A STABLE ITERATIVE ALGORITHM FOR EVALUATION OF UPPER AND LOWER APPROXIMATIONS TO ULTIMATE RUIN PROBABILITIES (*)

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1. INTRODUCTION

According to a classical result in the collective theory of risk (see Gerber, 1979), when claims follow a compound Poisson process with rate λ and premiums are paid continuously at rate c then the infinite time probability of ruin, $\psi(u)$, for an initial risk reserve of u, satisfies the Volterra integral equation

$$\psi(u) = \frac{\lambda}{c} \int_u^{\infty} (1 - F(t))dt + \frac{\lambda}{c} \int_0^u (1 - F(u-t))\,\psi(t)\,dt \tag{1.1}$$

where $c = \lambda\mu\,(1+\theta)$, μ is the expected claim size, $\theta > 0$ is the safety loading factor and F is the cumulative distribution function (c.d.f.) of the claim amounts. Modifying the classical surplus process by the inclusion of deterministic interest, δ, on the insurer's surplus leads to the equation

$$\bar{\psi}_0(u) = \frac{c}{c + \delta\, u} + \frac{1}{c + \delta\, u} \int_0^u (\delta + \lambda\ (1 - F(u-t)))\bar{\psi}_0(t)\,dt \tag{1.2}$$

where $\bar{\psi}_0 = \bar{\psi}(u) / \bar{\psi}(0)$ and $\bar{\psi} = 1 - \psi$ is the probability of ultimate survival (see Delbaen and Haezendonck, 1987). In case $\delta = 0$ the last equation reduces to

$$\bar{\psi}_0(u) = 1 + \frac{\lambda}{c} \int_0^u (1 - F(u-t))\bar{\psi}_0(t)\,dt$$

that can be used in place of equation (1.1) to obtain ψ.

Methods for numerically solving the above equations have been studied by several authors (see Dickson and Waters, 1999, for a recent survey). In particular, methods providing numerical upper and lower approximations to the solution,

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and hence rigorous error bounds, have been proposed, based on the discretization of the interval (0, u) and suitable recursive formulas for the computation of the approximate solutions (see Goovaerts and De Vylder, 1984, Sundt and Teugels, 1995). From the computational point of view, such methods have been proved to be stable, that is, there is no cumulative effect of the propagation of errors; however, both present two basic drawbacks. First, the evaluation of the stop loss transform of the distribution F, namely,

$$h_F(u) = \int_u^\infty (1 - F(t))dt$$

(Goovaerts and De Vylder, 1983), is required, but there are many cases of practical interest where it can not be expressed in closed form. Second, the methods involve repeated subtractions of nearly equal terms with consequent loss of significant digits and the modifications of the recursion in order to avoid such effects lead to procedures that may violate the upper and lower bound property (see Ramsay, 1992).

In this context, the purpose of the present contribution is to propose an alternative approach yielding numerical upper and lower approximations as well, based on an iterative scheme which eliminates the flaws described above. Numerical results are reported to show the effectiveness of the proposed method.

2. THE PROPOSED METHOD

Let us first gather some results from Numerical Functional Analysis: the reader can refer to Krasnosel'skii *et al.* (1972) and, in particular, to Rall (1965) and Casadei (1972) for further details and some applications to problems of computational physics.

Given the linear integral equation

$$y(x) = g(x) + \int_a^b K(x, t)\, y(t)\, \mathrm{d}t$$

where

$$\max_x \int_a^b K(x, t)\, \mathrm{d}t < 1, \tag{2.1}$$

let us divide the interval [a, b] into N subintervals so that for

$$K_{ij} = \max K(x, t), \qquad x_{i-1} \le x \le x_i\,, \quad t_{j-1} \le t \le t_j$$

one has

$$\max_i S_i = \sum_{j=1}^{N} K_{ij}\,(t_j - t_{j-1}) < 1. \tag{2.2}$$

Moreover, let

$k_{ij} = \min K(x, t),\ x_{i-1} \le x \le x_i ,\ \ t_{j-1} \le t \le t_j$

$s_i = \sum_{j=1}^{N} k_{ij}\ (t_j - t_{j-1})$

$G_i = \max g(x),\ g_i = \min g(x),\ x_{i-1} \le x \le x_i$

and let

$M^{(0)} = \mathbf{1} \max_i\ G_i /(1 - S_i)$

$m^{(0)} = \mathbf{1} \min_i\ g_i /(1 - s_i)$

where $\mathbf{1}$ is the N - dimensional unit vector. Then we have

$m^{(0)} \le y(x) \le M^{(0)}$

and letting

$$M_i^{(k+1)} = G_i + \sum_{j=1}^{N} K_{ij} M_j^{(k)} (t_j - t_{j-1})$$

$$m_i^{(k+1)} = g_i + \sum_{j=1}^{N} k_{ij} m_j^{(k)} (t_j - t_{j-1})$$

we obtain, with obvious notation,

$m^{(0)} \le m^{(1)} \le \ldots.. \le y(x) \le \ldots.. \le M^{(1)} \le M^{(0)}$

i.e. two sequences of improved approximations to the solution from above and from below. If

$M = \lim_k M^{(k)},\ m = \lim_k m^{(k)}$,

then M and m are the best bounds provided by the above discretization and can be used to compute an approximate solution as $(M + m)/2$ and an upper bound for the absolute error as $(M - m)/2$ (in practice, of course, the iterations are stopped when some convergence criterion is met). If the resulting error turns out to be larger than some prescribed tolerance, one can refine the subdivision of the given interval and repeat, assuming M and m as "initial" guesses: it is clear that, at least in principle, any desired degree of accuracy can be achieved.

As a final remark, we note that, in actual computations, if rounding is done in such a way as not to decrease $M_i^{(k+1)}$ and not to increase $m_i^{(k+1)}$, then the above inequalities will hold with the values obtained numerically.

Now, turning to our problem and focusing on equation (1.2), we first point out that condition (2.1) is satisfied, since

$$\frac{1}{c+\delta u}\int_0^u(\delta+\lambda(1-F(u-t)))dt<\frac{\delta\ u+\lambda\mu}{c+\delta\ u}<1$$

for all $u > 0$. Moreover, taking into account the monotonicity of F and the fact that $K(x, t) = 0$, $t > x$, we immediately obtain

$$M_i^{(k+1)}=\frac{c}{c+\delta(i-1)\Delta}+\frac{1}{c+\delta(i-1)\Delta}\sum_{j=1}^{i}(\delta+\lambda(1-F((i-j)\Delta)))M_j^{(k)}\Delta$$

$$m_i^{(k+1)}=\frac{c}{c+\delta i\Delta}+\frac{1}{c+\delta i\Delta}\sum_{j=1}^{i}(\delta+\lambda(1-F((i-j+1)\Delta)))m_j^{(k)}\Delta$$

$i = 1,\ldots,N$, where Δ is the common length of the subintervals. As mentioned before, these formulas require to evaluate the c.d.f. F in place of its transform h_F and do not involve subtractions.

We may note that the procedure can be modified (*à la* Gauss-Seidel) using $M_j^{(k+1)}$ and $m_j^{(k+1)}$, $j = 1, \ldots , i-1$, to compute $M_i^{(k+1)}$ and $m_i^{(k+1)}$ respectively:

$$M_i^{(k+1)}=\frac{c}{c+\delta(i-1)\Delta}+\frac{1}{c+\delta(i-1)\Delta}\ (\sum_{j=1}^{i-1}(\delta+\lambda(1-F((i-j)\Delta)))M_j^{(k+1)}+(\delta+\lambda)M_i^{(k)})\Delta$$

$$m_i^{(k+1)}=\frac{c}{c+\delta i\Delta}+\frac{1}{c+\delta i\Delta}\ (\sum_{j=1}^{i-1}(\delta+\lambda(1-F((i-j+1)\Delta)))m_j^{(k+1)}+(\delta+\lambda(1-F(\Delta))m_i^{(k)})\Delta$$

so eventually improving the speed of convergence.

3. NUMERICAL RESULTS

A number of numerical experiments were carried out using examples in the literature: from the entire study we have selected the problems illustrated below, which provide a significant picture of the performance of the proposed method in cases where the c.d.f. can be computed only approximately and hence the traditional recursive methods based on the use of the transform h_F are inherently unable to provide two-sided approximations. We may note that in case of Pareto or exponential claim size distributions, often considered in the literature for numerical comparisons, such *recursive* methods can provide very satisfactory two-sided approximations, so that no appreciable improvement can be obtained using an (obviously less efficient) *iterative* scheme.

Example 1. We assume that individual claim amounts for the basic process have an inverse Gaussian distribution with mean 1 and variance 5, and let $\theta = 2.5$ (see Cai and Garrido, 1999, example 4). Fig. 1 gives the results obtained with different step sizes, $\Delta = 0.05$ and $\Delta = 0.005$ respectively.

In Table 1 we show, for the values of u indicated, a lower bound (col. 1), an upper bound (col. 2), the arithmetic mean (col. 3) and a bound for the absolute error (col. 4) produced by the present method with $\Delta = 0.005$, and the corresponding quantities (col. 5 to 8) obtained using the results reported in Cai and Garrido, cit.. From the inspection of the results we see that the present method can provide sharper bounds for all values of the initial reserve u less than the 0.1% quantile (the largest value of u for which $0.001 < \psi(u)$), while giving comparable bounds for the tail of ψ.

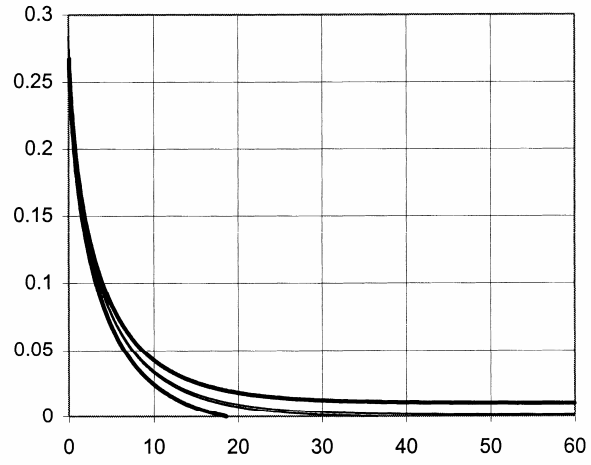

Figure 1 – Inverse Gaussian, with mean 1, variance 5 and θ=2.5, using Δ=0.05 and Δ=0.005.

TABLE 1

Inverse Gaussian, with mean 1*, variance* 5 *and* $\theta = 2.5$, *using* $\Delta = 0.005$

u	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
1	0.1849580	0.1854476	0.1852028	0.0002448	0.1816754	0.1989151	0.1902953	0.0086199
5	0.0763735	0.0771642	0.0767689	0.0003954	0.0688665	0.0955412	0.0822039	0.0133374
10	0.0331910	0.0340958	0.0336434	0.0004524	0.0273207	0.0479647	0.0376427	0.0103220
20	0.0076854	0.0086610	0.0081732	0.0004878	0.0056672	0.0144194	0.0100433	0.0043761
30	0.0017561	0.0027492	0.0022527	0.0004966	0.0013919	0.0047061	0.0030490	0.0016571
40	0.0001618	0.0011599	0.0006609	0.0004991	0.0003757	0.0015901	0.0009829	0.0006072
50	0	0.0007013	0.0002015	0.0003507	0.0001087	0.0005482	0.0003285	0.0002198
60	0	0.0005633	0.0000632	0.0002817	0.0000315	0.0001897	0.0001106	0.0000791

Example 2. We assume that individual claim amounts have a lognormal distribution with mean 1 and variance 25.53372, and let $\theta = 1$ (see Thorin and Wikstad, 1977, Grandell, 1991, Ramsay and Usabel, 1997). The appropriate results, obtained with $\Delta = 0.01$, are reported in Table 2 and illustrated in Fig. 2.

TABLE 2

Lognormal, with mean 1, *variance* 25.53372 *and* $\theta = 1$, *using* $\Delta = 0.01$

u	lower bound	upper bound	arithmetic mean	error bound
1	0.3917950	0.3946384	0.3932167	0.0014217
5	0.2589595	0.2639675	0.2614635	0.0025040
10	0.1888733	0.1950716	0.1919725	0.0030992
20	0.1214117	0.1288418	0.1251268	0.0037151
30	0.0870403	0.0951393	0.0910898	0.0040495
50	0.0519708	0.0607856	0.0563782	0.0044074
70	0.0344636	0.0436493	0.0390565	0.0045929
100	0.0205734	0.0300596	0.0253165	0.0047431
150	0.0097978	0.0195198	0.0146588	0.0048610
200	0.0047131	0.0145467	0.0096299	0.0049168

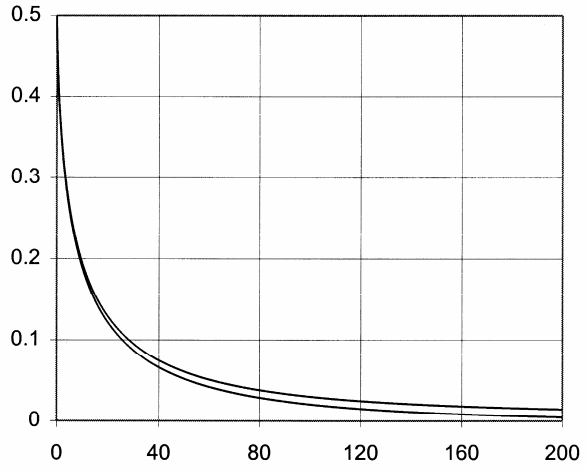

Figure 2 – Lognormal, with mean 1, variance 25.53372 and θ=1, using Δ=0.01.

Example 3. Same as Example 1 in case the reserve is invested at a constant interest rate 5% *per annum*, using $\Delta = 0.01$, see Table 3 and Fig. 3.

TABLE 3

Inverse Gaussian, with mean 1, *variance* 5, $\theta = 2.5$ *and* $\delta = \ln(1+0.05)$, *using* $\Delta = 0.01$

u	lower bound	upper bound	arithmetic mean	error bound
1	0.1685297	0.1706612	0.1695955	0.0010658
5	0.0622682	0.0655962	0.0639322	0.0016640
10	0.0233547	0.0270885	0.0252216	0.0018669
20	0.0030898	0.0070353	0.0050626	0.0019728
30	0	0.0031775	0.0011846	0.0015888
40	0	0.0023071	0.0003097	0.0011536
50	0	0.0020933	0.0000948	0.0010467
60	0	0.0020378	0.0000391	0.0010189

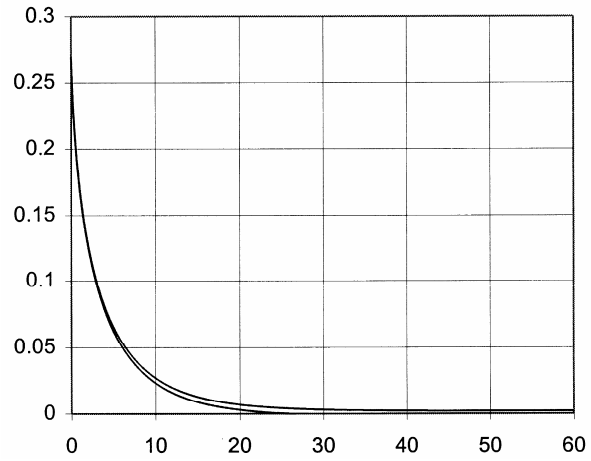

Figure 3 – Inverse Gaussian, with mean 1, variance 5, θ=2.5 and δ=ln(1+0.05), using Δ=0.01.

Comments.

1. Condition (2.2) is satisfied for all the above examples: the value of the left hand side is 0.28 (for example 1), 0.5 (for example 2) and 0.61 (for example 3) respectively.

2. In order to preserve the required upper and lower bound property, upper and lower approximations to the c.d.f.'s have to be used for the computation of $M_i^{(k+1)}$ and $m_i^{(k+1)}$ respectively. Now, since the c.d.f.'s in the above examples can be expressed in terms of the standard normal distribution, all we need is to employ suitable upper and lower approximations to the normal itself (see e.g. Zelen and Severo, 1965, ch. 26); for our calculations we have employed the approximation used by Ramsay and Usabel, cit., plus and, respectively, minus the maximum error $7.5\ 10^{-8}$.

3. In all our tests the "Gauss-Seidel" version of the algorithm, described at the end of Section 1, showed a remarkably faster convergence (2 or 3 versus up to 12 iterations) than the original Rall's algorithm.

4. The present method may (obviously) be computationally intensive in case "small" step sizes are used. However, thanks to its very fast convergence, it is clear that the overall computational cost turns out to be related mainly to the number of evaluations of the kernel of the integral equation, as for any method based on discretizations, and not to its iterative nature.

On the basis of our experience we can thus conclude that, as far as upper and lower approximations are a goal, the proposed method indeed provides an easy and effective tool, allowing to treat cases where numerical two-sided bounds can not be obtained by the usual algorithms.

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RIASSUNTO

Un algoritmo iterativo per il calcolo di approssimazioni per difetto e per eccesso della probabilità di rovina

Nel presente lavoro viene proposto un algoritmo iterativo per il calcolo di approssimazioni per difetto e per eccesso della probabilità di rovina nell'ambito della teoria collettiva del rischio. Il metodo può essere impiegato con successo anche nel caso di distribuzioni per le quali gli algoritmi noti non possono essere utilizzati, come illustrato negli esempi presentati.

SUMMARY

A stable iterative algorithm for evaluation of upper and lower approximations to ultimate ruin probabilities

In this contribution we present an iterative algorithm for the calculation of two-sided numerical approximations to the probability of ultimate ruin for the classical risk model, which can be directly used in case of compounding assets as well. Examples involving claim size distributions for which the existing recursive algorithms are inherently unable to provide numerical upper and lower bounds are illustrated, showing the merits of the proposed approach.