6. Repeat the steps 2 to 5 M times, and collect a sufficient number of replicates.

Bayes estimates of the parameters α, β, and p with respect to the SELF, ELF, and LLF are given, respectively, by

$$\hat{p}_{\text{self}} = \left(\frac{1}{M-T_0}\left[\sum_{I=T_0+1}^{M}(p^I)\right]\right),$$

$$\hat{p}_{\text{elf}} = \left(\frac{1}{M-T_0}\left[\sum_{I=T_0+1}^{M}(p^I)^{-c}\right]\right)^{-\frac{1}{c}},$$

and

$$\hat{p}_{_{\text{llf}}} = -\frac{1}{h}\log\left(\frac{1}{M-T_0}\left[\sum_{I=T_0+1}^{M} e^{-hp^I}\right]\right),$$

where T_0 is the number of iterations of the burn-in-period. The credible interval with $100(1-\zeta)\%\ 0<\zeta<1$ is $\left(\alpha_{((M-T_0)(\zeta/2))}, \alpha_{((M-T_0)(1-\zeta/2))}\right)$.

Retracing the above steps, Bayes estimates of α and β can be derived.

5. Prediction

Dey *et al.* (2018) discussed the prediction of censored data and the accompanying prediction intervals using progressive censoring. In this section, we discuss one and two sample prediction estimation as well as estimation of future observations.

5.1. *One sample prediction*

In this section, we calculate one sample predictive estimates and predictive bounds for censored samples. Suppose that $y=(y_1,y_2,\ldots,y_m)$ denotes an informative sample of size m observed using the progressive censoring scheme $r=(r_1,r_2,\ldots,r_m)$. Further assume that $x_i=(x_{i1},x_{i2},\ldots,x_{ir_i})$ denotes a sample censored at the i^{th} failure. The censored observation $x=(x_{id}, i=1,2,\ldots,m; d=1,2,\ldots,r_i)$ is then predicted. The conditional distribution of x given $(y_1,y_2,\ldots,y_m)$ is given by

$$g(x|y,\eta) = d\binom{r_i}{d}\sum_{j=0}^{d-1}(-1)^{d-1-j}\binom{d-1}{j}(1-F(y_i))^{j-r_i}(1-F(x))^{r_i-1-j}f(x),$$

where $\eta=(\alpha,\beta,p)$. In the observed data, the posterior predictive density of x is given by

$$g^*(x|y) = \int_\Theta g(x|y,\eta)\,\Pi(\eta|y)\,\mathrm{d}\eta,$$

where $\Theta=\{(\alpha,\beta,p):\alpha>0,\beta>0,0<p<1\}$. Hence, the predictive values of x using the SELF, ELF, and LLF are

$$\begin{aligned}
\hat{x}_{_{\text{self}}} &= \int_{y_i}^{\infty} x\, g^*(x|y,\eta)\,\mathrm{d}x \\
&= \int_0^1\int_0^\infty\int_0^\infty I_1(y_d|\alpha,\beta)\,\Pi(\alpha,\beta,p|y)\,\mathrm{d}\alpha,\mathrm{d}\beta\,\mathrm{d}p \\
&= \frac{1}{M-T_0}\sum_{d=T_0+1}^{M} I_1(y_d|\alpha_d,\beta_d), \qquad (33)
\end{aligned}$$

$$
\begin{aligned}
\hat{x}_{\text{elf}} &= \left[\int_{y_i}^{\infty} x^{-c}\, g^*(x|y,\eta)\, \mathrm{d}x\right]^{-\frac{1}{c}} \\
&= \left[\int_0^1 \int_0^\infty \int_0^\infty I_2(y_d|\alpha,\beta)\, \Pi(\alpha,\beta,p|y)\, \mathrm{d}\alpha, \mathrm{d}\beta\, \mathrm{d}p\right]^{-\frac{1}{c}} \\
&= \left[\frac{1}{M-T_0} \sum_{d=T_0+1}^{M} I_2(y_d|\alpha_d,\beta_d)\right]^{-\frac{1}{c}}, \qquad (34)
\end{aligned}
$$

and

$$
\begin{aligned}
\hat{x}_{\text{llf}} &= -\frac{1}{h}\log\left[\int_{y_i}^{\infty} e^{-hx}\, g^*(x|y,\eta)\, \mathrm{d}x\right] \\
&= -\frac{1}{h}\log\left[\int_0^1 \int_0^\infty \int_0^\infty I_3(y_d|\alpha,\beta)\, \Pi(\alpha,\beta|y)\, \mathrm{d}\alpha, \mathrm{d}\beta, \mathrm{d}p\right] \\
&= -\frac{1}{h}\log\left[\frac{1}{M-T_0} \sum_{d=T_0+1}^{M} I_3(y_d|\alpha_d,\beta_d)\right]. \qquad (35)
\end{aligned}
$$

In the following, we compute $I_i(y_d|\alpha_d,\beta_d)$, for $i=1,2,3$, as follows:

$$
\begin{aligned}
I_1(y_d|\alpha,\beta) &= \int_{y_i}^{\infty} x\; g^*(x|y,\eta)\mathrm{d}x \\
&= \left(\frac{\alpha}{\beta^\alpha}\right) d \begin{pmatrix} r_i \\ d \end{pmatrix} \sum_{k=0}^{d-1} (-1)^{d-1+j} \\
&\times \int_{y_i}^{\infty} x \left[1-\left(\frac{y_i}{\beta}\right)^{\alpha}\right]^{j-r_i} \left[1-\left(\frac{x}{\beta}\right)^{\alpha}\right]^{r_i-1-j} x^{\alpha-1}\mathrm{d}x, \qquad (36)
\end{aligned}
$$

$$
\begin{aligned}
I_2(y_d|\alpha,\beta) &= \int_{y_i}^{\infty} x^{-c}\, g^*(x|y,\eta)\mathrm{d}x \\
&= \left(\frac{\alpha}{\beta^\alpha}\right) d \begin{pmatrix} r_i \\ d \end{pmatrix} \sum_{k=0}^{d-1} (-1)^{d-1+j} \\
&\times \int_{y_i}^{\infty} x^{-c} \left[1-\left(\frac{y_i}{\beta}\right)^{\alpha}\right]^{j-r_i} \left[1-\left(\frac{x}{\beta}\right)^{\alpha}\right]^{r_i-1-j} x^{\alpha-1}\mathrm{d}x, \qquad (37)
\end{aligned}
$$

and

$$
\begin{aligned}
I_3(y_d|\alpha,\beta) &= \int_{y_i}^{\infty} e^{-hx} g^*(x|y,\eta)\mathrm{d}x \\
&= \left(\frac{\alpha}{\beta^{\alpha}}\right) d \binom{r_i}{d} \sum_{d=0}^{d-1} (-1)^{d-1+j} \\
&\times \int_{y_i}^{\infty} e^{-hx}\left[1-\left(\frac{y_i}{\beta}\right)^{\alpha}\right]^{j-r_i}\left[1-\left(\frac{x}{\beta}\right)^{\alpha}\right]^{r_i-1-j} x^{\alpha-1}\mathrm{d}x. \quad (38)
\end{aligned}
$$

To determine the prediction bounds, we need the survival function, which is derived as

$$
\begin{aligned}
S(x|y,\eta) &= \frac{P(x>t|y,\eta)}{P(x>y_i|y,\eta)} = \frac{\int_1^{\infty} g(x|y,\eta)\mathrm{d}u}{\int_{x_i}^{\infty} g(x|y,\eta)\mathrm{d}u} \\
&= \frac{\sum_{j=0}^{d-1}(-1)^{d-1-j}\binom{d-1}{j}\frac{(1-F(y_j))^{j-r_i}(1-F(x))^{r_i-j}}{r_i-j}}{\sum_{j=0}^{k-1}\binom{d-1}{j}\frac{(-1)^{d-1-j}}{r_i-j}}. \quad (39)
\end{aligned}
$$

Also, the posterior survival function is defined as

$$
S^*(x|y) = \int_{\Theta} S(x|y,\eta)\varphi(\eta|y)\mathrm{d}\eta.
$$

The $100(1-\zeta)\%$ $0<\zeta<1$ prediction bounds (L_1,U_1) for x are obtained as $S^*(L_1|x) = 1-\frac{\zeta}{2}$ and $S^*(U_1|x) = \frac{\zeta}{2}$, where L_1 and U_1 are the lower bounds and upper bounds, respectively.

5.2. *Two sample prediction*

In this section, we assume that $y=(y_1,y_2,\ldots,y_m)$ denote the progressive type-II censored data under the binomial removal scheme $r=(r_1,r_2,\ldots,r_m)$. We write $U^* = U_1,U_2,\ldots,U_{M_1}$ to represent a future sample taken from the PFD. The marginal density of U_j^* is

$$
H(u_j|\eta) = j\binom{M_1}{j}\sum_{d=0}^{j-1}(-1)^{j-1-d}\binom{j-1}{k}(1-F(u_j))^{M_1-1-d}f(u_j), \quad u_j>0.
$$

The Bayesian predictive density is given as

$$
g(u_j|y) = \int_{\Theta} H(u_j|\eta)\,\Pi(\eta|y)\mathrm{d}\eta, \quad (40)
$$

where $\Theta=\{(\alpha,\beta,p):\alpha>0,\beta>0,0<p<1\}$. The Bayes prediction estimators of the U_j^* using SELF, ELF, and LLF are j^{th} future observed samples and are obtained as

$$\hat{u}_{\text{self}}=\int_0^\infty u_j\, g(u_j|y)\,\mathrm{d}u_j=\frac{1}{M-T_0}\sum_{i=T_0+1}^{M} I_4(\alpha_i,\beta_i),$$

$$\hat{u}_{\text{elf}}=\left[\int_0^\infty u_j^{-c} g(u_j|y)\,\mathrm{d}u_j\right]^{-\frac{1}{c}}=\left[\frac{1}{M-T_0}\sum_{i=T_0+1}^{M} I_5(\alpha_i,\beta_i)\right]^{-\frac{1}{c}},$$

and

$$\hat{u}_{\text{llf}}=-\frac{1}{h}\log\left[\int_0^\infty e^{-hu_j} g(u_j|y)\,\mathrm{d}u_j\right]=-\frac{1}{h}\log\left[\frac{1}{M-T_0}\sum_{i=T_0+1}^{M} I_6(\alpha_i,\beta_i)\right],$$

where

$$\begin{aligned}I_4(y_i|\alpha,\beta) &= \left(\frac{\alpha}{\beta^\alpha}\right)j\binom{M_1}{j}\sum_{d=0}^{j-1}(-1)^{j-1-d}\binom{j-1}{d}\\ &\times \int_0^\infty u_j\left[1-\left(\frac{u_j}{\beta}\right)^\alpha\right]^{M_1-1-d} u_j^{\alpha-1}\mathrm{d}u_j,\end{aligned} \tag{41}$$

$$\begin{aligned}I_5(y_i|\alpha,\beta) &= \left(\frac{\alpha}{\beta^\alpha}\right)j\binom{M_1}{j}\sum_{d=0}^{j-1}(-1)^{j-1-d}\binom{j-1}{d}\\ &\times \int_0^\infty u_j^{-c}\left[1-\left(\frac{u_j}{\beta}\right)^\alpha\right]^{M_1-1-d} u_j^{\alpha-1}\mathrm{d}u_j,\end{aligned} \tag{42}$$

and

$$\begin{aligned}I_6(y_i|\alpha,\beta) &= \left(\frac{\alpha}{\beta^\alpha}\right)j\binom{M_1}{j}\sum_{k=0}^{j-1}(-1)^{j-1-d}\binom{j-1}{d}\\ &\times \int_0^\infty e^{-hu_j}\left[1-\left(\frac{u_j}{\beta}\right)^\alpha\right]^{M_1-1-d} u_j^{\alpha-1}\mathrm{d}u_j.\end{aligned} \tag{43}$$

6. SIMULATION STUDY

In this section, we present a simulation study of the performance of the estimators proposed in the previous sections. For the simulation study, we choose the sample sizes as $n = 15$, 18, 20 and $m = 5$, 8, 10, 15, 18 using various values of the parameters such as $\alpha = 0.80$, 1.5, $\beta = 0.5, 0.75$, and $p = 0.35$, 0.50, 0.80. Table 2 presents the values of MSE, ACI, and HPD corresponding to MLEs.

TABLE 2
MSE for MLE, ACI, and HPD for α, β, and p

(n,m)	$(r_1, r_2, \ldots, r_m)$	(α, β, p)	$\hat{\alpha}_{\text{mle}}$	$\hat{\beta}_{\text{mle}}$	$\hat{p}_{\text{mle}}$	ACI	HPD AIL	HPD CP
(15,5)	(0,…,0,2)	(0.80,0.5,0.35)	0.02	0.00	0.56	(0.10,0.11,0.46)	(0.10,0.26,0.32)	(0.93,0.92,0.94)
(15,8)	(2,2,0,…,0)	(1.5,0.75,0.35)	0.02	0.02	0.45	(0.18,0.28,0.48)	(0.10,0.28,0.36)	(0.93,0.91,0.92)
(18,10)	(3,0,…,0)	(0.80,0.5,0.50)	0.02	0.04	0.42	(0.11,0.24,0.48)	(0.10,0.18,0.36)	(0.99,0.95,0.93)
(18,15)	(5,0,…,0,5)	(1.5,0.75,0.50)	0.02	0.09	0.36	(0.13,0.23,0.36)	(0.23,0.16,0.35)	(0.93,0.90,0.91)
(20,15)	(5,0,…,0)	(0.80,0.5,80)	0.01	0.02	0.33	(0.11,0.26,0.36)	(0.32,0.26,0.22)	(0.91,0.95,0.93)
(20,18)	(2,2,2,…,1,3)	(1.5,0.75,80)	0.00	0.01	0.32	(0.17,0.23,0.52)	(0.24,0.28,0.25)	(0.95,0.93,0.94)

Bayes estimates using the Lindley approximation, importance sampling, and one and two sample predictions are presented, respectively, in Tables 3-5.

TABLE 3
MSE of the Bayes estimators of α, β, and p using the Lindley approximation method.

(n,m)	$(r_1, r_2, \ldots, r_m)$	(α, β, p)	SELF			ELF			LLF		
			$\hat{\alpha}$	$\hat{\beta}$	$\hat{p}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{p}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{p}$
(15,5)	(0,…,0,2)	0.80,0.5,0.35	0.00	0.00	0.02	0.01	0.00	0.04	0.00	0.00	0.01
(15,8)	(2,2,0,…,0)	1.5,0.75,0.35	0.00	0.02	0.02	0.01	0.02	0.03	0.00	0.00	0.01
(18,10)	(0,…,0,3)	0.80,0.5,0.50	0.00	0.02	0.02	0.02	0.02	0.03	0.00	0.00	0.01
(18,15)	(5,0,…,0,5)	1.5,0.75,0.50	0.01	0.02	0.03	0.02	0.02	0.03	0.00	0.00	0.01
(20,15)	(5,0,…,0)	0.80,0.5,0.80	0.01	0.05	0.05	0.02	0.02	0.03	0.01	0.00	0.02
(20,18)	(2,2,2,…,1,3)	1.5,0.75,0.80	0.01	0.09	0.05	0.02	0.02	0.03	0.01	0.01	0.02

TABLE 4
MSE of the Bayes estimators of α, β, and p using the importance sampling procedure.

(n,m)	$(r_1, r_2, \ldots, r_m)$	(α, β, p)	SELF			ELF			LLF		
			$\hat{\alpha}$	$\hat{\beta}$	$\hat{p}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{p}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{p}$
(15,5)	(0,…,0,2)	0.80,0.5,0.35	0.01	0.00	0.03	0.00	0.01	0.05	0.00	0.02	0.02
(15,8)	(2,2,0,…,0)	1.5,0.75,0.35	0.01	0.00	0.06	0.00	0.01	0.05	0.00	0.02	0.02
(18,10)	(0,…,0,3)	0.80,0.5,0.50	0.01	0.00	0.07	0.00	0.03	0.05	0.01	0.02	0.02
(18,15)	(5,0,…,0,5)	1.5,0.75,0.50	0.01	0.00	0.05	0.01	0.02	0.05	0.01	0.02	0.03
(20,15)	(5,0,…,0)	0.80,0.5,0.80	0.02	0.00	0.05	0.02	0.02	0.06	0.01	0.02	0.04
(20,18)	(2,2,2,…,1,3)	1.5,0.75,0.80	0.02	0.00	0.08	0.03	0.02	0.06	0.01	0.03	0.04

TABLE 5
MSE of one and two sample predictions of Y.

(n,m)	$(r_1, r_2, \ldots, r_m)$	p	i	k	One sample prediction			Two sample prediction		
					$\hat{x}_{\text{self}}$	$\hat{x}_{\text{elf}}$	$\hat{x}_{\text{llf}}$	$\hat{u}_{\text{self}}$	$\hat{u}_{\text{elf}}$	$\hat{u}_{\text{llf}}$
(15,5)	(0,...,0,2)	0.35	1	1	0.05	0.06	0.01	0.03	0.02	0.01
(15,8)	(2,2,0,...,0)	0.35	1	2	0.05	0.06	0.01	0.03	0.02	0.01
(18,10)	(0,...,0,3)	0.50	1	1	0.05	0.07	0.01	0.03	0.02	0.02
(18,15)	(5,0,...,0,5)	0.50	1	2	0.05	0.08	0.02	0.03	0.02	0.02
(20,15)	(5,0,...,0)	0.80	1	1	0.05	0.08	0.02	0.03	0.02	0.02
(20,18)	(2,2,2,...,1,3)	0.80	1	2	0.05	0.08	0.02	0.03	0.02	0.02

Based on the simulation results reported in Tables 2-5, we can list the following conclusions.

1. The MSE of the MLE decreases when the number of censored samples increases.
2. The average length of the approximate confidence interval and HPD are decreases as the sample size increases.
3. The value of the MSE decreases when the sample size increases for Bayes estimators.

7. Data analysis

In this section, we consider two real data sets: the duration of remission of 13 leukemia patients treated with a single drug (Balakrishnan and Cramer (2014)) and the breaking strength of jute fibres under gauge lengths of 15mm and 20mm (Chaturvedi *et al.* (2018)).

The first real data set, which includes the duration of remission of 13 leukemia patients, is reported in Table 6:

TABLE 6
First real data set.

Duration of remission of 13 leukemia patients:
1.013, 1.034, 1. 109, 1.266, 1.509, 1.533, 1.563, 1.929, 1.965, 2.061, 2.344, 2.546, 2.626.

The second real data set consists of two data sets, each of which contains gauge length of 15mm and 20mm of 30 fibres (Table 7).

TABLE 7
Second real data set.

	Gauge length of 30 fibres:
15mm	594.40, 202.75, 168.37, 574.86, 225.65, 76.38, 156.67, 127.81, 813.87, 562.39, 468.47, 135.09, 72.24, 497.94, 355.56, 569.07, 640.48, 200.76, 550.42, 748.75, 489.66, 678.06, 457.71, 106.73, 716.30, 42.66, 80.40, 339.22, 70.09, 193.42.
20mm	71.46, 419.02, 284.64, 585.57, 456.60, 113.85, 187.85, 688.16, 662.66, 45.58, 578.62, 756.70, 594.29, 166.49, 99.72, 707.36, 765.14, 187.13, 145.96, 350.70, 547.44, 116.99, 375.81, 581.60, 119.86, 48.01, 200.16, 36.75, 244.53, 83.55.

Table 8 shows the test values obtained by fitting PFD to the data set. The Kolmogorov-Smirnov (K-S) and Anderson-Darling (A-D) statistics and the corresponding p-values are given. As a result, we generate point and interval estimates of the unknown parameters using the binomial removal pattern based on progressive type-II censored samples. For the Bayes estimation, we choose the values of the hyperparameters as $a_1 = 1.6$, $b_1 = 3.2$, $a_2 = 2$, $b_2 = 2$, $a_3 = 2.5$, and $b_3 = 2.5$. Bayesian estimates using LLF and ELF are evaluated by fixing $h = c = 2$.

TABLE 8
Fitting the PFD to the real data-sets.

Data set	Parameters		K-S Test		A-D Test	
	α	β	Statistic	p-value	Statistic	p-value
Lekumia patients	0.38	2.14	0.21	0.55	1.26	0.24
Gauge length 15mm	0.00	0.90	0.15	0.44	1.40	0.20
Gauge length 20mm	0.00	0.86	0.15	0.44	1.03	0.34

The MLEs of α, β, and p and are given in Table 9, and the Bayes estimates of α, β and p under the SELF, LLF, and ELF are given in Table 10.

TABLE 9
MLE of α, β, and p using the duration of remission of the leukemia data-set.

(n,m)	$\hat{\alpha}_{\text{mle}}$	$\hat{\beta}_{\text{mle}}$	$\hat{p}_{\text{mle}}$
(8,5)	0.14	0.14	0.13
(8,8)	0.18	0.16	0.28
(10,6)	0.19	0.18	0.36
(10,8)	0.29	0.22	0.46
(12,10)	0.39	0.38	0.48

TABLE 10
Estimators of α, β, and p using the duration of remission of the leukemia data set using the $M-H$ algoritham.

(n,m)	M-H								
	$\hat{\alpha}_{\text{self}}$	$\hat{\beta}_{\text{self}}$	$\hat{p}_{\text{self}}$	$\hat{\alpha}_{\text{elf}}$	$\hat{\beta}_{\text{elf}}$	$\hat{p}_{\text{elf}}$	$\hat{\alpha}_{\text{llf}}$	$\hat{\beta}_{\text{llf}}$	$\hat{p}_{\text{llf}}$
(8,5)	0.10	0.12	0.31	0.34	0.12	0.13	0.42	0.33	0.44
(8,8)	0.15	0.15	0.35	0.42	0.14	0.18	0.44	0.40	0.45
(10,6)	0.19	0.19	0.36	0.44	0.19	0.18	0.47	0.48	0.45
(10,8)	0.21	0.26	0.41	0.45	0.28	0.27	0.50	0.49	0.45
(12,10)	0.21	0.36	0.52	0.49	0.32	0.33	0.55	0.58	0.62

Table 11 contains the details of one or two sample predictive estimates and predictive intervals using different loss functions.

TABLE 11
Prediction of Y using duration of remission of leukemia data-set.

(n, m)	p	i	k	N_1	j	Prediction						Prediction interval	
						One sample			Two sample				
						$\hat{x}_{\text{self}}$	$\hat{x}_{\text{elf}}$	$\hat{x}_{\text{llf}}$	$\hat{u}_{\text{self}}$	$\hat{u}_{\text{elf}}$	$\hat{u}_{\text{llf}}$	One sample	Two sample
(8,5)	0.2	2	1	8	1	0.12	0.26	0.18	0.39	0.23	0.20	(0.16,0.72)	(0.20,0.76)
(8,8)	0.5	2	2	8	2	0.12	0.36	0.28	0.38	0.52	0.31	(0.44,1.00)	(0.23,0.75)
(10,6)	0.7	2	1	10	1	0.12	0.33	0.46	0.42	0.31	0.40	(0.22,0.73)	(0.47,0.98)
(10,8)	0.9	2	2	10	2	0.12	0.29	0.38	0.44	0.44	0.57	(0.46,0.98)	(0.25,0.75)
(12,10)	1.2	2	1	12	2	0.12	0.33	0.49	0.48	0.53	0.39	(0.24,0.74)	(0.46,1.00)

The MLE and Bayes estimators of the parameters are given in Tables 12 and 13.

TABLE 12
Bayes estimators using the real data-set of gauge length 15mm.

(n, m)	Estimators	Estimation Methods				Prediction	
		MLE	LAM	ISP	M-H	One sample	Two sample
(30,20)	$\hat{\alpha}_{\text{mle}}$	0.22	-	-	-	-	-
	$\hat{\beta}_{\text{mle}}$	0.34	-	-	-	-	-
	$\hat{p}_{\text{mle}}$	0.58	-	-	-	-	-
	$\hat{\alpha}_{\text{self}}$	-	0.52	0.44	0.56	-	-
	$\hat{\beta}_{\text{self}}$	-	0.36	0.28	0.47	-	-
	$\hat{p}_{\text{self}}$	-	0.42	0.46	0.58	-	-
	$\hat{\alpha}_{\text{elf}}$	-	0.25	0.51	0.45	-	-
	$\hat{p}_{\text{elf}}$	-	0.55	0.48	0.36	-	-
	$\hat{\beta}_{\text{elf}}$	-	0.87	0.62	0.75	-	-
	$\hat{\alpha}_{\text{llf}}$	-	0.23	0.27	0.44	-	-
	$\hat{\beta}_{\text{llf}}$	-	0.68	0.46	0.63	-	-
	$\hat{p}_{\text{llf}}$	-	0.56	0.53	0.49	-	-
	$\hat{x}_{\text{self}}$	-	-	-	-	0.25	-
	$\hat{x}_{\text{elf}}$	-	-	-	-	0.35	-
	$\hat{x}_{\text{llf}}$	-	-	-	-	0.29	-
	$\hat{u}_{\text{self}}$	-	-	-	-	-	0.33
	$\hat{u}_{\text{elf}}$	-	-	-	-	-	0.36
	$\hat{u}_{\text{llf}}$	-	-	-	-	-	0.30

TABLE 13
Bayes estimators using the real data-set of gauge length 20mm.

(n, m)	Estimators	Estimation Methods				Prediction	
		MLE	LAM	ISP	M-H	One sample	Two sample
(30,25)	$\hat{\alpha}_{\text{mle}}$	0.29	-	-	-	-	-
	$\hat{\beta}_{\text{mle}}$	0.49	-	-	-	-	-
	$\hat{p}_{\text{mle}}$	0.52	-	-	-	-	-
	$\hat{\alpha}_{\text{self}}$	-	0.26	0.31	0.38	-	-
	$\hat{\beta}_{\text{self}}$	-	0.38	0.23	0.42	-	-
	$\hat{p}_{\text{self}}$	-	0.47	0.53	0.55	-	-
	$\hat{\alpha}_{\text{elf}}$	-	0.48	0.32	0.39	-	-
	$\hat{\beta}_{\text{elf}}$	-	0.45	0.52	0.51	-	-
	$\hat{p}_{\text{elf}}$	-	0.54	0.56	0.58	-	-
	$\hat{\alpha}_{\text{llf}}$	-	0.18	0.26	0.46	-	-
	$\hat{\beta}_{\text{llf}}$	-	0.10	0.49	0.59	-	-
	$\hat{p}_{\text{llf}}$	-	0.48	0.36	0.65	-	-
	$\hat{x}_{\text{self}}$	-	-	-	-	0.13	-
	$\hat{x}_{\text{elf}}$	-	-	-	-	0.52	-
	$\hat{x}_{\text{llf}}$	-	-	-	-	0.47	-
	$\hat{u}_{\text{self}}$	-	-	-	-	-	0.18
	$\hat{u}_{\text{elf}}$	-	-	-	-	-	0.31
	$\hat{u}_{\text{llf}}$	-	-	-	-	-	0.56

MLE and Bayes estimators of α, β, and p using different loss functions and the varying values of m are plotted in Figures 1-6.

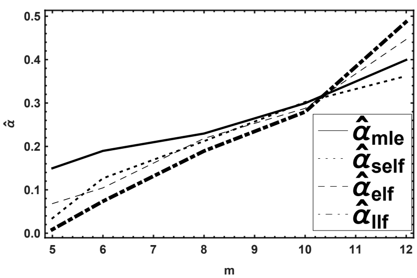

Figure 1 – MLE and Bayes estimators of α under SELF, ELF and LLF using Lindley approximation methods

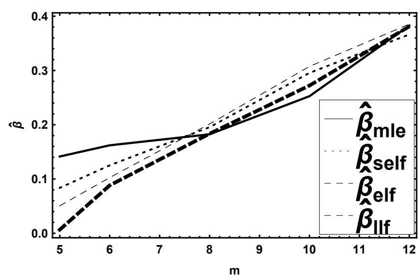

Figure 2 – MLE and Bayes estimators of β under SELF, ELF and LLF using Lindley approximation methods

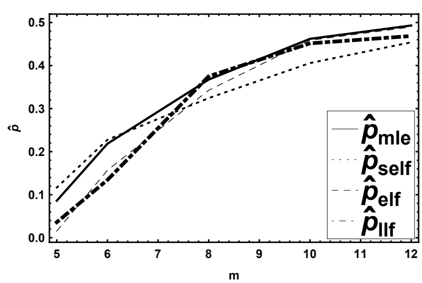

Figure 3 – MLE and Bayes estimators of p under SELF, ELF, and LLF using the Lindley approximation method

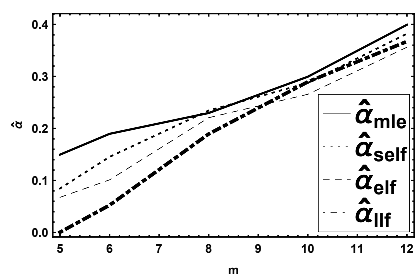

Figure 4 – MLE and Bayes estimators of α under SELF, ELF, and LLF using the importance sampling procedure

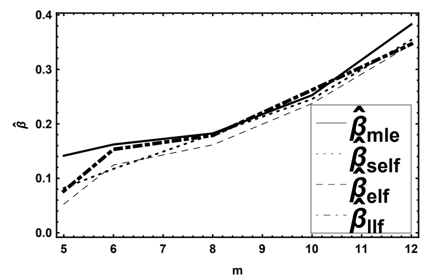

Figure 5 – MLE and Bayes estimators of β under SELF, ELF, and LLF using the importance sampling procedure

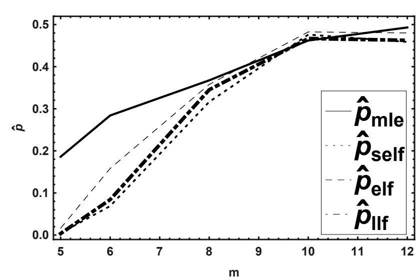

Figure 6 – MLE and Bayes estimators of p under SELF, ELF, and LLF using the importance sampling procedure

8. Conclusions

This paper discusses the Bayes estimation of the unknown parameters of the PFD using progressive type-II censored data with binomial removal scheme. The MLEs of the parameters α, β, and p are obtained. The Bayes estimates are obtained using different loss functions such as SELF, ELF, and LLF. To evalute the Bayes estimates, the MCMC method has been applied. Further, with the help of the posterior density and using the importance sampling procedure, we also computed highest posterior density credible intervals of the parameters α, β, and p. Also, we discussed one and two sample prediction and its confidence intervals. Based on the simulation study, Bayes estimation provides better results than the MLE in terms of the MSE. The estimation techniques described in this paper are demonstrated using two real data sets. Moreover, we showed that as m increases, the values of the estimator also increase.

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