ON THE COMPUTATION OF UPPER APPROXIMATIONS TO ULTIMATE RUIN PROBABILITIES IN CASE OF DFR CLAIMSIZE DISTRIBUTIONS

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1. INTRODUCTION

In the classical continuous time model of the collective theory of risk, see Gerber (1979), under which premiums are paid continuously at rate c, the aggregate claims process is a compound Poisson process with parameter λ, individual claims are independent of each other and of the number of claims, and identically distributed with distribution function F, and μ is the expected claim size, the infinite time probability of ruin, $\psi(u)$, for an initial risk reserve of u, satisfies the Volterra integral equation

$$\psi(u) = \frac{\lambda}{c} \int_u^\infty (1-F(t))dt + \frac{\lambda}{c} \int_0^u (1-F(u-t))\, \psi(t)\, dt \tag{1}$$

In the present contribution we assume that the claimsize distribution is DFR (decreasing failure rate) so that, according to well known queueing results, see Szekli (1986), ψ can be proved to be convex. It will be shown that this property, together with the convexity of the stop loss transform of the distribution F, namely,

$$h(u) = \int_u^\infty (1-F(t))dt$$

can be exploited to construct an efficient and stable recursive procedure yielding upper approximations to ψ, with a remarkable improvement over analogous existing methods. It may be noted that several distributions of actuarial interest, e.g. Pareto, exponential, gamma, etc., belong to the DFR class.

2. THE PROPOSED ALGORITHM

For our purpose, let us first cast equation (1) in the form

$$\psi(u)=\frac{\lambda}{c}h(u)-\frac{\lambda}{c}\int_0^u \psi(u-x)h'(x)dx \tag{2}$$

whence, integrating by parts,

$$(1-\frac{\lambda\mu}{c})\psi(u)=\frac{\lambda}{c}(1-\frac{\lambda\mu}{c})h(u)+\frac{\lambda}{c}\int_0^u -\psi'(u-x)h(x)dx\,. \tag{3}$$

Now, let us divide the interval $[0, u]$ into N subintervals of common length d and note that, thanks to the convexity of h, we have

$$h(x)\le a_i+b_i x \quad , \quad x\in[(i-1)d,id] \quad , \quad i=1,...,N, \tag{4}$$

where we have denoted

$$a_i=ih_{i-1}-(i-1)h_i \quad , \quad b_i=\frac{h_i-h_{i-1}}{d} \quad , \quad h_i=h(id).$$

As a consequence, from (3) we obtain

$$(1-\frac{\lambda\mu}{c})\psi(u)\le\frac{\lambda}{c}(1-\frac{\lambda\mu}{c})h(u)+\frac{\lambda}{c}\sum_{i=1}^{N}\int_{(i-1)d}^{id}-\psi'(u-x)(a_i+b_i x)dx \tag{5}$$

whence, integrating by parts,

$$\begin{aligned}(1-\frac{\lambda\mu}{c})\psi(u)\le{}&\frac{\lambda}{c}(1-\frac{\lambda\mu}{c})h(u)+\\&+\frac{\lambda}{c}\sum_{i=1}^{N}\{[(a_i+b_i x)\psi(u-x)]_{(i-1)d}^{id}-b_i\int_{(i-1)d}^{id}\psi(u-x)dx\}.\end{aligned} \tag{6}$$

Now, the convexity of ψ yields

$$\int_{(i-1)d}^{id}\psi(u-x)dx\le\frac{d}{2}[\psi(u-id)+\psi(u-(i-1)d)]\,; \tag{7}$$

therefore, the inequality (6) becomes:

$$\begin{aligned}(1-\frac{\lambda\mu}{c})\psi(u)\le{}&\frac{\lambda}{c}(1-\frac{\lambda\mu}{c})h(u)+\\&+\frac{\lambda}{c}\sum_{i=1}^{N}\{(a_i+b_i(i-\frac{1}{2})d)[\psi(u-id)-\psi(u-(i-1)d)]\}\end{aligned} \tag{8}$$

and, rearranging, we obtain the following inequality:

$$(1-\frac{\lambda}{2c}(\mu-h_1))\psi(u)\le\frac{\lambda}{c}(1-\frac{\lambda\mu}{c})h_N+ \\ +\frac{\lambda}{2c}\frac{\lambda\mu}{c}h_{N-1}+\frac{\lambda}{2c}\sum_{i=1}^{N-1}(h_{i-1}-h_{i+1})\psi(u-id). \tag{9}$$

Finally, let $\psi_0^+=\psi(0)=\frac{\lambda\mu}{c}$ and let

$$\psi_N^+=[1-\frac{\lambda}{2c}(\mu-h_1)]^{-1}\cdot \\ \cdot[\frac{\lambda}{c}(1-\frac{\lambda\mu}{c})h_N+\frac{\lambda}{2c}\frac{\lambda\mu}{c}h_{N-1}+\frac{\lambda}{2c}\sum_{i=1}^{N-1}(h_{i-1}-h_{i+1})\psi_{N-i}^+]. \tag{10}$$

Then we have, for N = 1, 2, ...,

$$\psi(u)\le\psi_N^+. \tag{11}$$

We point out that the coefficients of the recursion are positive, so that the procedure is strongly stable, see Panjer and Wang (1993).

Remark. For N = 1 the sum in the r.h.s of (9) vanishes; then we obtain the upper bound

$$\psi(u)\le[1-\frac{\lambda}{2c}(\mu-h(u))]^{-1}[\frac{\lambda^2\mu^2}{2c^2}+\frac{\lambda}{c}(1-\frac{\lambda\mu}{c})h(u)]. \tag{12}$$

It is a matter of tedious calculations to show that it is sharper than the bound

$$\psi(u)\le\frac{\lambda^2\mu^2}{c^2}+\frac{\lambda}{c}(1-\frac{\lambda\mu}{c})h(u) \tag{13}$$

established by Corradi (1991) without any assumption on the claimsize distribution.

3. SOME NUMERICAL RESULTS

The advantage of the proposed procedure will be illustrated in the following examples, where we compare numerical results produced by formula (10) with those produced by Goovaerts' and De Vylder's (1984) recursive algorithm.

Example 1 (see Dickson *et al.*, 1995).

We assume that individual claim amounts have an exponential distribution with mean 1 and that $\lambda = 1/(1.1)$; we take the interval of discretization to be 1.

TABLE 1

Exponential, with mean 1, $\lambda=(1.1)^{-1}$, d=1

u	(1)	(2)	(3)	(4)
0	0.9090909	0.0000000	0.9090909	0.0000000
2	0.7683947	0.0104375	0.8076102	0.0496530
4	0.6494734	0.0175244	0.7174577	0.0855087
6	0.5489571	0.0220678	0.6373688	0.1104795
8	0.4639973	0.0247019	0.5662201	0.1269247
10	0.3921863	0.0259225	0.5030137	0.1367499
20	0.1691911	0.0216270	0.2783250	0.1307609
40	0.0314882	0.0075355	0.0959186	0.0612586
60	0.0058603	0.0019723	0.0260881	0.0222001
80	0.0010907	0.0004596	0.0079871	0.0073560
100	0.0002030	0.0001005	0.0024453	0.0023428

The columns in Table 1 show for the values of u indicated:
(1) an approximation of $\psi(u)$ based on formula (10);
(2) the difference between the approximation in (1) and the exact value;
(3) an approximation of $\psi(u)$ calculated by the Goovaerts' and De Vylder's algorithm;
(4) the difference between the approximation in (3) and the exact value.

The better performance of the present algorithm is so strikingly evident that very little needs to be added to the mere inspection of the reported results, see also Fig. 1.

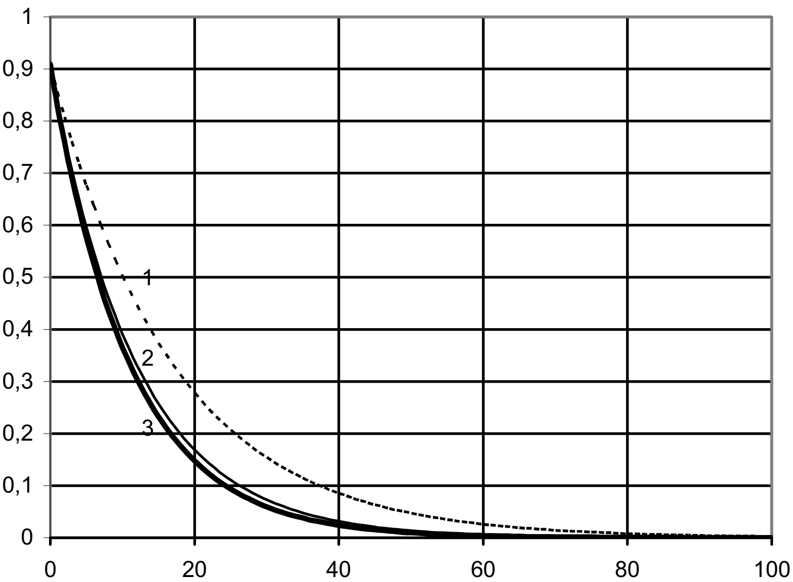

Figure 1 – Exponential, with mean 1, λ=1/1.1, *d*=1.
1. Goovaerts' and De Vylder's approximation.
2. Approximation based on formula (10).
3. Exact value.

Example 2 (see Ramsay and Usabel, 1997).

Now we assume that individual claim amounts have a Pareto (2,1) distribution and that $\lambda = 1/(1.1)$; the relevant interval is divided into 320 intervals of equal length.

The results are illustrated in Figure 2 and in Table 2, where we show for the values of *u* indicated:

(1) an approximation of $\psi(u)$ based on formula (10);

(2) an approximation of $\psi(u)$ calculated by the Goovaerts' and De Vylder's algorithm.

TABLE 2

Pareto, with mean 1, $\lambda=(1.1)^{-1}$, using 320 discretization points

u	(1)	(2)
0	0.9090909	0.9090909
12.5	0.6383196	0.7095667
25	0.4970101	0.5815087
50	0.3337626	0.4138032
75	0.2416450	0.3084746
100	0.1837325	0.2376186
200	0.0829193	0.1050071
300	0.0494909	0.0594826
500	0.0259990	0.0289254
700	0.0173531	0.0185869
1000	0.0115109	0.0120156

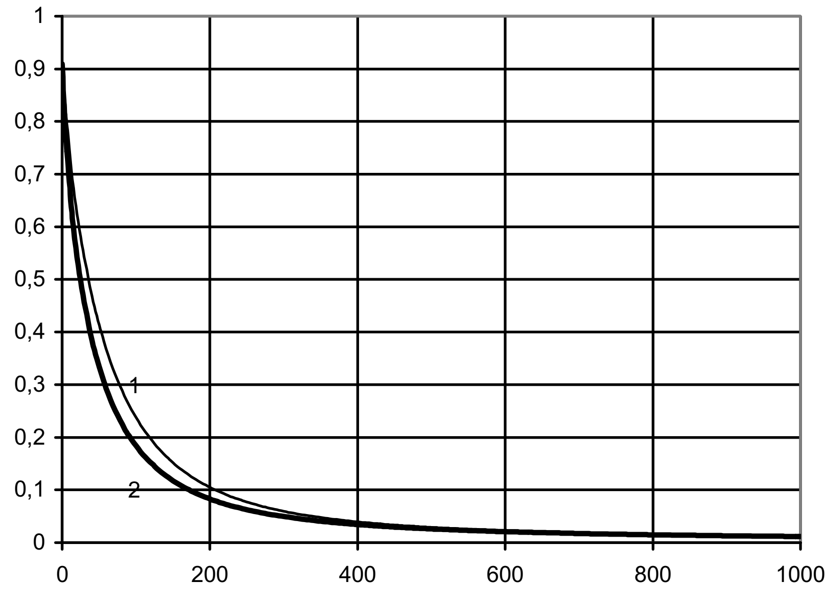

Figure 2 – Pareto, with mean 1, λ=1/1.1, using 320 discretization points.
1. Goovaerts' and De Vylder's approximation.
2. Approximation based on formula (10).

From the foregoing results we can unambiguously conclude that the present method provides a remarkable improvement over the traditional Goovaerts' and De Vylder's algorithm with the same computational effort.

As a final remark, we may point out that, in principle, the above idea can be exploited to obtain a lower approximation as well. Indeed, it is easy to see that from the monotonicity and the convexity of *h* and ψ it follows

$$a_{i+1} + b_{i+1}x \leq h(x), \quad x \in [(i-1)d, id], \quad i = 1, \ldots, N-1 \tag{14}$$

and

$$\frac{d}{2}[3\psi(u-(i-1)d) - \psi(u-(i-2)d)] \leq \int_{(i-1)d}^{id} \psi(u-x)dx \tag{15}$$

so that, replacing (4) and (7) with (14) and (15) respectively, and proceeding in the same way, we readily obtain the corresponding recursione for ψ_N^-

However, experimental results indicate that the gain over the traditional methods is not so marked as for the upper approximation, and in general it is practically negligible.

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RIASSUNTO

Sull'approssimazione per eccesso della probabilità di rovina nel caso di funzione di ripartizione di un singolo danno con tasso di eliminazione decrescente (DFR)

In questa nota si considera il modello classico della teoria collettiva del rischio nell'ipotesi che la funzione di ripartizione di un singolo danno appartenga alla famiglia delle distribuzioni DFR (*decreasing failure rate*). Sfruttando la conseguente convessità della probabilità di rovina viene sviluppato un procedimento ricorsivo che permette di ottenere, con la medesima complessità computazionale, una maggiorazione della probabilità di rovina notevolmente più accurata di quella fornita da metodi tradizionali, come mostrato attraverso esempi numerici.

SUMMARY

On the computation of upper approximations to ultimate ruin probabilities in case of DFR claimsize distributions

In the present note we consider the classical continuous time model of the collective theory of risk under the assumption that the claimsize distribution is DFR (decreasing failure rate) so that, according to well known queueing results, the ultimate ruin probability turns out to be convex. This property is exploited to develop a stable recursive formula for the calculation of a numerical upper approximation to the ultimate ruin probability with a remarkable improvement over analogous existing algorithms. Numerical results are reported to show the merits of the proposed approach.