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ON A BIVARIATE XGAMMA DISTRIBUTION DERIVED FROM COPULA

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1. Introduction

The classical probability distributions plays a significant role in the various fields of applied sciences like reliability, economics, medical sciences, and other related areas. The gamma and exponential distributions are widely used probability distributions for analyzing lifetime data. Various generalizations of the gamma and exponential distributions and their mixture have been proposed and studied in literature, which are successfully employed for modeling and explaining the various lifetime phenomena (see Johnson *et al.*, 1995; Sarhan and Kundu, 2009; Sen *et al.*, 2016). The classical distributions have limitations in dealing with a wide class of real data and provides motivation for construction of new families of flexible distributions.

In recent years, several techniques for constructing bivariate distributions using classical univariate distributions have been proposed. For the analysis of bivariate lifetime data, several distributions have been discussed which generalizes numerous popular univariate distributions such as exponential , Weibull, Pareto, gamma, and log-normal distribution (see, for example, Gumbel, 1960; Marshall and Olkin, 1967; Sankaran and Nair, 1993; Kundu and Gupta, 2009; Sarhan *et al.*, 2011). Construction of bivariate distributions based on conditional and marginals is an important technique and has been extensively studied in recent years. Recently, various noble methods for constructing bivariate distributions via order statistics have also been proposed and studied, which

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have absolutely continuous components as well as singular components and may be useful in situations where ties arise in the data. For some recent references, one can refer to Dolati *et al.* (2014), Mirhosseini *et al.* (2015), Kundu *et al.* (2017) and Pathak and Vellaisamy (2020). Besides existing techniques, copula models have recently been employed for modeling the dependence between random variables. A copula is a function which connects the marginals to the joint distribution and has been extensively used for modeling dependence among random variables with applications in finance, biology, engineering, hydrology, and geophysics. A copula is a multivariate distribution function whose one dimensional margins are uniform on unit interval $[0,1]$. In this paper, we restrict our study on a bivariate copula. A formal definition of the bivariate copula is as follows:

DEFINITION 1. *A function $C:[0,1]\times[0,1]\longrightarrow[0,1]$ is a bivariate copula if it satisfies the following properties:*

(i) For every $u,v\in[0,1]$

$$C(u,0)=0=C(0,v) \tag{1}$$

and

$$C(u,1)=1 \text{ and } C(1,v)=v. \tag{2}$$

(ii) For every $u_1,u_2,v_1,v_2\in[0,1]$ such that $u_1\leq u_2$ and $v_1\leq v_2$

$$C(u_2,v_2)-C(u_2,v_1)-C(u_1,v_2)+C(u_1,v_1)\geq 0. \tag{3}$$

Let X_1 and X_2 be random variables with joint distribution function F, and marginals F_1 and F_2, respectively, then Sklar (1959) says that there exists a copula function C which connects marginals to the joint distribution via the relation $F(x_1,x_2)=P(X_1\leq x_1,X_2\leq x_2)=C(F_1(x_1),F_2(x_2))$. If X_1 and X_2 are continuous, then the copula C is unique; otherwise it is uniquely determined on $\text{Range}(F_1)\times\text{Range}(F_2)$. The associated joint density is $f(x_1,x_2)=c(F_1(x_1),F_2(x_2))f_1(x_1)f_2(x_2)$, where c is copula density. The copula approach provides a powerful tool for constructing a large class of multivariate distributions based on marginals from different families. Any joint distribution function may be represented through copula in which dependence structure and marginals are separately specified. For a good source on copulas, one may refer to Nelsen (2006) and Joe (2014). Copula methods could be a flexible approach for constructing a large class of bivariate lifetime distributions with the ability to cope with different kinds of data and perceive the two lifetimes for the same patient. For example, it may be of interest in the study of human organs associated with kidney or eyes, and times between the first and second hospitalization for a particular disease (see Rinne, 2008; Bhattacharjee and Misra, 2016).

In the statistical literature, many authors used copula structure to construct a number of bivariate distributions to analyze lifetime data. Dos Santos and Achcar (2010) constructed bivariate Weibull distributions using different copula functions and discussed

Bayesian analysis with application in censored data. Kundu and Gupta (2011) proposed an absolute continuous bivariate generalized exponential distribution via simple transformation from exchangeable distribution. The proposed distribution can be easily derived from the Clayton copula with generalized exponential distribution marginals. Several statistical properties of the proposed distribution are discussed using copula techniques. A bivariate generalized exponential distribution based on Farlie-Gumbel-Morgenstern (FGM) copula has been proposed and studied by Achcar *et al.* (2015). Recently, Kundu and Gupta (2017) proposed the bivariate Birnbaum-Saunders distribution from Gaussian copula and investigated its several reliability and dependence properties. Abd Elaal and Jarwan (2017) considered bivariate generalized exponential distributions derived from FGM and Plackett copula functions and demonstrated their applications using real data sets. A bivariate modified Weibull distribution embedded by Peres *et al.* (2018) via FGM copula. Popović *et al.* (2018) discussed statistical properties of a bivariate Dagum distribution through copula. Nair *et al.* (2018) proposed a bivariate model for lifetime data analysis based on copula functions. Samanthi and Sepanski (2019) proposed a new bivariate extension of the beta-generated distributions using Archimedean copulas and discussed its applications in financial risk management. Shih *et al.* (2019) introduced a bivariate FGM copula model for bivariate meta-analysis and develop a maximum likelihood estimator for the common mean. de Oliveira Peres *et al.* (2020) proposed bivariate standard Weibull lifetime distributions using different copula functions and utilized them in real applications. Ota and Kimura (2021) discussed an effective algorithm for estimating the parameters of the multivariate FGM copula by using inference functions for the margins method. Several other bivariate distributions using copula have been proposed and studied in the literature. Some important references includes Saraiva *et al.* (2018), Taheri *et al.* (2018), Najarzadegan *et al.* (2019) and Almetwally *et al.* (2020).

The aim of this paper is to introduce a new bivariate XGamma (BXG) distribution and explore its various statistical properties with an application in real data. This paper is organized as follows. In Section 2, we review some basics of the univariate XGamma distribution. With the help of the univariate XGamma distribution, we define a new family of bivariate XGamma (BXG) distribution using the FGM copula. In Section 3, we derive the expressions for conditional density, conditional distribution, product moments, and regression function for the proposed BXG distribution. In Section 4, we present some concepts of reliability and obtain some measures of the local dependence and their important properties for the BXG distribution. We also established interconnection between various measures of dependence. In Section 5, we estimate parameters of the BXG distribution using maximum likelihood estimation and two-stage estimation procedures. Section 6, demonstrate data generation and several numerical experiments. Finally, an application to real data is demonstrated in Section 7.

2. Bivariate XGamma distribution

A continuous random variable X is said to follow an XGamma distribution with parameter θ, denoted by XG(θ), if its probability density function (pdf) is (Sen *et al.*, 2016)

$$f(x) = \frac{\theta^2}{1+\theta}\left(1+\frac{\theta}{2}x^2\right)e^{-\theta x}, \tag{4}$$

for $\theta > 0$ and $x > 0$.

The Xgamma distribution is a finite mixture of exponential and gamma distributions and may be an alternative of exponential distribution with wide applications in lifetime data. The associated cumulative distribution function (cdf) of X is given by

$$F(x) = 1 - \frac{\left(1+\theta+\theta x+\frac{\theta^2 x^2}{2}\right)e^{-\theta x}}{(1+\theta)}, \tag{5}$$

for $\theta > 0$ and $x > 0$.

If $X \sim$ XG(θ), then the hazard rate function is given by

$$h(x) = \frac{f(x)}{S(x)} = \frac{\theta^2\left(1+\frac{\theta}{2}x^2\right)}{\left(1+\theta+\theta x+\frac{\theta^2 x^2}{2}\right)}, \tag{6}$$

where $S(x) = 1 - F(x)$ is the survival function of the random variable X. The hazard rate function includes the two polynomials, hence is more flexible over the exponential distribution in analysis of lifetime data. One can easily show that $h(0)$ is constant. For $x > \sqrt{2/\theta}$, the hazard rate function $h(x)$ is Increasing Failure Rate (IFR), and for $x < \sqrt{2/\theta}$, the $h(x)$ is Decreasing Failure Rate (DFR) (see Sen *et al.*, 2018).

For the different values of the parameter θ, plots of the hazard rate function are shown in Figure 1. From the Figure, one can easily see that the XGamma distribution possesses the non-monotonic hazard rate function and may be useful in areas of medical sciences. The hazard rate of XGamma is more flexible than exponential distribution as its clear from Figure 1 and the hazard formula.

FGM copula is one of the most popular parametric families of copulas and has been widely used in literature due to its simple structure. Morgenstern (1956) proposed the FGM family, which was later studied by Gumbel (1958, 1960) using normal and exponential marginals. Farlie (1960) extended this family, derived its correlation structure and hence termed as the FGM family of distributions. The bivariate FGM copula is given by

$$C(u,v) = uv[1+\delta(1-u)(1-v)], \quad \delta \in [-1,1]. \tag{7}$$

In order to achieve the wider applications of the FGM copula in real applications, a large number of the generalized FGM copulas have been proposed and studied in literature. Some of the recent references includes Amblard and Girard (2009) and Pathak and Vellaisamy (2016a,b).

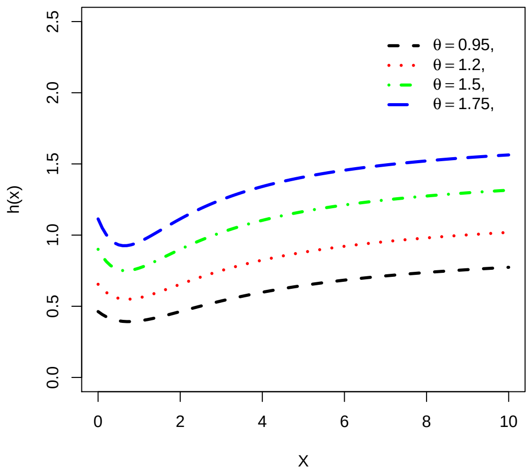

Figure 1 – Hazard rate function for XG(θ) for different values of the parameter θ.

The bivariate distribution determined by FGM copula is

$$F(x_1,x_2)=F_1(x_1)F_2(x_2)[1+\delta(1-F_1(x_1))(1-F_2(x_2))],\quad \delta\in[-1,1]. \tag{8}$$

A new family of bivarite XGamma distribution via FGM copula is given by

$$\begin{aligned}F_1(x_1,x_2)= &\left(1-\frac{\left(1+\theta_1+\theta_1x_1+\frac{\theta_1^2x_1^2}{2}\right)e^{-\theta_1x_1}}{(1+\theta_1)}\right)\left(1-\frac{\left(1+\theta_2+\theta_2x_2+\frac{\theta_2^2x_2^2}{2}\right)e^{-\theta_2x_2}}{(1+\theta_2)}\right)\\ &\left[1+\delta\left(\frac{\left(1+\theta_1+\theta_1x_1+\frac{\theta_1^2x_1^2}{2}\right)e^{-\theta_1x_1}}{(1+\theta_1)}\right)\left(\frac{\left(1+\theta_2+\theta_2x_2+\frac{\theta_2^2x_2^2}{2}\right)e^{-\theta_2x_2}}{(1+\theta_2)}\right)\right].\end{aligned} \tag{9}$$

A random vector (X_1,X_2) is said to have a bivarite XGamma distribution with parameters θ_1, θ_2 and δ if its distribution function is given by Eq. (9). It is denoted by BXG(θ_1,θ_2,δ). This family includes a mixture of exponential and gamma distributions and may be useful in a wide class of real data.

The joint density of the bivariate XGamma distribution $f(x_1,x_2)$ defined in Eq. (9) is

$$f(x_1,x_2)=\frac{\theta_1^2\theta_2^2}{(1+\theta_1)(1+\theta_2)}\left(1+\frac{\theta_1}{2}x_1^2\right)\left(1+\frac{\theta_2}{2}x_2^2\right)e^{-(\theta_1x_1+\theta_2x_2)}$$

$$\times\left[1+\delta\left(\frac{2\left(1+\theta_1+\theta_1x_1+\frac{\theta_1^2x_1^2}{2}\right)e^{-\theta_1x_1}}{(1+\theta_1)}-1\right)\left(\frac{2\left(1+\theta_2+\theta_2x_1+\frac{\theta_2^2x_2^2}{2}\right)e^{-\theta_2x_2}}{(1+\theta_2)}-1\right)\right]. \tag{10}$$

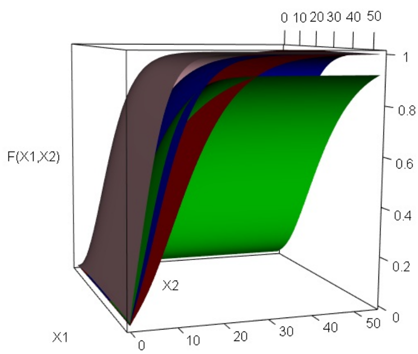

Figure 2 – CDF BXG distribution.

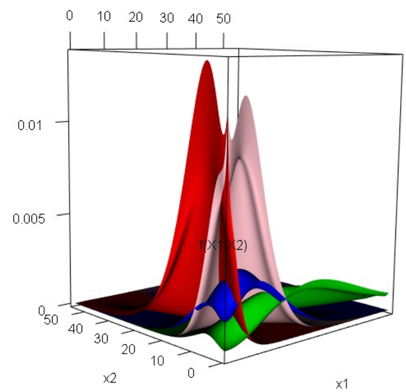

Figure 3 – PDF BXG distribution.

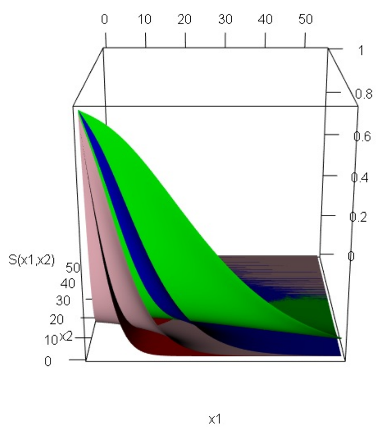

Figure 4 – Survival BXG distribution.

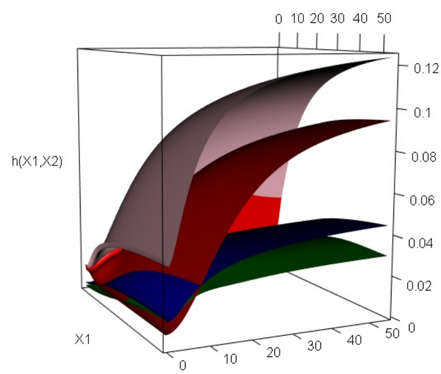

Figure 5 – Hazard BXG distribution.

Figures 2, 3, 4 and 5 demonstrate three dimension graphs for the cdf, pdf, survival function and hazard function of the BXG distribution given in Eq. (9) for different values of the parameter θ and δ. In graphs, we have taken $\theta_1 = 0.1$ (pink), 0.2 (red), 0.3 (blue), 0.4 (green), $\theta_2 = 0.5$ (pink), 0.2 (red), 0.4 (blue), 0.3 (green) and for copula parameter $\delta = 0.1$ (pink), 0.7 (red), -0.1 (blue), -0.7 (green).

3. Various properties of the XBG distribution

In this Section, we derive the expressions for some important properties of the BXG distribution. We obtain the expression for marginals, conditional distribution, regression function, product moments, and coefficient of correlation for the BXG distribution. The following result can be easily derived.

Theorem 2. *Let $(X_1, X_2) \sim BXG(\theta_1, \theta_2, \delta)$. Then*

(i) the $X_1 \sim XG(\theta_1)$ and $X_2 \sim XG(\theta_2)$,

(ii) the conditional density of X_1 given $X_2 = x_2$ is

$$f(x_1|x_2) = \frac{\theta_1^2}{1+\theta_1}\left(1+\frac{\theta_1}{2}x_1^2\right)e^{-\theta_1 x_1}$$

$$\times\left[1+\delta\left[\frac{2\left(1+\theta_1+\theta_1 x_1+\frac{\theta_1^2 x_1^2}{2}\right)e^{-\theta_1 x_1}}{(1+\theta_1)}-1\right]\left[\frac{2\left(1+\theta_2+\theta_2 x_1+\frac{\theta_2^2 x_2^2}{2}\right)e^{-\theta_2 x_2}}{(1+\theta_2)}-1\right]\right], \tag{11}$$

(iii) the conditional distribution of X_1 given $X_2 = x_2$ is

$$F(x_1|x_2) = \left(1-\frac{\left(1+\theta_2+\theta_2 x_2+\frac{\theta_2^2 x_2^2}{2}\right)e^{-\theta_2 x_2}}{(1+\theta_2)}\right)$$

$$\times\left[1+\delta\left(1-\frac{(1+\theta_2+\theta_2 x_2+\frac{\theta_2^2 x_2^2}{2})e^{-\theta_2 x_2}}{(1+\theta_2)}\right)\left(\frac{2\left(1+\theta_1+\theta_1 x_1+\frac{\theta_1^2 x_1^2}{2}\right)e^{-\theta_1 x_1}}{(1+\theta_1)}-1\right)\right], \tag{12}$$

(iv) the conditional survival function of X_1 given $X_2 = x_2$ is

$$S(x_1|x_2) = \frac{\left(1+\theta_2+\theta_2 x_2+\frac{\theta_2^2 x_2^2}{2}\right)e^{-\theta_2 x_2}}{(1+\theta_2)} + \delta\left[\frac{\left(1+\theta_2+\theta_2 x_2+\frac{\theta_2^2 x_2^2}{2}\right)e^{-\theta_2 x_2}}{(1+\theta_2)}-1\right]\times$$
$$\left(1-\frac{(1+\theta_2+\theta_2 x_2+\frac{\theta_2^2 x_2^2}{2})e^{-\theta_2 x_2}}{(1+\theta_2)}\right)\left(\frac{2\left(1+\theta_1+\theta_1 x_1+\frac{\theta_1^2 x_1^2}{2}\right)e^{-\theta_1 x_1}}{(1+\theta_1)}-1\right). \tag{13}$$

In the next results, we obtain the expression of regression function and product moments for the BXG distribution.

THEOREM 3. *If* $(X_1, X_2) \sim BXG(\theta_1, \theta_2, \delta)$, *then the regression function of* X_1 *on* $X_2 = x_2$ *is given by*

$$E(X_1|X_2 = x_2) = \frac{e^{-\theta_2 x_2}}{16\theta_1(1+\theta_1)^2(1+\theta_2)}\Big[16\big(3+4\theta_1+\theta_1^2\big)(1+\theta_2)e^{\theta_2 x_2} + \\ +\big(15+36\theta_1+8\theta_1^2\big)\delta\Big(-2+e^{\theta_2 x_2}-\theta_2^2 x_2^2+\theta_2\big(e^{\theta_2 x_2}-2(1+x_2)\big)\Big)\Big]. \tag{14}$$

PROOF. The proof is given in Appendix. □

THEOREM 4. *If* $(X_1, X_2) \sim BXG(\theta_1, \theta_2, \delta)$, *then the product moment of* (r, s)*-th order is*

$$\frac{2^{-12-r-s}\Gamma(1+r)\Gamma(1+s)}{(1+\theta_1)^2(1+\theta_2)^2\theta_1^r\theta_2^s}\Big[A\delta+\big[-64(-1+2^r)\theta_1^2+(1+r)(2+r)\big(-32(-1+2^r) \\ +r(11+r)\big)+16\theta_1\big(8+5r+r^2-2^{1+r}(4+r(3+r))\big)\big]+ \\ +2^{5+r}(1+\theta_2)B\big[2^{5+r}(1+\theta_1)(1+\delta)\big(2\theta_1+(1+r)(2+r)\big)+ \\ -\delta\big(64\theta_1^2+16\theta_1(8+r(5+r))+(1+r)(2+r)(32+r(11+r))\big)\big]\Big], \tag{15}$$

where $A = \Big(64\theta_2^2 + 16\theta_2\big(8+s(5+s)\big) + (1+s)(2+s)\big(32+s(11+s)\big)\Big)$ *and* $B = \big(2\theta_2 + (1+s)(2+s)\big)$.

PROOF. The proof is given in Appendix. □

REMARK 5. *The following observations can be made.*

(i) For $r = s = 1$, *Eq. (15) reduces to*

$$E(X_1X_2) = \frac{(4\theta_1(2\theta_1+9)+15)(4\theta_2(2\theta_2+9)+15)\delta+256(\theta_1+1)(\theta_1+3)(\theta_2+1)(\theta_2+3)}{256\theta_1(\theta_1+1)^2\theta_2(\theta_2+1)^2}. \tag{16}$$

(ii) Consider $X_1 \sim XG(\theta_1)$, *then its* r*-th moment about the origin is given by (see* Sen et al., *2016)*

$$E(X_1^r) = \frac{r!(\theta_1+r+b_r)}{\theta_1^r(1+\theta_1)}, \tag{17}$$

where $b_r = b_{r-1} + r$, *with* $r = 1, 2, \cdots$ *and initial values* $b_0 = 0$ *and* $b_1 = 2$. *A similar expression for* X_2 *can also be derived.*

(iii) For $r = s = 1$ with the help of Eq. (15) and Eq. (17) trough simple algebra, the coefficient of correlation for the BXG distribution is given by

$$\rho = \frac{(4\theta_1(2\theta_1+9)+15)(4\theta_2(2\theta_2+9)+15)\delta}{256\theta_1(\theta_1+1)^2\theta_2(\theta_2+1)^2\left(\frac{(\theta_1(\theta_1+8)+3)(\theta_2(\theta_2+8)+3)}{\theta_1^2(\theta_1+1)^2\theta_2^2(\theta_2+1)^2}\right)^{1/2}}. \tag{18}$$

The graphs for the correlation coefficient are illustrated in Figure 6 for the following values of θ_1 and θ_2: (i) $\theta_1 = 0.5$ and $\theta_2 = 0.9$, (ii) $\theta_1 = 1.2$ and $\theta_2 = 3.8$ (iii) $\theta_1 = 2.5$ and $\theta_2 = 4.5$ (iv) $\theta_1 = 7.5$ and $\theta_2 = 9.5$.

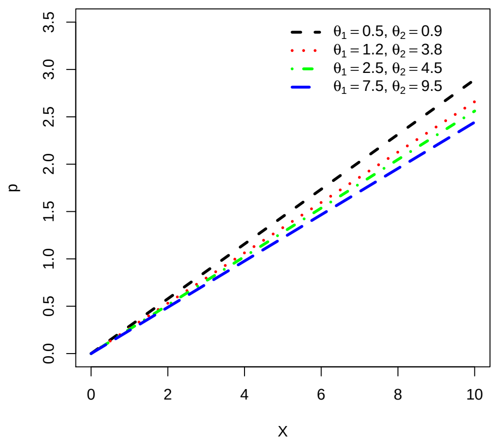

Figure 6 – The correlation cofficient ρ of θ_1 and θ_2 as a function of δ.

4. Reliability and dependence

Let (X_1, X_2) be a bivariate random vector with joint density $f(x_1, x_2)$ and survival function $S(x_1, x_2) = P(X_1 > x_1, X_2 > x_2)$. Then the bivariate hazard rate function is defined as (see Basu, 1971)

$$h(x_1, x_2) = \frac{f(x_1, x_2)}{S(x_1, x_2)}. \tag{19}$$

If $(X_1, X_2) \sim \text{BXG}(\theta_1, \theta_2, \delta)$, then we have $h(x_1, x_2)$

$$h(x_1,x_2)=\frac{\theta_1^2\left(\theta_1 {x_1}^2+2\right)\theta_2^2\left(\theta_2 x_2^2+2\right)\left[\delta(\theta_1-2\mathrm{A}+1)(\theta_2-2\mathrm{B}+1)+(\theta_1+1)(\theta_2+1)\right]}{4\mathrm{AB}\,e^{\theta_1 x_1+\theta_2 x_2}\left[\delta(\theta_1-\mathrm{A}+1)(\theta_2-\mathrm{B}+1)+(\theta_1+1)(\theta_2+1)\right]}, \tag{20}$$

where $\mathrm{A}=e^{-\theta_1 x_1}\left(\frac{\theta_1^2 x_1^2}{2}+\theta_1 x_1+\theta_1+1\right)$ and $\mathrm{B}=e^{-\theta_2 x_2}\left(\frac{\theta_2^2 x_2^2}{2}+\theta_2 x_2+\theta_2+1\right)$.

4.1. Hazard gradient functions

Consider a bivariate random vector (X_1, X_2) with joint density $f(x_1, x_2)$ and survival function $S(x_1, x_2)$, then the hazard components are defined as (see Johnson and Kotz, 1975)

$$\eta_1(x_1,x_2)=-\frac{\partial}{\partial x_1}\ln S(x_1,x_2) \tag{21}$$

and

$$\eta_2(x_1,x_2)=-\frac{\partial}{\partial x_2}\ln S(x_1,x_2). \tag{22}$$

The vector $(\eta_1(x_1,x_2), \eta_2(x_1,x_2))$ is termed hazard gradient of a bivariate random vector (X_1, X_2). Hence, for the BXG distribution the hazard gradient is

$$\eta_1(x_1,x_2)=-\frac{\partial}{\partial x_1}\ln S(x_1,x_2) \tag{23}$$

and

$$\eta_1(x_1,x_2)=\frac{2\theta_1^2\left(\theta_1 x_1^2+2\right)}{(\theta_1(x_1(\theta_1 x_1+2)+2)+2)}\times\frac{\left[2(\theta_1+1)(\theta_2+1)e^{\theta_1 x_1+\theta_2 x_2}+\delta\left(-\theta_1\left(x_1(\theta_1 x_1+2)-e^{\theta_1 x_1}+2\right)+e^{\theta_1 x_1}-2\right)A\right]}{4(\theta_1+1)(\theta_2+1)e^{\theta_1 x_1+\theta_2 x_2}+\left[\delta\left(-\theta_1^2 x_1^2-2\theta_1 x_1+2(\theta_1+1)e^{\theta_1 x_1}-2\theta_1-2\right)A\right]}, \tag{24}$$

where $A=\left(-\theta_2^2 x_2^2-2\theta_2 x_2+2(\theta_2+1)e^{\theta_2 x_2}-2\theta_2-2\right)$.

Moreover, we have

$$\eta_2(x_2,x_1)=\frac{2\theta_2^2\left(\theta_2 x_2^2+2\right)}{(\theta_2(x_2(\theta_2 x_2+2)+2)+2)}\times\frac{\left[2(\theta_1+1)(\theta_2+1)e^{\theta_1 x_1+\theta_2 x_2}+\delta\left(-\theta_2\left(x_2(\theta_2 x_2+2)-e^{\theta_2 x_2}+2\right)+e^{\theta_2 x_2}-2\right)A\right]}{4(\theta_1+1)(\theta_2+1)e^{\theta_1 x_1+\theta_2 x_2}+\left[\delta\left(-\theta_2^2 x_2^2-2\theta_2 x_2+2(\theta_2+1)e^{\theta_2 x_2}-2\theta_2-2\right)A\right]}, \tag{25}$$

where $A=\left(-\theta_1^2 x_1^2-2\theta_1 x_1+2(\theta_1+1)e^{\theta_1 x_1}-2\theta_1-2\right)$.

In next Sections, we discuss some measures of the local dependences for the BXG distribution and discuss its important properties.

4.2. *A local dependence function $\gamma(x_1, x_2)$*

In order to study the dependence between random variables X_1 and X_2, Holland and Wang (1987) proposed a local dependence function $\gamma(x_1, x_2)$ and is defined as

$$\gamma(x_1, x_2) = \frac{\partial^2}{\partial x_1 \partial x_2} \ln f(x_1, x_2). \tag{26}$$

This dependence function provides a powerful tool to study the totally positive of order 2 (TP2) property of a bivariate distribution. A detailed properties of the $\gamma(x_1, x_2)$ has been studied in Holland and Wang (1987) and Balakrishnan and Lai (2009).

PROPOSITION 6. *Let $(X_1, X_2) \sim BXG(\theta_1, \theta_2, \delta)$. Then*

$$\gamma(x_1, x_2) = \frac{\theta_1^2(\theta_1+1)\theta_2^2(\theta_2+1)\delta\left(\theta_1 x_1^2+2\right)\left(\theta_2 x_2^2+2\right)e^{\theta_1 x_1+\theta_2 x_2}}{\left[\delta\left(\theta_1\left[B-e^{\theta_1 x_1}+2\right]-e^{\theta_1 x_1}+2\right)\left(\theta_2\left[C-e^{\theta_2 x_2}+2\right]-e^{\theta_2 x_2}+2\right)+A\right]^2}, \tag{27}$$

where $A = (1+\theta_1)(1+\theta_2)e^{\theta_1 x_1+\theta_2 x_2}$, $B = x_1(\theta_1 x_1 + 2)$ and $C = x_2(\theta_2 x_2 + 2)$.

It may be noticed that, when $\delta = 0$, then $\gamma(x_1, x_2) = 0$, which leads to the independence of X_1 and X_2. Holland and Wang (1987) established that a bivariate density $f(x_1, x_2)$ will possesses the TP2 property if and only if $\gamma(x_1, x_2) \geq 0$.

Then, we have the following result:

PROPOSITION 7. *Let $(X_1, X_2) \sim BXG(\theta_1, \theta_2, \delta)$. Then for $\delta \geq 0$ the density $f(x_1, x_2)$ given in Eq. (10) is TP2.*

It is important to note that TP2 is a stronger concept of dependence. It has been already studied that TP2 implies the stochastically increasing (SI), right-tail increasing (RTI), association, positive quadrant dependence (PQD) (see Nelsen, 2006; Balakrishnan and Lai, 2009). Hence, for $0 \leq \delta \leq 1$, the BXG distribution have all these properties of dependence.

4.3. *Clayton-Oakes association measure*

Clayton (1978) and Oakes (1989) discussed a local dependence function via survival function which is defined as:

$$l(x_1, x_2) = \frac{f(x_1, x_2)S(x_1, x_2)}{S_1(x_1, x_2)S_2(x_1, x_2)}, \tag{28}$$

where $S_1(x_1, x_2) = \frac{\partial}{\partial x_1} S(x_1, x_2)$ and $S_1(x_1, x_2) = \frac{\partial}{\partial x_2} S(x_1, x_2)$.

PROPOSITION 8. *Let* $(X_1, X_2) \sim BXG(\theta_1, \theta_2, \delta)$. *Then*

$$l_1(x_1,x_2) = \frac{\left(4(\theta_1+1)(\theta_2+1)e^{\theta_1 x_1+\theta_2 x_2}+AB\delta\right)\left((\theta_1+1)(\theta_2+1)e^{\theta_1 x_1+\theta_2 x_2}+\delta CD\right)}{\left(2(\theta_1+1)(\theta_2+1)e^{\theta_1 x_1+\theta_2 x_2}+A\delta D\right)\left(2(\theta_1+1)(\theta_2+1)e^{\theta_1 x_1+\theta_2 x_2}+B\delta C\right)}, \tag{29}$$

where
$A = -\theta_1^2 x_1^2 - 2\theta_1 x_1 + 2(\theta_1+1)e^{\theta_1 x_1} - 2\theta_1 - 2,$
$B = -\theta_2^2 x_2^2 - 2\theta_2 x_2 + 2(\theta_2+1)e^{\theta_2 x_2} - 2\theta_2 - 2,$
$C = -\theta_1\left(x_1(\theta_1 x_1 + 2) - e^{\theta_1 x_1} + 2\right) + e^{\theta_1 x_1} - 2$ *and*
$D = -\theta_2\left(x_2(\theta_2 x_2 + 2) - e^{\theta_2 x_2} + 2\right) + e^{\theta_2 x_2} - 2.$

It can be easily verified that for $l_1(x_1,x_2) = 1$, the random variables X_1 and X_2 are independent. From Eq. (29), the random variables X_1 and X_2 are independent for $\delta = 0$.

4.4. *Conditional probability measure* ψ

Anderson *et al.* (1992) defined a measure of association using conditional probability as follows:

$$\psi(x_1,x_2) = \frac{P(X_1 > x_1 | X_2 > x_2)}{P(X_1 > x_1)} = \frac{S(x_1,x_2)}{S(x_1,0)S(0,x_2)}. \tag{30}$$

We say that the random variable X_1 and X_2 are independent if and only if $\psi(x_1,x_2) = 1$ and PQD if $\psi(x_1,x_2) > 1$ for all (x_1,x_2). For the BXG distribution, we have the following result.

PROPOSITION 9.

$$\psi_1(x_1,x_2) = \frac{e^{-(\theta_1 x_1+\theta_2 x_2)}\left(\delta AB + 4(\theta_1+1)(\theta_2+1)e^{\theta_1 x_1+\theta_2 x_2}\right)}{4(\theta_1+1)(\theta_2+1)}, \tag{31}$$

where
$A = -\theta_1^2 x_1^2 - 2\theta_1 x_1 + 2(\theta_1+1)e^{\theta_1 x_1} - 2\theta_1 - 2$ *and*
$B = -\theta_2^2 x_2^2 - 2\theta_2 x_2 + 2(\theta_2+1)e^{\theta_2 x_2} - 2\theta_2 - 2$

REMARK 10. *From Eq. (31), we see that when* $\delta = 0$, *we get* $\psi(x_1,x_2) = 1$. *Hence,* X_1 *and* X_2 *are independent. Similarly, for* $\delta > 0$, X_1 *and* X_2 *are PQD.*

5. PARAMETER ESTIMATION OF THE BXG DISTRIBUTION

In this Section, we obtain the estimates of the unknown parameters θ_1, θ_1, and δ for the BXG$(\theta_1,\theta_2,\delta)$. We adopt the two-stage estimation (inference function of margins(IFM)) procedure proposed by Joe and Xu (1996) to estimate the parameters of the BXG distribution. In order to obtain the maximum likelihood estimates (MLE) for the BXG distribution, the likelihood function involves three unknown parameters and is complicated in nature. The two-stage estimation procedure simplifies the computational difficulties and also is convenient in dealing with different copula models. In the first step, we obtain the estimate of the parameters θ_1 and θ_2 of the univariate marginal from the marginal log-likelihood, and in the second step we plug-in the estimates of the marginals into the copula log-likelihood and obtain the estimate of the dependence parameter δ. The log-likelihood function of the BXG model can be written as

$$\ln \mathrm{L}(\theta_1,\theta_2,\delta)=\sum_{i=1}^{n}\ln f_1(x_1,\theta_1)+\sum_{i=1}^{n}\ln f_2(x_2,\theta_2)+\sum_{i=1}^{n}\ln c(F_1(x_1;\theta_1),F_2(x_2;\theta_2),\delta). \tag{32}$$

In the first step, we obtain the estimates of θ_1 and θ_2 by differentiating Eq. (32) partially with respect to θ_1 and θ_2 and equating it to zero as follows:

$$\begin{aligned} L(\theta_j) &= \Pi_{i=1}^{n}\frac{\theta_j^2}{1+\theta_j}\left(1+\frac{\theta_j}{2}x_ij^2\right)e^{-\theta_j x_i j} \\ &= \frac{\theta_j^{2n}}{(1+\theta_j)^n}\sum_{i=1}^{n}\left(1+\frac{\theta_j}{2}x_{ij}^2\right)e^{-\theta_j \sum_{i=1}^{n} x_i j} \quad \text{with} \quad j=1,2; \end{aligned} \tag{33}$$

$$\ln L(\theta_j)=2n\ln(\theta_j)-n\ln(1+\theta_j)+\sum_{i=1}^{n}\ln\left(1+\frac{\theta_j}{2}x_{ij}^2\right)-\theta_j\sum_{i=1}^{n}x_{ij}; \tag{34}$$

$$\frac{\partial \ln L(\theta_j)}{\partial\theta_j}=\frac{2n}{\theta_j}-\frac{n}{(1+\theta_j)}+\sum_{i=1}^{n}\frac{x_{ij}^2}{(2+x_{ij}^2\theta_j)}-\sum_{i=1}^{n}x_{ij}=0 \quad \text{with} \quad j=1,2. \tag{35}$$

The MLE $(\widehat{\theta}_1,\widehat{\theta}_2)$ can be obtained by solving simultaneously the likelihood equations, the estimate of $\widehat{\theta}_1,\widehat{\theta}_2$ are handled numerically through statistical software using iterative method.

$$\hat{\theta}_j=\frac{2n}{\frac{n}{(1+\theta_j)}+\sum_{i=1}^{n}\frac{x_{ij}^2}{(2+x_{ij}^2\theta_j)}+\sum_{i=1}^{n}x_{ij}} \quad \text{with} \quad j=1,2. \tag{36}$$

In the second step, the copula parameter is estimated by using the $\widehat{\theta}_1$ and $\widehat{\theta}_2$ from the first step. For the FGM copula

$$\begin{aligned} L_1=\sum_{i=1}^{n}\ln c(F_1(x_1;\widehat{\theta}_1),F_2(x_2;\widehat{\theta}_2),\delta)=\sum_{i=1}^{n}\ln\Bigg[1+\delta\left(\frac{2(1+\widehat{\theta}_1+\widehat{\theta}_1x_1+\frac{\widehat{\theta}_1^2x_1^2}{2})e^{-\widehat{\theta}_1x_1}}{1+\widehat{\theta}_1}-1\right) \\ \times\left(\frac{2(1+\widehat{\theta}_2+\widehat{\theta}_2x_2+\frac{\widehat{\theta}_2^2x_2^2}{2})e^{-\widehat{\theta}_2x_2}}{1+\widehat{\theta}_2}-1\right)\Bigg] \end{aligned} \tag{37}$$

$$\frac{\partial L_1}{\partial \delta} = \sum_{i=1}^{n} \frac{\left(\frac{e^{-\theta_1 x_1}(\theta_1^2 x_1^2 + 2\theta_1 x_1 + 2\theta_1 + 2)}{1+\theta_1} - 1\right)\left(\frac{e^{-\theta_2 x_2}(\theta_2^2 x_2^2 + 2\theta_2 x_2 + 2\theta_2 + 2)}{1+\theta_2} - 1\right)}{1 + \delta\left(\frac{e^{-\theta_1 x_1}(\theta_1^2 x_1^2 + 2\theta_1 x_1 + 2\theta_1 + 2)}{1+\theta_1} - 1\right)\left(\frac{e^{-\theta_2 x_2}(\theta_2^2 x_2^2 + 2\theta_2 x_2 + 2\theta_2 + 2)}{1+\theta_2} - 1\right)} = 0. \quad (38)$$

Since there is no closed expression in Eq. (38), the estimate of the parameter δ is obtained numerically using a non-linear optimization algorithm. By considering the full log-likelihood function in Eq. (32), we obtain the MLEs of the parameters θ_1, θ_2 and δ via usual numerical optimization techniques.

6. SIMULATION STUDY

In this Section, we report simulation study for the BXG distribution derived using FGM copula. First, we describe the random sample generation from BXG distribution. We employed the conditional procedure for random sample generation which has been reported in Nelsen (2006). Let X_1, X_2 be a random sample having BXG distribution determined by FGM copula C. The copula C is a joint distribution of a bivariate vector (U, V) with marginals as uniform $U(0,1)$. The conditional distribution of the vector (U, V) is given as $P(V \leq v | U = u) = \frac{\partial}{\partial u} C_1(u, v) = v[1 + \delta(1 - v)(1 - 2u)]$. Using the conditional distribution approach, random numbers (x_1, x_2) from the BXG can be generated using the following algorithm:

1. From uniform $U(0,1)$ generate two independent sample u and t.
2. Set $t = \frac{\partial}{\partial u} C(u, v)$ and solved for v.
3. Find $x_1 = F^{-1}(u; \theta_1)$ and $x_2 = F^{-1}(v; \theta_2)$; where F^{-1} is the inverse of XGamma.
4. Finally, the desired random sample is (x_1, x_2).

For parameters estimation we used two-stage estimation and maximum likelihood methods. For two-stage estimation, we considered the Inference Functions for Margins (IFM) technique as discussed by Joe and Xu (1996) in which, we first estimate marginals parameter and in the second step, plug these estimators into the log-likelihood and obtain the maximum likelihood estimate of the copula parameter. A simulation study was carried out based on the following data generated from BXG distribution. The value of the parameters θ_1 and θ_2 is chosen with different value of the copula parameter δ and different sizes of sample (n = 15, 30, 50, 100), as shown for the following cases for the random variables generating from BXG distribution:

Case 1: $\theta_1 = 0.5, \theta_2 = 0.5, \delta = -0.3$,

Case 2: $\theta_1 = 1, \theta_2 = 1.5, \delta = 0.3$,

Case 3: $\theta_1 = 2.5, \theta_2 = 2, \delta = 0.7$,

Case 4: $\theta_1 = 3, \theta_2 = 2.5, \delta = -0.7$.

The simulations in this study were repeated 10,000 times. The estimate of parameters by the two-stage (IFM) and MLE methods along with the mean squared error (MSE) and confidence interval (CI) are summarized in Table 1 to 8. From the reported Tables, we concluded the following. In the simulation study, if the sample size increases, the value of mean square error and length of the confidence interval decreases in both the considered methods i.e. IFM and MLE. In the simulations Tables, when the initial value of parameters increases, the corresponding mean square error increases for the small sample, and after that it decreases gradually by increasing the sample size as observed from the corresponding MSE for different values of the parameters. In general, the effect of marginal parameters has a little effect on estimating the copula parameters as shown in the Table. There is no substantive difference between the two methods IFM and MLE, based on the MSE criterion. The simulation study was carried out using the R software (R 3.5.3).

TABLE 1
Simulation study of the parameters of BXG distribution (Case 1 - IFM).

n	Parameter	Estimate	MSE	Lower CI	Upper CI
15	$\theta_1(0.5)$	0.5179	0.0082	0.3724	0.7146
	$\theta_2(0.5)$	0.5151	0.0083	0.3740	0.7195
	$\delta(-0.3)$	-0.4669	0.1048	-0.9655	-0.0240
30	$\theta_1(0.5)$	0.5073	0.0037	0.4034	0.6449
	$\theta_2(0.5)$	0.5091	0.0037	0.4064	0.6427
	$\delta(-0.3)$	-0.4549	0.0963	-0.9540	-0.0268
50	$\theta_1(0.5)$	0.5037	0.0021	0.4227	0.6045
	$\theta_2(0.5)$	0.5053	0.0021	0.4229	0.6059
	$\delta(-0.3)$	-0.4241	0.0816	-0.9364	-0.0221
100	$\theta_1(0.5)$	0.5023	0.0010	0.4420	0.5686
	$\theta_2(0.5)$	0.5022	0.0010	0.4438	0.5724
	$\delta(-0.3)$	-0.3766	0.0573	-0.8720	-0.0224

7. Application to real data

To demonstrate a real application of the considered BXG distribution, we consider the UEFA Champion's League data set reported in Meintanis (2007). The considered data represents the time (in minutes) of the first kick goal scored by any team (X_1), and the

TABLE 2
Simulation study of the parameters of BXG distribution (Case 2 - IFM).

n	Parameter	Estimate	MSE	Lower CI	Upper CI
15	$\theta_1(1)$	1.0403	0.0398	0.7414	1.4912
	$\theta_2(1.5)$	1.5754	0.1098	1.0832	2.3483
	$\delta(0.3)$	0.4698	0.1056	0.0278	0.9620
30	$\theta_1(1)$	1.0199	0.0176	0.7947	1.3157
	$\theta_2(1.5)$	1.5338	0.0453	1.1796	2.0140
	$\delta(0.3)$	0.4544	0.0974	0.0264	0.9497
50	$\theta_1(1)$	1.0117	0.0104	0.8351	1.2284
	$\theta_2(1.5)$	1.5215	0.0256	1.2478	1.8630
	$\delta(0.3)$	0.4211	0.0791	0.0248	0.9308
100	$\theta_1(1)$	1.0046	0.0048	0.8825	1.1521
	$\theta_2(1.5)$	1.5052	0.0119	1.3086	1.7418
	$\delta(0.3)$	0.3838	0.0571	0.0274	0.8568

TABLE 3
Simulation study of the parameters of BXG distribution (Case 3 - IFM).

n	Parameter	Estimate	MSE	Lower CI	Upper CI
15	$\theta_1(2.5)$	2.6452	0.3360	1.7854	3.9522
	$\theta_2(2)$	2.0989	0.2085	1.4325	3.1452
	$\delta(0.7)$	0.5229	0.1087	0.0279	0.9781
30	$\theta_1(2.5)$	2.5672	0.1532	1.9347	3.4446
	$\theta_2(2)$	2.0599	0.0892	1.5752	2.7164
	$\delta(0.7)$	0.5494	0.0951	0.0457	0.9764
50	$\theta_1(2.5)$	2.5395	0.0826	2.0628	3.1734
	$\theta_2(2)$	2.0276	0.0474	1.6482	2.4718
	$\delta(0.7)$	0.5838	0.0779	0.0687	0.9745
100	$\theta_1(2.5)$	2.5169	0.0394	2.1656	2.9358
	$\theta_2(2)$	2.0099	0.0226	1.7300	2.3191
	$\delta(0.7)$	0.6358	0.0521	0.1556	0.9753

TABLE 4
Simulation study of the parameters of BXG distribution (Case 4 - IFM).

n	Parameter	Estimate	MSE	Lower CI	Upper CI
15	$\theta_1(3)$	3.1746	0.5268	2.1264	4.8756
	$\theta_2(2.5)$	2.6320	0.3517	1.7722	3.9734
	$\delta(-0.7)$	-0.5258	0.1064	-0.9762	-0.0375
30	$\theta_1(3)$	3.0822	0.2237	2.3159	4.1160
	$\theta_2(2.5)$	2.5659	0.1492	1.9563	3.4312
	$\delta(-0.7)$	-0.5605	0.0900	-0.9746	-0.0429
50	$\theta_1(3)$	3.0468	0.1258	2.4577	3.8250
	$\theta_2(2.5)$	2.5423	0.0850	2.0454	3.1893
	$\delta(-0.7)$	-0.5856	0.0786	-0.9766	-0.0618
100	$\theta_1(3)$	3.0212	0.0617	2.5743	3.5477
	$\theta_2(2.5)$	2.5195	0.0400	2.1662	2.9513
	$\delta(-0.7)$	-0.6319	0.0532	-0.9724	-0.1562

TABLE 5
Simulation study of the parameters of BXG distribution (Case 1 - MLE).

n	Parameter	Estimate	MSE	Lower CI	Upper CI
15	$\theta_1(0.5)$	0.5169	0.0084	0.3717	0.7212
	$\theta_2(0.5)$	0.5146	0.0082	0.3722	0.7110
	$\delta(-0.3)$	-0.4658	0.1034	-0.9641	-0.0246
30	$\theta_1(0.5)$	0.5080	0.0036	0.4047	0.6409
	$\theta_2(0.5)$	0.5064	0.0037	0.4030	0.6398
	$\delta(-0.3)$	-0.4548	0.0980	-0.9547	-0.0235
50	$\theta_1(0.5)$	0.5040	0.0022	0.4219	0.6090
	$\theta_2(0.5)$	0.5044	0.0021	0.4234	0.6060
	$\delta(-0.3)$	-0.4261	0.0842	-0.9411	-0.0231
100	$\theta_1(0.5)$	0.5021	0.0010	0.4441	0.5704
	$\theta_2(0.5)$	0.5033	0.0010	0.4431	0.5714
	$\delta(-0.3)$	-0.3826	0.0577	-0.8541	-0.0231

TABLE 6
Simulation study of the parameters of BXG distribution (Case 2 - MLE).

n	Parameter	Estimate	MSE	Lower CI	Upper CI
15	$\theta_1(1)$	1.0426	0.0422	0.7347	1.5168
	$\theta_2(1.5)$	1.5762	0.1070	1.0953	2.3218
	$\delta(0.3)$	0.4705	0.1069	0.0226	0.9669
30	$\theta_1(1)$	1.0198	0.0176	0.7997	1.3099
	$\theta_2(1.5)$	1.5319	0.0430	1.1812	1.9848
	$\delta(0.3)$	0.4522	0.0964	0.0233	0.9515
50	$\theta_1(1)$	1.0097	0.0104	0.8336	1.2304
	$\theta_2(1.5)$	1.5209	0.0256	1.2520	1.8618
	$\delta(0.3)$	0.4222	0.0810	0.0227	0.9284
100	$\theta_1(1)$	1.0079	0.0048	0.8801	1.1501
	$\theta_2(1.5)$	1.5083	0.0118	1.3117	1.7402
	$\delta(0.3)$	0.3795	0.0575	0.0269	0.8583

TABLE 7
Simulation study of the parameters of BXG distribution (Case 3 - MLE).

n	Parameter	Estimate	MSE	Lower CI	Upper CI
15	$\theta_1(2.5)$	2.6537	0.3496	1.7802	4.0218
	$\theta_2(2)$	2.1142	0.2146	1.4524	3.1685
	$\delta(0.7)$	0.5141	0.1112	0.0327	0.9721
30	$\theta_1(2.5)$	2.5645	0.1507	1.9399	3.4182
	$\theta_2(2)$	2.0537	0.0879	1.5704	2.7214
	$\delta(0.7)$	0.5523	0.0939	0.0425	0.9746
50	$\theta_1(2.5)$	2.5398	0.0815	2.0685	3.1789
	$\theta_2(2)$	2.0337	0.0498	1.6549	2.5122
	$\delta(0.7)$	0.5873	0.0779	0.0664	0.9738
100	$\theta_1(2.5)$	2.5214	0.0383	2.1699	2.9366
	$\theta_2(2)$	2.0139	0.0231	1.7388	2.3366
	$\delta(0.7)$	0.6391	0.0523	0.1517	0.9706

TABLE 8
Simulation study of the parameters of BXG distribution (Case 4 - MLE).

n	Parameter	Estimate	MSE	Lower CI	Upper CI
15	$\theta_1(3)$	3.1579	0.5210	2.1039	4.8434
	$\theta_2(2.5)$	2.6337	0.3558	1.7834	4.0169
	$\delta(-0.7)$	-0.5213	0.1102	-0.9751	-0.0346
30	$\theta_1(3)$	3.0849	0.2312	2.3059	4.1513
	$\theta_2(2.5)$	2.5573	0.1406	1.9373	3.3920
	$\delta(-0.7)$	-0.5576	0.0922	-0.9760	-0.0422
50	$\theta_1(3)$	3.0428	0.1243	2.4487	3.8022
	$\theta_2(2.5)$	2.5447	0.0840	2.0531	3.2010
	$\delta(-0.7)$	-0.5834	0.0792	-0.9785	-0.0679
100	$\theta_1(3)$	3.0226	0.0612	2.5787	3.5520
	$\theta_2(2.5)$	2.5201	0.0401	2.1716	2.9438
	$\delta(-0.7)$	-0.6307	0.0532	-0.9713	-0.1500

time of first goal of any type scored by the home team (X_2), respectively. In order to illustrate the marginal distribution behavior for the considered data set, we first fit X_1 and X_2 using XG distribution separately. We calculate Kolmogorov-Smirnov (K-S) goodness of fit test statistic which are presented in Table 9. Figure 7 and 8 and Table 9 suggest that XG distribution is suitable for the marginal data sets. In order to demonstrate the nature of the UEFA Champion's League data set, a total time test (TTT) plots is presented in Figure 9, separately for X_1 and X_2. The TTT plots of considered data sets indicates that hazard rate function is increasing except in a small portion of data. So, based on features of the XGamma distribution, we use it for modeling the UEFA Champion's League data set.

TABLE 9
Kolmogrove-Smirnov goodness of fit test statistic with p-value and maximized log-likelihood (LL) for the marginal of XG distribution.

Model	X_1			X_2		
	K-S	p-value	LL	K-S	p-value	LL
BXG	0.1081	0.9821	-164.1034	0.1622	0.7154	-164.9335

Next, we fit the BXG distribution for the considered bivariate data using IMF and MLE methods. One of the most common criteria for model selection and differentiate between models is Akaike information criterion (AIC) and Bayesian information criterion (BIC) values. The AIC and BIC are defined as AIC $= -2\text{LL} + 2p$ and BIC $=$

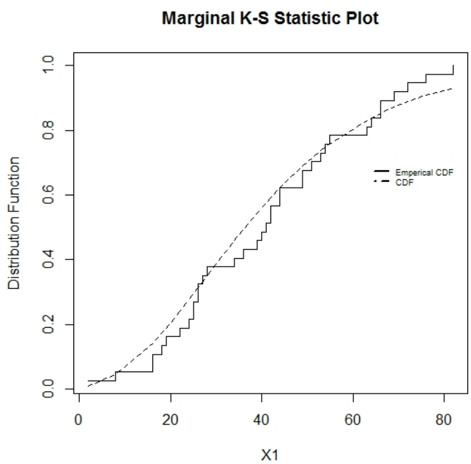

Figure 7 – K-S plot for data set X_1.

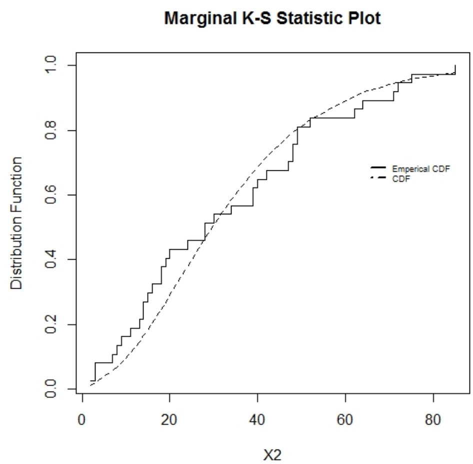

Figure 8 – K-S plot for data set X_2.

$-2\text{LL} + p\log(n)$, where p is the number of parameters in the model and n is the number of observations. For a detailed discussion on AIC and BIC, one may refer to a recent article by Pathak and Vellaisamy (2020) and references therein. The model with minimum AIC and BIC is better than other models fitted for the considered data set. The estimated values, standard error (S.E.) (in parentheses), AIC, and BIC values are given in Table 10. From Table 10, it is clear that there is no substantial difference between the two estimation procedures.

TABLE 10
MLE and IFM estimate for BXG disribution.

Method	$\hat{\theta}_1$ (S.E.)	$\widehat{\theta}_2$ (S.E.)	$\hat{\delta}$ (S.E.)	LL	AIC	BIC
MLE	0.0689 (0.0064)	0.0861 (0.0084)	$\hat{\delta} = 0.9998$ (0.3648)	-326.2689	658.5378	663.3706
IFM	0.0703 (0.0067)	0.0853 (0.0083)	$\hat{\delta}$ =0.9885 (0.3693)	-326.3074	658.6148	664.6128

We also compare our results based on the BXG with Marshall-Olkin (MO) bivariate exponential model and bivariate generalized exponential (BGE) model studied by Meintanis (2007) and Mirhosseini *et al.* (2015), respectively. We also used their estimates for compression purpose. The estimated results, AIC, BIC, and the standard error (S.E.) of the estimated values are reported in Table 11. By comparing the values of AIC and BIC

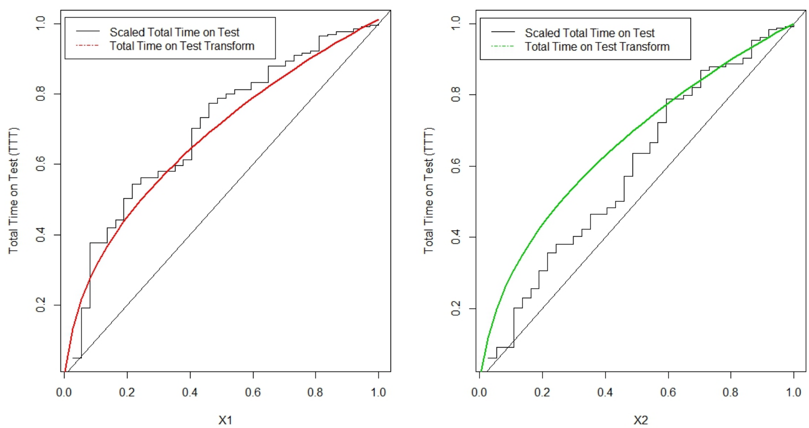

Figure 9 – TTT-plot of UEFA Champion's League data.

TABLE 11
The MLEs of the parameters, copula parameter estimate, the log-likelihood values, AIC and BIC values.

Model	MLE	LL	AIC	BIC
BXG	$\hat{\theta}_1 = 0.0689$, $\hat{\theta}_2 = 0.0861$, $\hat{\delta} = 0.9998$	-326.2689	658.5378	664.6128
BGE	$\hat{a}_1 = 0.0244$, $\hat{a}_2 = 0.0304$, $\hat{\theta} = 0.9999$	-340.5234	687.0468	691.8795
MO	$\hat{\lambda}_1 = 0.012$, $\hat{\lambda}_2 = 0.014$, $\hat{\lambda}_3 = 0.022$	-339.006	684.012	688.8448

in Table 11, we conclude that the BXG distribution provides a better fit over MO and BGE models for the considered data set in this paper.

8. Conclusions

In this paper, we have introduced a new bivariate XGamma (BXG) distribution derived from FGM copula whose univariate marginals follows Xgamma distribution. We derive the expressions for conditional distribution, regression function, product moments, and some concepts related to reliability for the BXG distribution. Some local dependence measures are derived and the concept of the TP2 is also studied. For the copula parameter δ, it has been seen that for $0 \leq \delta \leq 1$, the BXG distribution exhibits TP2 property, which is a powerful property of dependence. Parameters were estimated using two different methods namely MLE and IFM. Several numerical experiments are also reported in this study. Finally, an application to a real data shows that the BXG distribution work well and we anticipate that the BXG distribution may be useful in various piratical applications.

APPENDIX

A. PROOFS

PROOF. of Theorem 3

We have

$$E(X_1|X_2=x_2)=\int_0^\infty x_1 f(x_1|x_2)dx_1=\frac{1}{f_2(x_2)}\int_0^\infty x_1 f(x_1,x_2)dx_1. \tag{39}$$

Using Eq. (10) in Eq. (39), we get

$$E(X_1|X_2=x_2)=\int_0^\infty x_1\frac{\theta_1^2}{1+\theta_1}\left(1+\frac{\theta_1}{2}x_1^2\right)e^{-\theta_1x_1}$$
$$\times\left[1+\delta\left(\frac{2(1+\theta_1+\theta_1x_1+\frac{\theta_1^2x_1^2}{2})e^{-\theta_1x_1}}{(1+\theta_1)}-1\right)\left(\frac{2(1+\theta_2+\theta_2x_1+\frac{\theta_2^2x_2^2}{2})e^{-\theta_2x_2}}{(1+\theta_2)}-1\right)\right]dx_1. \tag{40}$$

After a simple integration of Eq. (40), we get Theorem 3. □

PROOF. of Theorem 4

By definition, we have

$$E(X_1^rX_2^s)=\int_0^\infty\int_0^\infty x_1^rx_2^sf(x_1,x_2)dx_1dx_2. \tag{41}$$

Using Eq. (10) in Eq. (41), we get

$$E(X_1^rX_2^s)=\int_0^\infty\int_0^\infty\frac{x_1^rx_2^s\theta_1^2\theta_2^2e^{-\theta_1x_1-\theta_2x_2}}{(1+\theta_1^2)(1+\theta_2^2)}\left(1+\frac{\theta_1}{2}x_1^2\right)\left(1+\frac{\theta_2}{2}x_2^2\right)$$
$$\left[1+\delta\left(\frac{2(1+\theta_1+\theta_1x_1+\frac{\theta_1^2x_1^2}{2})e^{-\theta_1x_1}}{(1+\theta_1)}-1\right)\right.$$
$$\left.\left(\frac{2(1+\theta_2+\theta_2x_1+\frac{\theta_2^2x_2^2}{2})e^{-\theta_2x_2}}{(1+\theta_2)}-1\right)\right]dx_1dx_2. \tag{42}$$

Evaluation of double integral in Eq. (42) completes the proof of Theorem 4. □

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SUMMARY

In this paper, a new bivariate XGamma (BXG) distribution is presented using Farlie-Gumbel-Morgenstern (FGM) copula. We derive the expressions for conditional distribution, regression function and product moments for the BXG distribution. Concept of reliability and various measures of local dependence are also studied for the proposed model. Furthermore, estimation of the parameters of the BXG distribution is obtained through maximum likelihood estimation and inference function of margin estimation procedures. Finally, an application of the same is also demonstrated to a real data set.

Keywords: Bivariate XGamma distribution; Copulas; FGM copula; Maximum likelihood estimate; Inference function of margin.