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APPLICATION OF RANKED SET SAMPLING IN PARAMETER ESTIMATION OF CAMBANIS-TYPE BIVARIATE EXPONENTIAL DISTRIBUTION

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SUMMARY

Ranked set sampling (RSS) is an efficient technique for estimating parameters and is applicable whenever ranking on a set of sampling units can be done easily by a judgment method or based on an auxiliary variable. In this paper, we assume (X,Y) to have a Cambanis-type bivariate exponential (CTBE) distribution, where a study variable Y is difficult and/or expensive to measure and is correlated with an auxiliary variable X that is readily measurable. The auxiliary variable is used to rank the sampling units. This paper addresses the problem of estimation of the scale parameter associated with the Y-variable based on the RSS scheme and some of the other modified RSS schemes. Comparison between estimators is done through relative efficiency to find the best RSS scheme. The efficiency performance of the estimators under various RSS schemes is presented numerically and graphically through 2-D and 3-D plots. To study the performance of the proposed estimators through a simulation study we develop a Matlab function to simulate data from the CTBE distribution. The results are applied to a real data set on mercury concentration in large mouth bass from Florida.

Keywords: Ranked set sampling; Concomitants of order statistics; Cambanis-type bivariate exponential distribution; Best linear unbiased estimator.

1. INTRODUCTION

The ranked set sampling (RSS) technique is widely used for improving the precision of the sample mean, an estimator of the population mean. RSS was introduced by McIntyre (1952) as a cost-effective alternative to simple random sampling (SRS), and was applied to

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the problem of estimating the mean pasture yield. This method is applicable whenever a variable of interest is difficult and/or expensive to measure but a ranking on a small set of measurements is easily available. McIntyre (1952) uses the judgement method for ranking a set of sample units. As an alternative to McIntyre's method of RSS, Stokes (1977) uses an auxiliary variable to rank sampling units, which is assumed to be readily measurable. The procedure of RSS using an auxiliary variable described by Stokes (1977) is as follows.

Choose n^2 independent units, arrange them randomly into n sets, each with n units, and observe the value of the X-variate on each of these units. In the first set, the unit for which the measurement on X is the smallest is chosen. In the second set, the unit for which the measurement on X is the second smallest is chosen. The procedure is repeated until in the last set, the unit for which the measurement on X is the largest is chosen. Now make measurements on Y for the selected units. Let $X_{(r)r}$ be the measurement on the r^{th} unit for the auxiliary variable X from the r^{th} set, and $Y_{[r]r}$ be the measurement made on the study variable Y for the same unit for $r = 1, 2, \ldots, n$. Thus $(Y_{[1]1}, Y_{[2]2}, \ldots, Y_{[n]n})$ forms a ranked set sample. Here clearly $Y_{[r]r}$ is the concomitant of r^{th} order statistic arising from the r^{th} sample as coined by David and Nagaraja (2004).

The procedure of RSS described by Stokes (1977) has found many diverse applications in environmental, agricultural, and ecological studies. We discuss some of the examples in the literature where the RSS schemes are used. Bain (1978) and Chacko and Thomas (2008) considered the problem of oil pollution of sea water. They considered the oil pollution of sea water as study variable Y and the tar deposit in the nearby sea shore as an auxiliary variable X. These two variables are highly positively correlated, and here Y is really difficult and expensive to measure whereas X is easy to rank. Chacko and Ghosh (2016) discussed an example related to Confir trees. In this example, X represents the diameter (in cm) of the Confir tree at breast height which can be measured easily, and Y represents the height (in feet) of the tree which is somewhat difficult to measure. Mohsin *et al.* (2014) and Chacko (2017) considered the data set used by Lange *et al.* (1993) to study the influence of water chemistry on mercury concentration in largemouth bass from different Florida lakes. Here X represents the amount of alkalinity (mg/l) in water sample and Y is the minimum mercury concentration $(\mu g/g)$ in sampled fish.

The bivariate set up encourages the researcher to assume a suitable distribution for (X, Y). For the bivariate normal distribution, Stokes (1977) have proposed a ranked set sample mean as an estimator for the mean of the study variate Y, when an auxiliary variable X is used for ranking the sample units, whereas Barnett and Moore (1997) obtained the BLUE of the mean of Y based on a ranked set sample. Various modifications of RSS schemes have been proposed in the literature. Stokes (1980) introduced a modified RSS procedure in which only the smallest or the largest judgment ranked unit is chosen for quantification from each set which is later on known as Lower RSS (LRSS) or Upper RSS (URSS) scheme. Samawi *et al.* (1996) investigated the use of a variety of extreme RSS (ERSS) schemes for estimating the population mean. Some other variations in RSS schemes are Median RSS (MRSS) and Moving extreme RSS (MERSS). For details about these schemes refer to Muttlak (1998) and Al-Saleh and Al-Ananbeh (2007).

Specifically, the exponential distribution is found to be the most important one to model lifetimes of components in reliability systems. The bivariate exponential distribution can be used as a model to study the lifetimes of engineering system when component lifetimes depend on some of the other correlated variables. Several versions of bivariate exponential distributions are found in the literature with many real-life applications such as life testing, reliability theory, survival analysis, stress-strength models etc. Lam *et al.* (1994) used the RSS scheme to estimate the parameters of a two-parameter exponential distribution. Chacko and Thomas (2008) obtained the BLUE of a parameter associated with the study variable for a Morgenstern type bivariate exponential (MTBE) distribution by RSS and a censored RSS scheme. Al-Saleh and Diab (2009) obtained the estimators of parameters of Downton's bivariate exponential distribution using the RSS scheme. Tahmasebi and Jafari (2014) considered the Morgenstern type bivariate generalized exponential distribution and obtained an estimator for the population mean using several RSS schemes. Chacko (2016) investigates a new RSS scheme called ordered extreme RSS scheme and obtains an estimator of the parameter for MTBE distribution. Chacko (2017) obtains a Bayes estimator for the mean of the study variate for the MTBE distribution. Estimation of parameters of various distributions has been carried out using RSS. Some other recent work in this direction is by Shaibu and Muttlak (2004), Chacko and Thomas (2007), Singh and Mehta (2013), Koshti and Kamalja (2017), Kamalja and Koshti (2019), Samuh *et al.* (2020), and Koshti and Kamalja (2021a).

Many researchers have studied bivariate distributions from the Morgenstern family as well as from extensions of the Morgenstern family and dealt with the problem of estimation of parameter under RSS schemes. The Cambanis (1977) family is one of the generalizations of the Morgenstern family in which additional parameters are introduced. Nair *et al.* (2016) discussed distributional characteristics, nature of dependence, reliability properties, and applications of Cambanis family in modelling bivariate lifetime data. Koshti and Kamalja (2021b) obtained an estimator of scale parameter associated with a study variable based on different RSS schemes for the Cambanis-type bivariate uniform distribution.

In this paper we consider the problem of the estimation of the parameters of the Cambanis-type bivariate exponential (CTBE) distribution using various RSS schemes. We estimate the scale parameter associated with the study variable Y, when an auxiliary variable X is used for ranking the sampling units and (X, Y) have a CTBE distribution. This paper is organized as follows.

In Section 2, we explore some aspects of the CTBE distribution and concomitants of order statistics (COS) obtained by Thomas (2018). Section 3 presents a brief discussion on RSS schemes and estimation of the scale parameter under these RSS schemes for CTBE distribution. Section 4 covers estimation of all the parameters of the CTBE distribution by method of moments, some of which are to be used as substitutes of population parameters. In Section 5, we perform an efficiency comparison of the proposed RSS estimators numerically and present trends in efficiency with respect to parameters and sample size. In Section 6 we develop an algorithm to simulate data from the CTBE distribution and present some simulation results. Demonstrations about the performance of the proposed estimators for simulated and real-life data are presented in Section 7. Concluding remarks are given in Section 8.

2. Distributional properties and COS for the CTBE distribution

In this Section, we explore some aspects of distribution theory of the CTBE distribution and present a brief overview on COS.

2.1. *Distributional properties*

Morgenstern (1956) has provided a flexible family that can be used to construct bivariate distributions with specified marginal distributions. This family is also known in the literature as the Farlie-Gumbel-Morgenstern (FGM) family. One important limitation of the Morgenstern family is that its correlation coefficient is restricted to a narrow range $(-1/3, 1/3)$. Cambanis (1977) has introduced a modification to the classical Morgenstern family of distributions by introducing additional parameters which are helpful to enhance the range of correlation.

The distribution function $H_{X,Y}(x,y)$ of the Cambanis-type bivariate distribution with parameters $\alpha_1, \alpha_2, \alpha_3$, denoted by CTB$(\alpha_1, \alpha_2, \alpha_3)$, corresponding to the pair of random variables (X, Y) as given by Cambanis (1977) is

$$\begin{aligned} H_{X,Y}(x,y) = F_X(x)F_Y(y)[1 + \alpha_1(1 - F_X(x)) + \alpha_2(1 - F_Y(y)) + \\ \alpha_3(1 - F_X(x))(1 - F_Y(y))], \end{aligned} \tag{1}$$

where the parameters α_1, α_2 and α_3 are real constants satisfying the following conditions:

$$1 + \alpha_1 + \alpha_2 + \alpha_3 > 0, \qquad 1 + \alpha_1 - \alpha_2 - \alpha_3 > 0,$$

$$1 - \alpha_1 + \alpha_2 - \alpha_3 > 0, \qquad 1 - \alpha_1 - \alpha_2 + \alpha_3 > 0.$$

The marginal distributions of X and Y are given by

$$H_X(x) = F_X(x)\Big(1 + \alpha_1\big(1 - F_X(x)\big)\Big), \tag{2}$$

$$H_Y(y) = F_Y(y)\Big(1 + \alpha_2\big(1 - F_Y(y)\big)\Big). \tag{3}$$

The Morgenstern family of bivariate distributions is a special case of the Cambanis family when both α_1 and α_2 are zero. Further the two variables are independent when $\alpha_i = 0$, for $i = 1, 2, 3$.

We consider a Cambanis-type bivariate exponential distribution with parameters $\alpha_1, \alpha_2, \alpha_3, \theta_1, \theta_2$ and denote it as CTBE$(\alpha_1, \alpha_2, \alpha_3, \theta_1, \theta_2)$. The probability density function (pdf) of the CTBE$(\alpha_1, \alpha_2, \alpha_3, \theta_1, \theta_2)$ distribution is

$$h(x,y) = \frac{e^{-(\frac{x}{\theta_1}+\frac{y}{\theta_2})}}{\theta_1\theta_2}\Big[1+\alpha_1(2e^{\frac{-x}{\theta_1}}-1)+\alpha_2(2e^{\frac{-y}{\theta_2}}-1)+\alpha_3(2e^{\frac{-x}{\theta_1}}-1)(2e^{\frac{-y}{\theta_2}}-1)\Big], \quad (4)$$

where $x, y, \theta_1, \theta_2 > 0$, and

$$1+\alpha_1+\alpha_2+\alpha_3 > 0, 1+\alpha_1-\alpha_2-\alpha_3 > 0,$$
$$1-\alpha_1+\alpha_2-\alpha_3 > 0, 1-\alpha_1-\alpha_2+\alpha_3 > 0.$$

The marginal distributions of X and Y are as follows:

$$h_X(x) = (1-\alpha_1)\frac{e^{\frac{-x}{\theta_1}}}{\theta_1} + \alpha_1\frac{2e^{\frac{-2x}{\theta_1}}}{\theta_1}, \quad x > 0 \text{ and } \theta_1 > 0, \quad (5)$$

$$h_Y(y) = (1-\alpha_2)\frac{e^{\frac{-y}{\theta_2}}}{\theta_2} + \alpha_2\frac{2e^{\frac{-2y}{\theta_2}}}{\theta_2}, \quad y > 0 \text{ and } \theta_2 > 0. \quad (6)$$

Note that the marginal distributions are not exponential, but they are mixture of two exponential random variables. The marginal of $X(Y)$ is a mixture of exponential distributions with mean $\theta_1(\theta_2)$ and $\theta_1/2(\theta_2/2)$. The mean and variance of Y are

$$E(Y) = \theta_2\Big(1-\frac{\alpha_2}{2}\Big), \qquad \mathrm{Var}(Y) = \theta_2^2\Big(1-\frac{\alpha_2}{2}-\frac{\alpha_2^2}{4}\Big). \quad (7)$$

Further the bivariate $(r,s)^{th}$ product moment of the distribution in Eq. (1) is given by

$$\begin{aligned} E(X^rY^s) &= \Gamma((r+1)(s+1))\theta_1^r\theta_2^s\Big[1+\alpha_1\Big(\frac{1}{2^r}-1\Big)+\alpha_2\Big(\frac{1}{2^s}-1\Big)+ \\ &+ \alpha_3\Big(\frac{1}{2^r}-1\Big)\Big(\frac{1}{2^s}-1\Big)\Big], \quad r,s = 1,2,3,\ldots. \quad (8)\end{aligned}$$

The Hoeffding's formula for Cov(X,Y) is,

$$\mathrm{Cov}(X,Y) = \int\int [H_{X,Y}(x,y) - H_X(x)H_Y(y)]\mathrm{d}x\mathrm{d}y. \quad (9)$$

We obtain Cov(X,Y) for $(X,Y) \sim$ CTBE$(\alpha_1, \alpha_2, \alpha_3, \theta_1, \theta_2)$ distribution using this formula as

$$\mathrm{Cov}(X,Y) = \Big(\frac{\alpha_3-\alpha_1\alpha_2}{4}\Big)\theta_1\theta_2. \quad (10)$$

Consequently, the correlation between X and Y simplifies to

$$\rho(X,Y) = \frac{(\alpha_3 - \alpha_1\alpha_2)}{\sqrt{(\alpha_1^2 + 2\alpha_1 - 4)(\alpha_2^2 + 2\alpha_2 - 4)}}. \tag{11}$$

The variation in correlation with respect to $\alpha_1, \alpha_2, \alpha_3$ for CTBE$(\alpha_1, \alpha_2, \alpha_3, \theta_1, \theta_2)$ distribution is shown in Figure 1 and Figure 2. The trends in these Figures help to choose the feasible values of the association parameters α_2, α_3 for fixed α_1 and give an idea about the amount of correlation.

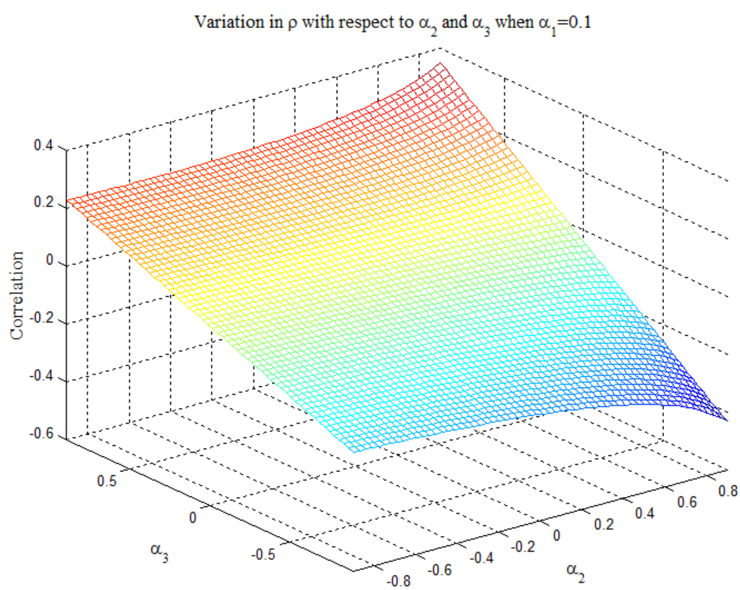

Figure 1 – Correlation w.r.t. α_2, α_3 for CTBE$(0.1, \alpha_2, \alpha_3, \theta_1, \theta_2)$ distribution.

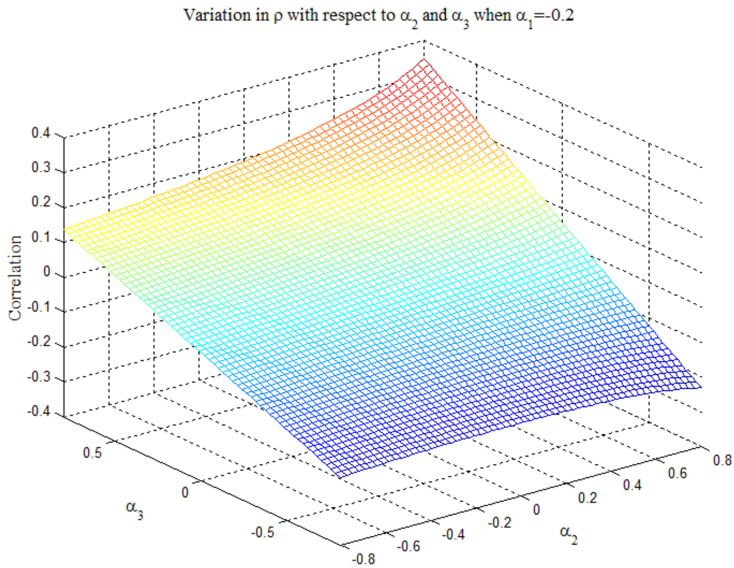

Figure 2 – Correlation w.r.t. α_2, α_3 for CTBE$(-0.2, \alpha_2, \alpha_3, \theta_1, \theta_2)$ distribution.

Nadarajah and Kotz (2006) discussed the calculation of $P(X < Y)$ for the class of bivariate exponential distributions. This probability is used to obtain component reliability. For the CTBE$(\alpha_1, \alpha_2, \alpha_3, \theta_1, \theta_2)$ distribution we obtain it as

$$\begin{aligned} P(X<Y) &= \frac{\theta_2}{\theta_1+\theta_2}+\alpha_1\left(\frac{2\theta_2}{\theta_1+2\theta_2}-\frac{\theta_2}{\theta_1+\theta_2}\right)+\alpha_2\left(\frac{2\theta_2}{2\theta_1+\theta_2}-\frac{\theta_2}{\theta_1+\theta_2}\right) \\ &+ \alpha_3\left(\frac{3\theta_2}{\theta_1+\theta_2}-\frac{2\theta_2}{\theta_1+2\theta_2}-\frac{2\theta_2}{2\theta_1+\theta_2}\right). \end{aligned} \tag{12}$$

2.2. *Review on concomitants of order statistics*

Scaria and Nair (1999) and Thomas (2018) investigated the distribution theory on COS from the Morgenstern family and the Cambanis family, respectively. Thomas (2018) obtains the COS for the general CTB$(\alpha_1, \alpha_2, \alpha_3)$ distribution and for the CTBE$(\alpha_1, \alpha_2, \alpha_3, \theta_1, \theta_2)$ distribution when $\alpha_1 = 0$. We present a brief summary on COS of the CTBE$(0, \alpha_2, \alpha_3, \theta_1, \theta_2)$ distribution, which is denoted as CTBE$(\alpha_2, \alpha_3, \theta_1, \theta_2)$ in the rest of the paper. Let (x_i, y_i), $i = 1, 2, \ldots, n$ be a random sample of size n from the CTBE$(\alpha_2, \alpha_3, \theta_1, \theta_2)$ distribution with pdf

$$h(x,y)=\frac{e^{-(\frac{x}{\theta_1}+\frac{y}{\theta_2})}}{\theta_1\theta_2}\left[1+\alpha_2\left(2e^{\frac{-y}{\theta_2}}-1\right)+\alpha_3\left(2e^{\frac{-x}{\theta_1}}-1\right)\left(2e^{\frac{-y}{\theta_2}}-1\right)\right], \tag{13}$$

where $x,y>0$, $\theta_1,\theta_2>0$, $|\alpha_2+\alpha_3|\leq 1$, $|\alpha_2-\alpha_3|\leq 1$.
The pdf $h_{Y_{[r]r}}(y)$ of the concomitant of the r^{th} order statistic from the sample of size n for the CTBE$(\alpha_2,\alpha_3,\theta_1,\theta_2)$ distribution for $1\leq r\leq n$ is

$$h_{Y_{[r]r}}(y)=\frac{e^{\frac{-y}{\theta_2}}}{\theta_2}\left[1+\left(\alpha_2+\alpha_3\frac{n-2r+1}{n+1}\right)\left(2e^{\frac{-y}{\theta_2}}-1\right)\right],\quad y>0,\ \theta_2>0. \tag{14}$$

For $1\leq r\leq n$, the k^{th} moment of $Y_{[r]r}$ is given by

$$\mu_{[r]r}^{(k)}=E(Y_{[r]r}^k)=\Gamma(k+1)\theta_2^k\left[1+\left(\alpha_2+\alpha_3\frac{n-2r+1}{n+1}\right)(2^{-k}-1)\right]. \tag{15}$$

Specifically, the mean and variance of $Y_{[r]r}$ are

$$E(Y_{[r]r})=\theta_2\xi_r,\qquad \mathrm{Var}(Y_{[r]r})=\theta_2^2\delta_r, \tag{16}$$

where

$$\xi_r=1-\frac{1}{2}\left(\alpha_2+\alpha_3\frac{n-2r+1}{n+1}\right),$$

$$\delta_r=1-\frac{1}{2}\left(\alpha_2+\alpha_3\frac{n-2r+1}{n+1}\right)-\frac{1}{4}\left(\alpha_2+\alpha_3\frac{n-2r+1}{n+1}\right)^2. \tag{17}$$

The covariance between $Y_{[r]r}$ and $Y_{[s]s}$ as given by Thomas (2018) is

$$\mathrm{Cov}(Y_{[r]r},Y_{[s]s})=\frac{r(n-s+1)}{(n+1)^2(n+2)}\alpha_3^2\theta_2^2,\qquad 1\leq r<s\leq n. \tag{18}$$

The above results for concomitants of the r^{th} order statistic $Y_{[r]r}$ would be used for estimating the scale parameter θ_2 under various RSS schemes.

3. Estimation of the scale parameter θ_2 based on different RSS schemes

In this Section, we consider estimation of the scale parameter θ_2 of the CTBE$(\alpha_2,\alpha_3,\theta_1,\theta_2)$ distribution based on RSS, LRSS, URSS, and ERSS schemes under the assumption that α_2 and α_3 are known. We discuss some of the modified RSS schemes in brief and then use these schemes for estimation.

3.1. Sample selection under various RSS schemes

Stokes (1980) introduces a modified RSS procedure in which only the smallest or the largest judgment ranked unit is chosen for quantification from each set. The procedure of the LRSS and URSS scheme begins with randomly choosing n sets each of n units from the population as in case of RSS. In the LRSS (URSS) scheme the measurement on Y is made on units for which the measurement on X is smallest (largest) in each of the n sets. The collection of the sample observations $Y_{[1]1}, Y_{[1]2}, \ldots, Y_{[1]n} (Y_{[n]1}, Y_{[n]2}, \ldots, Y_{[n]n})$ is referred as sample under the LRSS (URSS) scheme.

Koshti and Kamalja (2021b) used a variety of ERSS schemes. The samples under the ERSS scheme can be chosen in different ways, which are based on even or odd sample size. The ERSS scheme involves randomly choosing n sets with n units each from the population. For even sample size, from odd numbered samples (i.e. $r = 1, 3, \ldots, n-1$), select the smallest X, and from the even numbered samples (i.e. $r = 2, 4, \ldots, n$), select the largest X. Now measure the Y-variate associated with the X-variate selected as $X_{(1)1}, X_{(n)2}, X_{(1)3}, X_{(n)4}, \ldots, X_{(1)n-1}, X_{(n)n}$. Thus $Y_{[1]1}, Y_{[n]2}, \ldots, Y_{[1]n-1}, Y_{[n]n}$ forms a sample under the ERSS scheme and is denoted by ERSS_1.

For odd sample size, from odd numbered samples (i.e. $r = 1, 3, \ldots, n-2$), select the smallest X and from the even numbered samples (i.e. $r = 2, 4, \ldots, n-1$), select the largest X. Now measure the Y-variate associated with the X-variate selected as $X_{(1)1}, X_{(n)2}, X_{(1)3}, X_{(n)4}, \ldots, X_{(1)n-2}, X_{(n)n-1}$. This completes the procedure of selection of $n-1$ units of the sample. Now the n^{th} unit of the sample can be chosen from the n^{th} set by one of the following four ways.

i) The n^{th} unit is the average of the Y variates associated with the smallest and the largest X in the n^{th} set, i.e., $(Y_{[1]n} + Y_{[n]n})/2$.

ii) Choose Y associated with the median $X_{((n+1)/2)n}$ of n^{th} set, i.e., $Y_{[(n+1)/2]n}$.

iii) Choose the Y variate associated with the smallest X variate in the n^{th} set, i.e., $Y_{[1]n}$

iv) Choose Y associated with the largest X variate in the n^{th} set, i.e., $Y_{[n]n}$

The odd sized samples under the ERSS scheme under four different ways are denoted by ERSS_i, $i = 2, 3, 4, 5$, respectively, and can be represented as follows:

$$\text{ERSS}_i = \begin{cases} Y_{[1]1}, Y_{[n]2}, Y_{[1]3} \ldots, Y_{[n]n-1}, \frac{Y_{[1]n}+Y_{[n]n}}{2}, & i = 2 \\ Y_{[1]1}, Y_{[n]2}, Y_{[1]3} \ldots, Y_{[n]n-1}, Y_{[(n+1)/2]n}, & i = 3 \\ Y_{[1]1}, Y_{[n]2}, Y_{[1]3} \ldots, Y_{[n]n-1}, Y_{[1]n}, & i = 4 \\ Y_{[1]1}, Y_{[n]2}, Y_{[1]3} \ldots, Y_{[n]n-1}, Y_{[n]n}, & i = 5. \end{cases}$$

3.2. Estimation of scale parameter θ_2 based on RSS schemes

Using the samples under different RSS schemes, we wish to recommend the most efficient estimator of θ_2. To obtain unbiased estimators and their variances under these schemes we use mean, variance, and covariance of COS arising from the CTBE distribution as in Eq. (16) and (18). We obtain the BLUE of θ_2 when α_2 and α_3 are known using the generalized Gauss-Markov set up due to David and Nagaraja (2004) for the RSS samples under consideration except the ERSS_2 scheme (as the sample units under ERSS_2 scheme are not all independent). We discuss in brief the steps required to obtain the BLUE of θ_2 under the RSS schemes.

The mean vector and the dispersion matrix of $Y_{[n]}$, the vector of sample of size n, is given by

$$E(Y_{[n]}) = \theta_2 \xi, \qquad \text{Cov}(Y_{[n]}) = \theta_2^2 G, \tag{19}$$

where $\xi = [\xi_1, \xi_2, \ldots, \xi_n]'$ and $G = \text{diag}(\delta_1, \delta_2, \ldots, \delta_n)$. The BLUE $\hat{\theta}_{2,\text{Scheme}}$ of θ_2 and its variance under respective RSS scheme is given by

$$\hat{\theta}_{2,\text{Scheme}} = (\xi' G^{-1} \xi)^{-1} \xi' G^{-1} Y_{[n]}, \qquad \text{Var}(\hat{\theta}_{2,\text{Scheme}}) = (\xi' G^{-1} \xi)^{-1} \theta_2^2. \tag{20}$$

In Table 1 we summarize the estimators of θ_2 and their variances for the $\text{CTBE}(\alpha_2, \alpha_3, \theta_1, \theta_2)$ distribution under the schemes considered. The estimator of θ_2 under SRS is also included and is obtained using the marginal distribution of Y.

Note: We also obtained an estimator of the scale parameter under MRSS, we varied the set MERSS scheme and found that the usual RSS estimator overperforms with respect to the estimators under these schemes. Hence, we do not report the details of the MRSS and MERSS scheme here.

To compare all the estimation schemes let $E_{\text{RSS}}^{\text{SRS}}$ and $E_{\text{BLUE}}^{\text{RSS}}$ be the relative efficiency of $\hat{\theta}_{2,\text{RSS}}$ relative to $\hat{\theta}_{2,\text{SRS}}$ and $\hat{\theta}_{2,\text{BLUE}}$ relative to $\hat{\theta}_{2,\text{RSS}}$ respectively. Further let E_{Scheme} be the efficiency of $\hat{\theta}_{2,\text{Scheme}}$ relative to $\hat{\theta}_{2,\text{BLUE}}$ (see Table 2) and is given by

$$E_{\text{Scheme}} = \frac{\text{Var}(\hat{\theta}_{2,\text{BLUE}})}{\text{Var}(\hat{\theta}_{2,\text{Scheme}})}. \tag{21}$$

The numerical evaluations and graphical presentation of these efficiencies with respect to different sample sizes and association parameters are shown in Section 5.

TABLE 1
Summary of estimators of θ_2 and their variances.

Scheme	Sample	$\hat{\theta}_{2,\text{Scheme}}$: Unbiased estimator of θ_2	$\text{Var}(\hat{\theta}_{2,\text{Scheme}})$
SRS	$y_1, y_2, \ldots, y_n$	$\frac{\bar{y}}{\left(1-\frac{\alpha_2}{2}\right)}$	$\theta_2^2\left(1-\frac{\alpha_2}{2}-\frac{\alpha_2^2}{4}\right)/n\left(1-\frac{\alpha_2}{2}\right)^2$
RSS	$Y_{[1]1}, Y_{[2]2}, \ldots, Y_{[n]n}$	$\sum_{r=1}^{n} Y_{[r]r}/n\left(1-\frac{\alpha_2}{2}\right)$	$\theta_2^2\sum_{r=1}^{n}\delta_r/n^2\left(1-\frac{\alpha_2}{2}\right)^2$
RSS BLUE	$Y_{[1]1}, Y_{[2]2}, \ldots, Y_{[n]n}$	$\sum_{r=1}^{n}(\xi_r/\delta_r)Y_{[r]r}\,/\sum_{r=1}^{n}\xi_r^2/\delta_r$	$\theta_2^2/\sum_{r=1}^{n}\frac{\xi_r^2}{\delta_r}$
LRSS	$Y_{[1]1}, Y_{[1]2}, \ldots, Y_{[1]n}$	$\sum_{r=1}^{n} Y_{[1]r}/n\xi_1$	$\theta_2^2\delta_1/n\xi_1^2$
URSS	$Y_{[n]1}, Y_{[n]2}, \ldots, Y_{[n]n}$	$\sum_{r=1}^{n} Y_{[n]r}/n\xi_n$	$\theta_2^2\delta_n/n\xi_n^2$
ERSS$_1$ (n :even)	$Y_{[1]1}, Y_{[n]2}, Y_{[1]3}, \ldots, Y_{[1]n-1}, Y_{[n]n}$	$\frac{2}{n\left(\frac{\xi_1^2}{\delta_1}+\frac{\xi_n^2}{\delta_n}\right)}\left(\sum_{r=1}^{\frac{n}{2}}\left(\frac{\xi_1}{\delta_1}Y_{[1]2r-1}+\frac{\xi_n}{\delta_n}Y_{[n]2r}\right)\right)$	$2\theta_2^2/n\left(\frac{\xi_1^2}{\delta_1}+\frac{\xi_n^2}{\delta_n}\right)$
ERSS$_2$ (n :odd)	$Y_{[1]1}, Y_{[n]2}, Y_{[1]3}, \ldots, Y_{[n]n-1}, \frac{\left(Y_{[1]n}+Y_{[n]n}\right)}{2}$	$\frac{1}{n\left(1-\frac{\alpha_2}{2}\right)}\left(\sum_{r=1}^{\frac{n-1}{2}}\left(Y_{[1]2r-1}+Y_{[n]2r}\right)+\frac{Y_{[1]n}+Y_{[n]n}}{2}\right)$	$\frac{\theta_2^2\left(\left(\frac{2n-1}{4}\right)(\delta_1+\delta_n)+\frac{2\alpha_3^2}{(n+1)^2(n+2)}\right)}{\left(n\left(1-\frac{\alpha_2}{2}\right)\right)^2}$
ERSS$_3$ (n :odd)	$Y_{[1]1}, Y_{[n]2}, Y_{[1]3}, \ldots, Y_{[n]n-1}, Y_{\left[\frac{n+1}{2}\right]n}$	$\frac{\left(\frac{\xi_1}{\delta_1}\sum_{r=1}^{\frac{n-1}{2}}Y_{[1]2r-1}+\frac{\xi_n}{\delta_n}\sum_{r=1}^{\frac{n-1}{2}}Y_{[n]2r}+\frac{\xi_{\frac{n+1}{2}}}{\delta_{\frac{n+1}{2}}}Y_{\left[\frac{n+1}{2}\right]n}\right)}{\left(\frac{n-1}{2}\right)\left(\frac{\xi_1^2}{\delta_1}+\frac{\xi_n^2}{\delta_n}\right)+\frac{\xi_{\frac{n+1}{2}}^2}{\delta_{\frac{n+1}{2}}}}$	$\frac{\theta_2^2}{\left(\frac{n-1}{2}\right)\left(\frac{\xi_1^2}{\delta_1}+\frac{\xi_n^2}{\delta_n}\right)+\frac{\xi_{\frac{n+1}{2}}^2}{\delta_{\frac{n+1}{2}}}}$
ERSS$_4$ (n :odd)	$Y_{[1]1}, Y_{[n]2}, Y_{[1]3}, \ldots, Y_{[n]n-1}, Y_{[1]n}$	$\frac{\frac{\xi_1}{\delta_1}\sum_{r=1}^{\frac{n+1}{2}}Y_{[1]2r-1}+\frac{\xi_n}{\delta_n}\sum_{r=1}^{\frac{n-1}{2}}Y_{[n]2r}}{\left(\frac{n+1}{2}\right)\frac{\xi_1^2}{\delta_1}+\left(\frac{n-1}{2}\right)\frac{\xi_n^2}{\delta_n}}$	$\frac{\theta_2^2}{\left(\frac{n+1}{2}\right)\frac{\xi_1^2}{\delta_1}+\left(\frac{n-1}{2}\right)\frac{\xi_n^2}{\delta_n}}$
ERSS$_5$ (n :odd)	$Y_{[1]1}, Y_{[n]2}, Y_{[1]3}, \ldots, Y_{[n]n-1}, Y_{[n]n}$	$\frac{\frac{\xi_1}{\delta_1}\sum_{r=1}^{\frac{n-1}{2}}Y_{[1]2r}+\frac{\xi_n}{\delta_n}\sum_{r=1}^{\frac{n+1}{2}}Y_{[n]2r-1}}{\left(\frac{n-1}{2}\right)\frac{\xi_1^2}{\delta_1}+\left(\frac{n+1}{2}\right)\frac{\xi_n^2}{\delta_n}}$	$\frac{\theta_2^2}{\left(\frac{n-1}{2}\right)\frac{\xi_1^2}{\delta_1}+\left(\frac{n+1}{2}\right)\frac{\xi_n^2}{\delta_n}}$

TABLE 2
Efficiency comparison of estimators under RSS schemes.

Efficiency	Remark
$E_{\text{RSS}}^{\text{SRS}} = \frac{n\left(1-\frac{\alpha_2}{2}-\frac{\alpha_2^2}{4}\right)}{\sum_{r=1}^{n}\delta_r}$	Always > 1
$E_{\text{BLUE}}^{\text{RSS}} = \frac{\left(\sum_{r=1}^{n}\delta_r\right)\left(\sum_{r=1}^{n}\frac{\xi_r^2}{\delta_r}\right)}{n^2\left(1-\frac{\alpha_2}{2}\right)^2}$	Always > 1
$E_{\text{LRSS}} = \frac{n\xi_1^2/\delta_1}{\sum_{r=1}^{n}\xi_r^2/\delta_r}$	If $\alpha_3 < 0$, $E_{\text{LRSS}} > 1$
$E_{\text{URSS}} = \frac{n\xi_n^2/\delta_n}{\sum_{r=1}^{n}\xi_r^2/\delta_r}$	If $\alpha_3 > 0$, $E_{\text{URSS}} > 1$
$E_{\text{ERSS}_1} = \frac{n}{2\sum_{r=1}^{n}\xi_r^2/\delta_r}\left(\frac{\xi_1^2}{\delta_1}+\frac{\xi_n^2}{\delta_n}\right)$	Always > 1
$E_{\text{ERSS}_2} = \frac{\left(n\left(1-\frac{\alpha_2}{2}\right)\right)^2}{\left(\sum_{r=1}^{n}\frac{\xi_r^2}{\delta_r}\right)\left(\frac{2n-1}{4}\right)(\delta_1+\delta_n)+\frac{2\alpha_3^2}{(n+1)^2(n+2)}}$	Always > 1
$E_{\text{ERSS}_3} = \frac{\left(\frac{n-1}{2}\right)\left(\frac{\xi_1^2}{\delta_1}+\frac{\xi_n^2}{\delta_n}\right)+\frac{\xi_{\frac{n+1}{2}}^2}{\delta_{\frac{n+1}{2}}}}{\sum_{r=1}^{n}\xi_r^2/\delta_r}$	Always ≥ 1
$E_{\text{ERSS}_4} = \frac{\left(\frac{n+1}{2}\right)\frac{\xi_1^2}{\delta_1}+\left(\frac{n-1}{2}\right)\frac{\xi_n^2}{\delta_n}}{\sum_{r=1}^{n}\xi_r^2/\delta_r}$	Conditionally efficient
$E_{\text{ERSS}_5} = \frac{\left(\frac{n+1}{2}\right)\frac{\xi_n^2}{\delta_n}+\left(\frac{n-1}{2}\right)\frac{\xi_1^2}{\delta_1}}{\sum_{r=1}^{n}\xi_r^2/\delta_r}$	Conditionally efficient

4. Estimation of the parameters of the CTBE distribution by the method of moments

While estimating the scale parameter θ_2 under various RSS schemes, we assumed that the association parameters α_2, α_3 are known. Accordingly, to obtain the RSS estimates of θ_2 and its variance under a specific scheme, we will need the true values of α_2 and α_3. But in fact, as they would be unknown, we need to replace them by appropriate estimates. The simplest method of estimation is the method of moments. In this Section we discuss estimation of all the parameters of the CTBE$(\alpha_1,\alpha_2,\alpha_3,\theta_1,\theta_2)$ distribution by the method of moments.

Let (x_i, y_i), $i = 1,2,\ldots,n$ be a simple random sample from CTBE$(\alpha_1,\alpha_2,\alpha_3,\theta_1,\theta_2)$ distribution. Let m_1' and m_2' be the first and second sample moments, respectively, based on the y - observations, and let $\hat{\rho}$ be the sample correlation. The moment estimators of α_1, α_2, α_3, θ_1, θ_2 can be obtained through the following steps.

i) Consider the moment equation corresponding to the ratio of sample moments based on the y- observations m_2' and $m_1'^2$ as follows.

$$\frac{2-\frac{3}{2}\alpha_2}{1-\alpha_2+\frac{\alpha_2^2}{4}}=\frac{m_2'}{m_1'^2}. \tag{22}$$

This leads to the following quadratic moment equation in α_2.

$$m_2'\alpha_2^2+4\left(\frac{3}{2}m_1'^2-m_2'\right)\alpha_2+4\left(m_2'-2m_1'^2\right)=0. \tag{23}$$

Among two solutions of Eq. (23), let $\hat{\alpha}_2$ be the feasible one. Using similar moment equations based on moments of the x - observations, the moment estimator $\hat{\alpha}_1$ of α_1 can be obtained.

ii) To obtain the moment estimator of α_3, we use the moment equation based on the correlation between (X,Y) and replace α_1 and α_2 by their respective moment estimators $\hat{\alpha}_1$ and $\hat{\alpha}_2$.

$$\hat{\rho}=(\alpha_3-\hat{\alpha}_1\hat{\alpha}_2)[(\alpha_1^2+2\alpha_1-4)(\alpha_2^2+2\alpha_2-4)]^{-1/2}. \tag{24}$$

The moment estimator $\hat{\alpha}_3$ of α_3 along with feasibility condition is given by

$$\hat{\alpha}_3=\begin{cases}\max(-1-\hat{\alpha}_1-\hat{\alpha}_2,-1+\hat{\alpha}_1+\hat{\alpha}_2) & \text{if } \hat{\rho}<a,\\ \hat{\rho}\sqrt{\left(\hat{\alpha}_1^2+2\hat{\alpha}_1-4\right)\left(\hat{\alpha}_2^2+2\hat{\alpha}_2-4\right)}+\hat{\alpha}_1\hat{\alpha}_2 & \text{if } a\le\hat{\rho}\le b,\\ \min(1+\hat{\alpha}_1-\hat{\alpha}_2,1-\hat{\alpha}_1+\hat{\alpha}_2) & \text{if } \hat{\rho}>b,\end{cases}$$

where

$$a=\frac{\max(-1-\hat{\alpha}_1-\hat{\alpha}_2,-1+\hat{\alpha}_1+\hat{\alpha}_2)-\hat{\alpha}_1\hat{\alpha}_2}{\sqrt{\left(\hat{\alpha}_1^2+2\hat{\alpha}_1-4\right)\left(\hat{\alpha}_2^2+2\hat{\alpha}_2-4\right)}}, \tag{25}$$

and

$$b=\frac{\min(1+\hat{\alpha}_1-\hat{\alpha}_2,1-\hat{\alpha}_1+\hat{\alpha}_2)-\hat{\alpha}_1\hat{\alpha}_2}{\sqrt{\left(\hat{\alpha}_1^2+2\hat{\alpha}_1-4\right)\left(\hat{\alpha}_2^2+2\hat{\alpha}_2-4\right)}}. \tag{26}$$

Note that for $\alpha_1=0$ and $\alpha_2=0$ (i.e. $(X,Y)\sim\text{MTBE}(\alpha_3,\theta_1,\theta_2)$) the above moment estimator of α_3 reduces to the estimator of α_3 given by Chacko and Thomas (2008).

iii) The moment estimators $\hat{\theta}_1$ and $\hat{\theta}_2$ of θ_1 and θ_2, respectively, are obtained as

$$\hat{\theta}_1 = \frac{\bar{x}}{1-\frac{\hat{\alpha}_1}{2}}, \qquad \hat{\theta}_2 = \frac{\bar{y}}{1-\frac{\hat{\alpha}_2}{2}}. \tag{27}$$

The feasibility of all the moment estimators of $\alpha_1, \alpha_2, \alpha_3, \theta_1, \theta_2$ has been checked through a simulation study in Section 6 and it has been found that they are fairly reasonable to use. These moment estimates can also be used as the initial guess values of the parameters while obtaining maximum likelihood estimators. Specifically we use moment estimators of α_2 and α_3 as substitutes of the real parameters α_2 and α_3 in the RSS estimator of θ_2 and their variances.

5. Numerical study

This Section presents the comparison of the efficiencies of the estimator of θ_2 under the respective RSS schemes by numerically evaluating them, and it investigates the performance of these estimators graphically. We are interested in the study of performance of estimators with respect to sample size and association parameters. The comparisons are presented through numerical work and graphical trends in efficiency.

5.1. *Efficiency with respect to sample size and α_3 when α_2 is fixed*

We evaluate the efficiencies of the estimators of θ_2 numerically based on the RSS, BLUE, LRSS, URSS, and ERSS schemes for different sample sizes when $\alpha_2 = -0.25, 0.20$ and $\alpha_3 \in (-(1-\alpha_2), (1-\alpha_2))$. The efficiencies for $\alpha_2 = -0.25$ are presented in Table 3, whereas Table 4 shows these efficiencies for $\alpha_2 = 0.20$. The RSS, BLUE, URSS, LRSS, and ERSS_1 schemes are to be compared for even sample sizes, whereas for odd sample sizes all except the ERSS_1 scheme are to be compared.

TABLE 3
The efficiencies for $\alpha_2 = -0.25$.

n	α_3	$E_{\text{RSS}}^{\text{SRS}}$	$E_{\text{BLUE}}^{\text{RSS}}$	E_{LRSS}	E_{URSS}	E_{ERSS_1}	E_{ERSS_2}	E_{ERSS_3}	E_{ERSS_4}	E_{ERSS_5}
3	-0.75	1.0216	1.0011	1.2153	0.8069	—	1.2099	1.0000	1.0792	0.9430
	-0.50	1.0095	1.0005	1.1420	0.8679	—	1.2043	1.0000	1.0506	0.9593
	-0.25	1.0024	1.0001	1.0700	0.9325	—	1.2011	1.0000	1.0242	0.9783
	0.25	1.0024	1.0001	0.9325	1.0700	—	1.2011	1.0000	0.9783	1.0242
	0.50	1.0095	1.0005	0.8679	1.1420	—	1.2043	1.0000	0.9593	1.0506
	0.75	1.0216	1.0011	0.8069	1.2153	—	1.2099	1.0000	0.9430	1.0792
5	-0.75	1.0290	1.0016	1.3026	0.7577	—	1.1422	1.0182	1.0846	0.9756
	-0.50	1.0127	1.0007	1.1961	0.8305	—	1.1245	1.0080	1.0498	0.9767
	-0.25	1.0031	1.0002	1.0950	0.9116	—	1.1144	1.0020	1.0216	0.9850
	0.25	1.0031	1.0002	0.9116	1.0950	—	1.1144	1.0020	0.9850	1.0216
	0.50	1.0127	1.0007	0.8305	1.1961	—	1.1245	1.0080	0.9767	1.0498
	0.75	1.0290	1.0016	0.7577	1.3026	—	1.1422	1.0182	0.9756	1.0846
10	-0.75	1.0358	1.0022	1.3901	0.7200	1.0550	—	—	—	—
	-0.50	1.0156	1.0008	1.2485	0.7992	1.0239	—	—	—	—
	-0.25	1.0039	1.0002	1.1185	0.8933	1.0059	—	—	—	—
	0.25	1.0039	1.0002	0.8933	1.1185	1.0059	—	—	—	—
	0.50	1.0156	1.0008	0.7992	1.2485	1.0239	—	—	—	—
	0.75	1.0358	1.0022	0.7200	1.3901	1.0550	—	—	—	—
15	-0.75	1.0384	1.0024	1.4251	0.7077	—	1.1007	1.0593	1.0903	1.0425
	-0.50	1.0167	1.0009	1.2690	0.7882	—	1.0624	1.0256	1.0447	1.0126
	-0.25	1.0041	1.0002	1.1275	0.8866	—	1.0412	1.0063	1.0151	0.9990
	0.25	1.0041	1.0002	0.8866	1.1275	—	1.0412	1.0063	0.9990	1.0151
	0.50	1.0167	1.0009	0.7882	1.2690	—	1.0624	1.0256	1.0126	1.0447
	0.75	1.0384	1.0024	0.7077	1.4251	—	1.1007	1.0593	1.0425	1.0903
20	-0.75	1.0397	1.0025	1.4439	0.7017	1.0728	—	—	—	—
	-0.50	1.0173	1.0010	1.2800	0.7826	1.0313	—	—	—	—
	-0.25	1.0043	1.0002	1.1322	0.8831	1.0077	—	—	—	—
	0.25	1.0043	1.0002	0.8831	1.1322	1.0077	—	—	—	—
	0.50	1.0173	1.0010	0.7826	1.2800	1.0313	—	—	—	—
	0.75	1.0397	1.0025	0.7017	1.4439	1.0728	—	—	—	—

TABLE 4
The efficiencies for $\alpha_2 = 0.20$.

n	α_3	$E_{\text{RSS}}^{\text{SRS}}$	$E_{\text{BLUE}}^{\text{RSS}}$	E_{LRSS}	E_{URSS}	E_{ERSS_1}	E_{ERSS_2}	E_{ERSS_3}	E_{ERSS_4}	E_{ERSS_5}
3	-0.8	1.0309	1.0031	1.1795	0.8535	—	1.2118	1.0000	1.0708	0.9622
	-0.6	1.0171	1.0013	1.1355	0.8827	—	1.2070	1.0000	1.0512	0.9669
	-0.4	1.0075	1.0005	1.0900	0.9179	—	1.2032	1.0000	1.0327	0.9753
	-0.2	1.0019	1.0001	1.0445	0.9575	—	1.2008	1.0000	1.0155	0.9865
	0.2	1.0019	1.0001	0.9575	1.0445	—	1.2008	1.0000	0.9865	1.0155
	0.4	1.0075	1.0005	0.9179	1.0900	—	1.2032	1.0000	0.9753	1.0327
	0.6	1.0171	1.0013	0.8827	1.1355	—	1.2070	1.0000	0.9669	1.0512
	0.8	1.0309	1.0031	0.8535	1.1795	—	1.2118	1.0000	0.9622	1.0708
5	-0.8	1.0416	1.0061	1.2530	0.8430	—	1.1520	1.0292	1.0890	1.0070
	-0.6	1.0230	1.0022	1.1897	0.8609	—	1.1345	1.0153	1.0581	0.9924
	-0.4	1.0101	1.0007	1.1245	0.8971	—	1.1216	1.0065	1.0335	0.9880
	-0.2	1.0025	1.0001	1.0606	0.9447	—	1.1137	1.0016	1.0142	0.9911
	0.2	1.0025	1.0001	0.9447	1.0606	—	1.1137	1.0016	0.9911	1.0142
	0.4	1.0101	1.0007	0.8971	1.1245	—	1.1216	1.0065	0.9880	1.0335
	0.6	1.0230	1.0022	0.8609	1.1897	—	1.1345	1.0153	0.9924	1.0581
	0.8	1.0416	1.0061	0.8430	1.2530	—	1.1520	1.0292	1.0070	1.0890
10	-0.8	1.0516	1.0102	1.3252	0.8696	1.0974	—	—	—	—
	-0.6	1.0284	1.0033	1.2428	0.8525	1.0476	—	—	—	—
	-0.4	1.0124	1.0009	1.1577	0.8816	1.0197	—	—	—	—
	-0.2	1.0031	1.0002	1.0757	0.9338	1.0047	—	—	—	—
	0.2	1.0031	1.0002	0.9338	1.0757	1.0047	—	—	—	—
	0.4	1.0124	1.0009	0.8816	1.1577	1.0197	—	—	—	—
	0.6	1.0284	1.0033	0.8525	1.2428	1.0476	—	—	—	—
	0.8	1.0516	1.0102	0.8696	1.3252	1.0974	—	—	—	—
15	-0.8	1.0553	1.0121	1.3535	0.8955	—	1.1231	1.1120	1.1398	1.1092
	-0.6	1.0304	1.0038	1.2637	0.8531	—	1.0842	1.0523	1.0721	1.0447
	-0.4	1.0133	1.0010	1.1706	0.8768	—	1.0562	1.0212	1.0335	1.0139
	-0.2	1.0033	1.0002	1.0814	0.9299	—	1.0398	1.0051	1.0107	1.0006
	0.2	1.0033	1.0002	0.9299	1.0814	—	1.0398	1.0051	1.0006	1.0107
	0.4	1.0133	1.0010	0.8768	1.1706	—	1.0562	1.0212	1.0139	1.0335
	0.6	1.0304	1.0038	0.8531	1.2637	—	1.0842	1.0523	1.0447	1.0721
	0.8	1.0553	1.0121	0.8955	1.3535	—	1.1231	1.1120	1.1092	1.1398
20	-0.8	1.0573	1.0132	1.3686	0.9145	1.1416	—	—	—	—
	-0.6	1.0315	1.0040	1.2749	0.8545	1.0647	—	—	—	—
	-0.4	1.0137	1.0011	1.1775	0.8745	1.0260	—	—	—	—
	-0.2	1.0034	1.0002	1.0845	0.9279	1.0062	—	—	—	—
	0.2	1.0034	1.0002	0.9279	1.0845	1.0062	—	—	—	—
	0.4	1.0137	1.0011	0.8745	1.1775	1.0260	—	—	—	—
	0.6	1.0315	1.0040	0.8545	1.2749	1.0647	—	—	—	—
	0.8	1.0573	1.0132	0.9145	1.3686	1.1416	—	—	—	—

The efficiencies in Table 3 and Table 4 are used to get an idea about the gain in efficiency of the estimators of θ_2. The results indicate that it all depends on the sample size and the association parameters. We summarize the performance of the estimators in the following.

i) There is notable gain in efficiency of the estimator of θ_2 under the LRSS scheme over the BLUE under the RSS scheme as the sample size increases.

ii) For even sample sizes, the BLUE based on the LRSS (URSS) scheme outperforms the BLUE under the RSS, URSS (LRSS) and ERSS_1 schemes for $\alpha_3 < 0$ ($\alpha_3 > 0$). For n =5, 10, 15 and 20, the gain in efficiency for $\hat{\theta}_{2,\text{LRSS}}$ over $\hat{\theta}_{2,\text{BLUE}}$ is almost 30%, 39%, 43% and 44%, respectively, when $\alpha_2 = -0.25$ and $\alpha_3 = -0.75$, whereas the same amount of gain in efficiency for $\hat{\theta}_{2,\text{URSS}}$ over $\hat{\theta}_{2,\text{BLUE}}$ is observed with respect to n when $\alpha_2 = -0.25$ and $\alpha_3 = 0.75$.

iii) When the sample size is odd, no particular sampling scheme among the LRSS, URSS, ERSS_i, $i = 2,4,5$ schemes is found to be the most efficient unconditionally.

iv) It is observed that the BLUE based on LRSS/URSS and ERSS_i, $i = 2,4,5$ are close competitors, and the performance of efficiencies under the respective sampling schemes change with respect to the association parameters and the sample size. In most cases it is observed that for $\alpha_3 < 0$ ($\alpha_3 > 0$), the LRSS (URSS) scheme ranks first followed by the ERSS_2 scheme (except $n = 3$).

v) For the same values of n and α_2, ERSS_i, $i = 2,4,5$ outperforms URSS/LRSS for some values of α_3, whereas URSS/LRSS outperforms ERSS_i for other values of α_3. No specific condition on parameters is observed for the performance of these schemes one over the other.

In the following, we will examine the trends in the efficiencies of the estimators under various schemes with respect to n, α_2 and α_3 graphically.

5.2. *Efficiency across α_3 when the sample size and α_2 are fixed*

Figure 3 presents the trends in efficiencies of $\hat{\theta}_{2,\text{LRSS}}$, $\hat{\theta}_{2,\text{URSS}}$ and $\hat{\theta}_{2,\text{ERSS}_1}$ over $\hat{\theta}_{2,\text{BLUE}}$ across α_3 when $n = 6$ and $\alpha_2 = -0.2$. As the sample size is even, here the ERSS_i, $i = 2,3,4,5$ schemes are not under consideration.

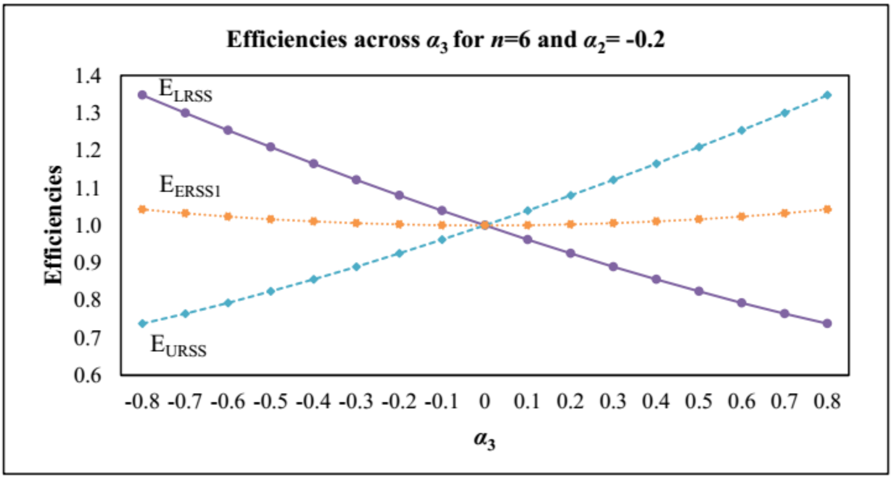

Figure 3 – Efficiencies $E_{\text{LRSS}}, E_{\text{URSS}}$ and E_{ERSS_1} for $n = 6$ and $\alpha_2 = -0.2$.

Figure 4 presents the trends in efficiencies of $\hat{\theta}_{2,\text{ERSS}_i}, i = 2, 3, 4, 5$ along with $\hat{\theta}_{2,\text{LRSS}}$, $\hat{\theta}_{2,\text{URSS}}$ over $\hat{\theta}_{2,\text{BLUE}}$ across α_3 for $n = 9$ and $\alpha_2 = 0.2$.

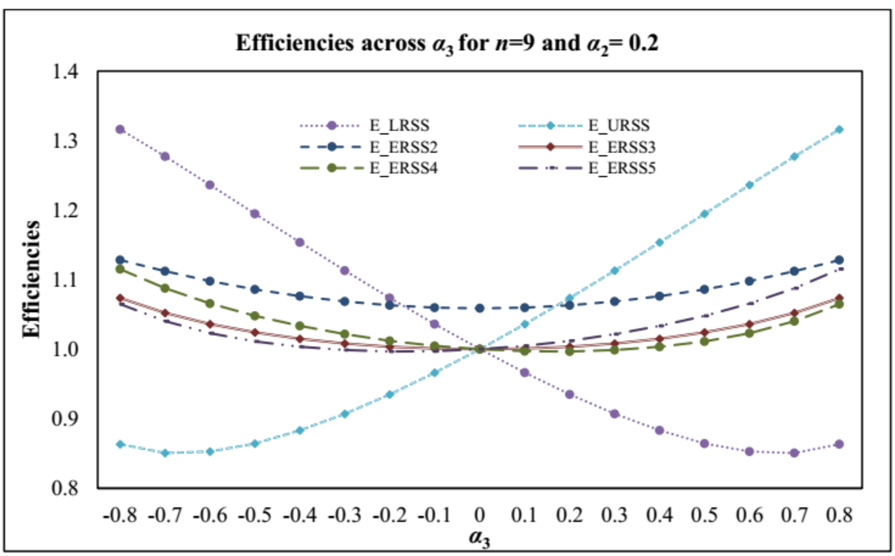

Figure 4 – Efficiencies $E_{\text{LRSS}}, E_{\text{URSS}}$ and E_{ERSS_i}, $i = 2,3,4,5$ for $n = 9$ and $\alpha_2 = 0.2$.

Here for $\alpha_3 < 0$, E_{LRSS} decreases quadratically as α_3 increases. The behavior of E_{URSS} is exactly opposite to that of E_{LRSS} with respect to α_3 as $\mathrm{Var}(\hat{\theta}_{2,\mathrm{LRSS}})|_{\alpha_3} = \mathrm{Var}(\hat{\theta}_{2,\mathrm{URSS}})|_{-\alpha_3}$. From Figure 4 it can be observed that for $\alpha_3 < -0.2$ ($\alpha_3 > 0.2$), the estimator of θ_2 under LRSS (URSS) scheme outperforms all other schemes when $\alpha_2 = 0.2$. We observe that efficiencies $E_{\mathrm{ERSS}_i}, i = 2,3,4,5$ vary in a small interval. Their performance is presented separately in Figure 5 for the same values of the parameters as in Figure 4.

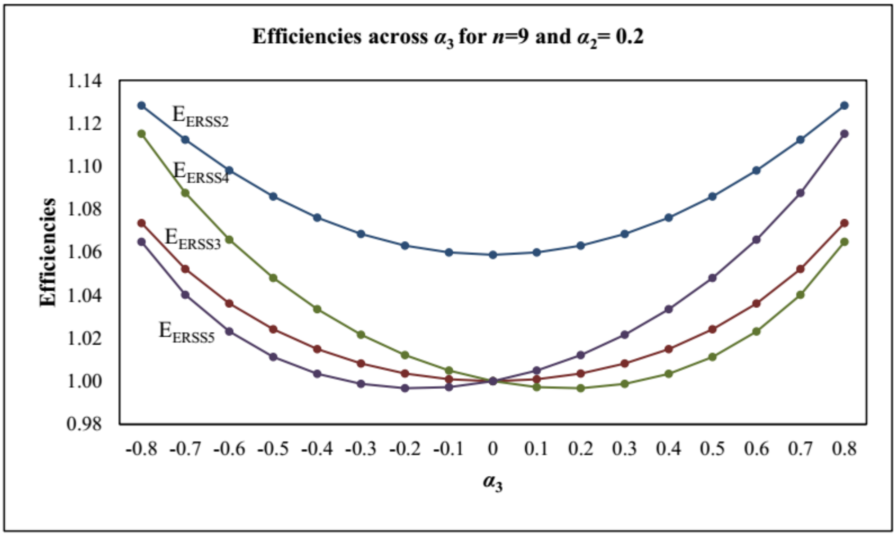

Figure 5 – Efficiencies E_{ERSS_i}, $i = 2,3,4,5$ for $n = 9$ and $\alpha_2 = 0.2$.

From Figure 5, observe that the estimator of θ_2 under ERSS_2 is more efficient than ERSS_i, $i = 3,4,5$. The efficiencies $E_{\mathrm{ERSS}_i}, i = 2,3,4,5$ increase quadratically as $|\alpha_3|$ increases. The behavior of E_{ERSS_4} is exactly opposite to that of E_{ERSS_5} with respect to α_3. It can also be observed that the rank of efficiencies of the estimators under these sampling schemes may change for different values of n, α_2 and α_3.

5.3. *Efficiency of $\hat{\theta}_{2,LRSS}$ and $\hat{\theta}_{2,URSS}$ across n and α_3*

As the LRSS and URSS schemes are conditionally best performing, we study their performance in detail. The variation in efficiencies of $\hat{\theta}_{2,\mathrm{LRSS}}$ and $\hat{\theta}_{2,\mathrm{URSS}}$ over $\hat{\theta}_{2,\mathrm{BLUE}}$ with respect to n and α_3 are presented in Figure 6 and 7, respectively, for $\alpha_2 = -0.25$. These graphs help us to study the effect of n and α_3 jointly on the efficiencies E_{LRSS} and E_{URSS}

for fixed α_2. The efficiency performance of $\hat{\theta}_{2,\text{LRSS}}$ for $\alpha_3 < 0$ is exactly the same as that of the performance of $\hat{\theta}_{2,\text{URSS}}$ for $\alpha_3 > 0$, thus the efficiency performances E_{LRSS} and E_{URSS} are complementary to each other. Further both the efficiencies E_{LRSS} and E_{URSS} increase quadratically with respect to n and stabilize for larger n. Also, for $\alpha_3 < 0$ ($\alpha_3 > 0$), E_{LRSS} (E_{URSS}) increases (decreases) irrespective of n for given α_2.

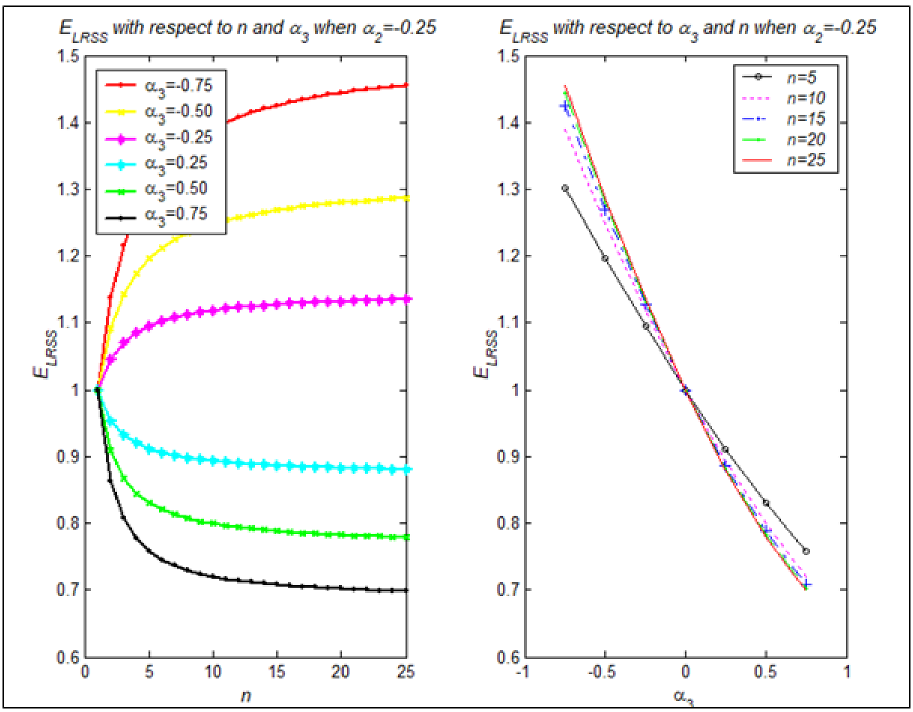

Figure 6 – Efficiency E_{LRSS} of $\hat{\theta}_{2,\text{LRSS}}$ over $\hat{\theta}_{2,\text{BLUE}}$ across n and α_3 when $\alpha_2 = -0.25$.

To visualize the gain in efficiencies of $\hat{\theta}_{2,\text{LRSS}}$ and $\hat{\theta}_{2,\text{URSS}}$ over $\hat{\theta}_{2,\text{BLUE}}$ with respect to α_2 and α_3 simultaneously, we present surface plots of E_{LRSS} and E_{URSS} for $n = 7$ in Figure 8.

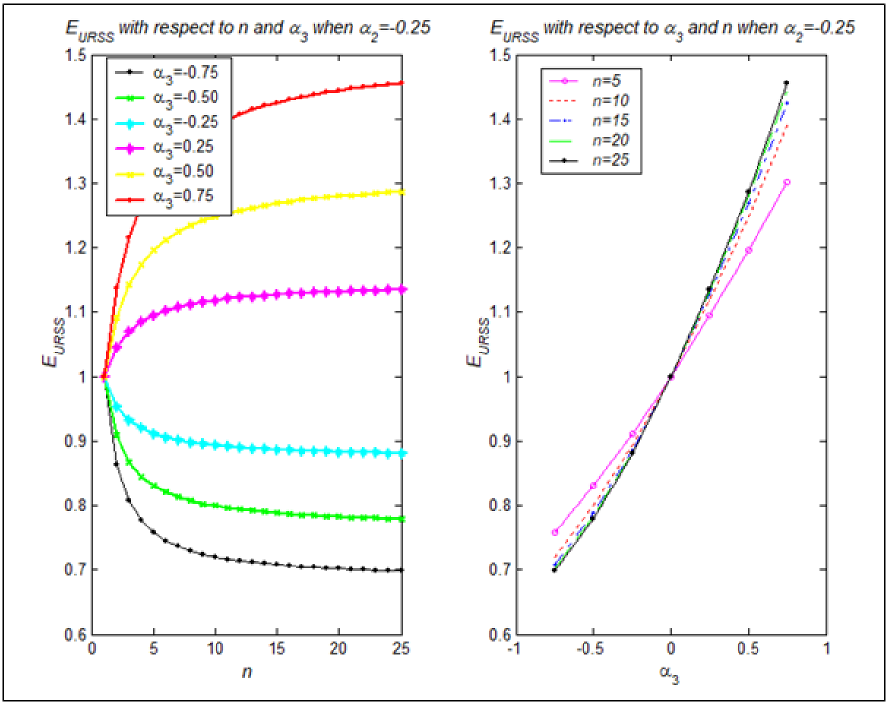

Figure 7 – Efficiency E_{URSS} of $\hat{\theta}_{2,\text{URSS}}$ over $\hat{\theta}_{2,\text{BLUE}}$ across n and α_3 when $\alpha_2 = -0.25$.

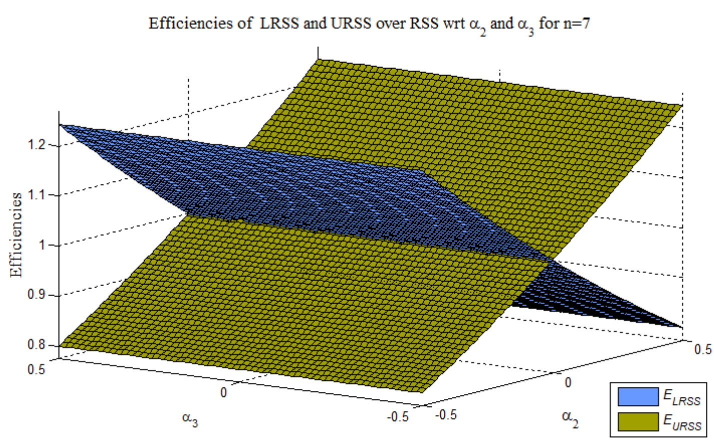

Figure 8 – Efficiency E_{LRSS} (E_{URSS}) of $\hat{\theta}_{2,\text{LRSS}}$ ($\hat{\theta}_{2,\text{URSS}}$) over $\hat{\theta}_{2,\text{BLUE}}$ with respect to α_2 and α_3 when $n = 7$.

The trends in efficiencies E_{LRSS} and E_{URSS} with respect to α_2 and α_3 can be seen marginally as well as jointly in Figure 8.

5.4. *Efficiency of $\hat{\theta}_{2,ERSS_2}$ and $\hat{\theta}_{2,ERSS_4}$ over $\hat{\theta}_{2,LRSS}$ and $\hat{\theta}_{2,URSS}$*

We observe that for odd sample sizes, the estimators $\hat{\theta}_{2,\text{LRSS}}$, $\hat{\theta}_{2,\text{URSS}}$, $\hat{\theta}_{2,\text{ERSS}_i}$, $i = 2, 4, 5$ compete for efficiency performance. Further as $\text{Var}(\hat{\theta}_{2,\text{ERSS}_4})|_{\alpha_3} = \text{Var}(\hat{\theta}_{2,\text{ERSS}_5})|_{-\alpha_3}$, among $\hat{\theta}_{2,\text{ERSS}_4}$ and $\hat{\theta}_{2,\text{ERSS}_5}$, we consider only $\hat{\theta}_{2,\text{ERSS}_4}$ for study. Hence, we consider efficiencies of $\hat{\theta}_{2,\text{ERSS}_2}$ and $\hat{\theta}_{2,\text{ERSS}_4}$ over $\hat{\theta}_{2,\text{LRSS}}$ and $\hat{\theta}_{2,\text{URSS}}$ for further study. Specifically, the efficiency performance with respect to α_2 and α_3 for given sample size is of interest. Figure 9 and Figure 10 present the efficiency performance of $\hat{\theta}_{2,\text{ERSS}_2}$ and $\hat{\theta}_{2,\text{ERSS}_4}$ over $\hat{\theta}_{2,\text{LRSS}}$, $\hat{\theta}_{2,\text{URSS}}$ for $n = 9$ respectively.

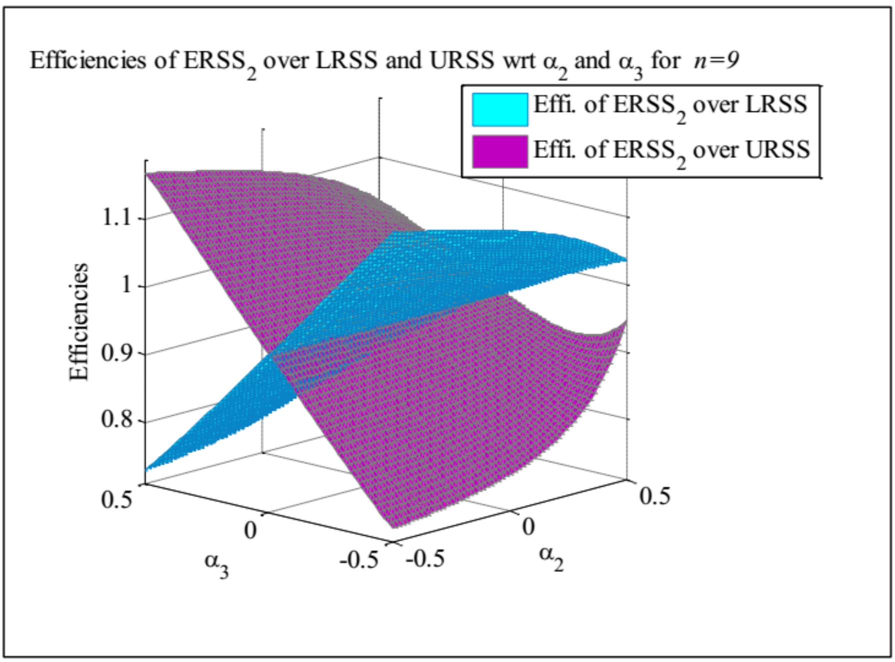

Figure 9 – Efficiency of $\hat{\theta}_{2,\text{ERSS}_2}$ over $\hat{\theta}_{2,\text{LRSS}}$ and $\hat{\theta}_{2,\text{URSS}}$ for $n = 9$.

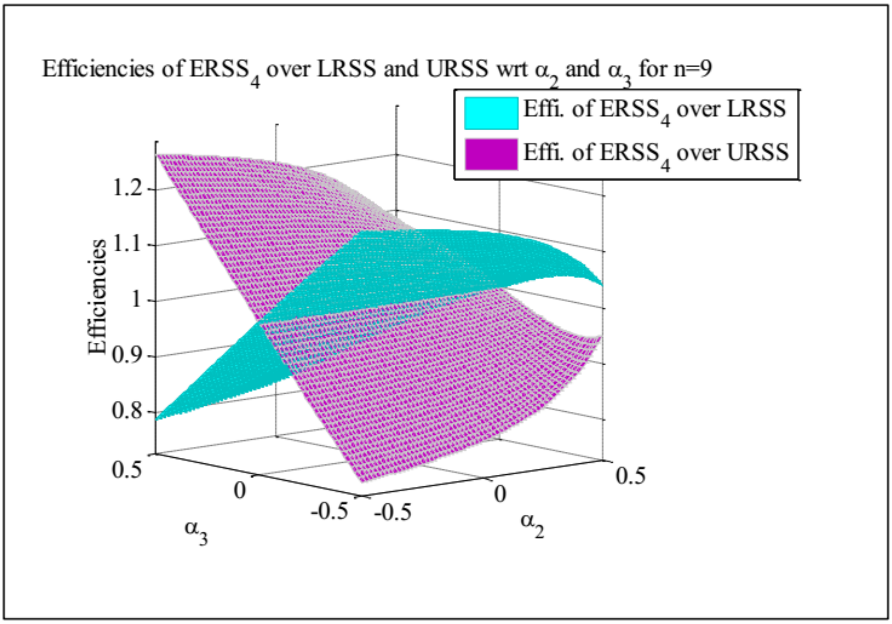

Figure 10 – Efficiency of $\hat{\theta}_{2,\text{ERSS}_4}$ over $\hat{\theta}_{2,\text{LRSS}}$ and $\hat{\theta}_{2,\text{URSS}}$ for $n=9$.

Figure 9 and Figure 10 help to understand the over- or underperformance of $\hat{\theta}_{2,\text{ERSS}_2}$ and $\hat{\theta}_{2,\text{ERSS}_4}$ over $\hat{\theta}_{2,\text{LRSS}}$ and $\hat{\theta}_{2,\text{URSS}}$ with respect to α_2 and α_3 for given sample size. These findings are usually affected by the sample size and association parameters.

6. Simulation from the CTBE $(\alpha_2, \alpha_3, \theta_1, \theta_2)$ distribution

The simulation study from the CTBE$(\alpha_2, \alpha_3, \theta_1, \theta_2)$ distribution will be helpful to verify and illustrate the developed results. But presently no software is known to provide simulations from the CTBE distribution. Hence to overcome this difficulty, we develop an algorithm to simulate data from the CTBE$(\alpha_2, \alpha_3, \theta_1, \theta_2)$ distribution and implement it in `Matlab`. We develop a `Matlab` function rctbe$(\alpha_2, \alpha_3, \theta_1, \theta_2, n)$ that generates a random sample of size n from the CTBE$(\alpha_2, \alpha_3, \theta_1, \theta_2)$ distribution. It is based on the following algorithm.

i) Simulate u_0 from the $U(0,1)$ distribution and use it as value of $F(x) = 1 - e^{-x/\theta_1}$. We get a realization of X by inverting $F(x) = u_0$ as $x_0 = -\theta_1 \log(1 - u_0)$.

ii) Generate another $U(0,1)$ random number v_0 and use it as a simulated value of

$F(y|x_0)$. The distribution function of $Y|X=x$ is

$$F(y|x)=\left(1-e^{\frac{y}{\theta_2}}\right)\left[1+\alpha_2 e^{\frac{-y}{\theta_2}}+\alpha_3\left(2e^{\frac{-x}{\theta_1}}-1\right)e^{\frac{-y}{\theta_2}}\right]. \quad (28)$$

Now solve the equation $v_0=\left(1-e^{\frac{y}{\theta_2}}\right)\left[1+\alpha_2 e^{\frac{-y}{\theta_2}}+\alpha_3\left(2e^{\frac{-x_0}{\theta_1}}-1\right)e^{\frac{-y}{\theta_2}}\right]$, for y, which leads to two solutions. Among these two solutions choose y such that $0<y<\infty$ and let it be y_0.

iii) The pair (x_0, y_0) represents a sample from CTBE$(\alpha_2,\alpha_3,\theta_1,\theta_2)$ distribution.

To check the validity of simulation methodology developed, samples of sizes 100 and 1000 are generated from the CTBE$(\alpha_2,\alpha_3,1,1)$ distribution for $(\alpha_2,\alpha_3)=(\pm 0.1,\pm 0.9)$ using the `rctbe()` function in `Matlab`, and various sample quantities are compared with the corresponding population quantities. The results are summarised in Table 5.

TABLE 5
Sample and population quantities associated with the simulated data.

(α_2,α_3)	Population quantities		Sample quantities	$n=100$	$n=1000$
(0.1, 0.9)	$E(X)$	1	$\overline{X}$	0.9303	1.0108
	$E(Y)$	0.9500	$\overline{Y}$	1.0573	0.9645
	Var(X)	1	S_X^2	0.8377	1.0355
	Var(Y)	0.9475	S_Y^2	1.2624	0.9954
	ρ	0.2312	r	0.5221	0.2452
(−0.1, 0.9)	$E(X)$	1	$\overline{X}$	0.9226	0.9966
	$E(Y)$	1.0500	$\overline{Y}$	1.2158	1.0978
	Var(X)	1	S_X^2	0.7505	1.0302
	Var(Y)	1.0475	S_Y^2	1.0607	1.1613
	ρ	0.2198	r	0.1891	0.2086
(0.1, −0.9)	$E(X)$	1	$\overline{X}$	0.9281	1.0005
	$E(Y)$	0.9500	$\overline{Y}$	0.9974	0.9480
	Var(X)	1	S_X^2	0.8647	0.9952
	Var(Y)	0.9475	S_Y^2	0.7314	0.9099
	ρ	-0.2312	r	-0.1583	-0.2383
(−0.1, −0.9)	$E(X)$	1	$\overline{X}$	0.9635	0.9987
	$E(Y)$	1.0500	$\overline{Y}$	1.0257	1.0788
	Var(X)	1	S_X^2	0.6830	1.0008
	Var(Y)	1.0475	S_Y^2	0.9895	1.0327
	ρ	-0.2198	r	-0.2360	-0.2254

Now we present one more simulation study which shows the reasonability of the estimators obtained under the method of moments in Section 4 and provide support to the use of the moment estimates as substitutes of population parameters when needed. The moment estimators of all parameters of the CTBE$(\alpha_2, \alpha_3, \theta_1, \theta_2)$ distribution proposed in Section 4 are evaluated for samples of sizes 500 and 1000 for some values of $\alpha_2, \alpha_3, \theta_1, \theta_2$. The results are presented in Table 6.

TABLE 6
Moment estimates of the parameter of the CTBE$(\alpha_2, \alpha_3, \theta_1, \theta_2)$ distribution for simulated data.

n	True value of parameter				Moment estimates of parameters			
	α_2	α_3	θ_1	θ_2	$\hat{\alpha}_2$	$\hat{\alpha}_3$	$\hat{\theta}_1$	$\hat{\theta}_2$
500	0.2	0.8	1	1	0.2963(0.9047)	0.7037	0.9758	1.0681
	0.1	-0.9	2	2.5	0.2455(0.9248)	-0.7545	1.9848	2.5925
1000	0.2	0.8	1	1	0.2188(0.9346)	0.7491	0.9417	0.9798
	0.1	-0.9	2	2.5	0.2237(0.9328)	-0.7763	2.1159	2.592

The values in the bracket from Table 6 indicate an infeasible solution of the quadratic moment equation for $\hat{\alpha}_2$ in Eq. (23) for respective samples. The next Section is devoted to illustrating the performance of the proposed estimators of θ_2 under different RSS schemes through a simulation study and real-life data.

7. An application

This Section illustrates applications of the developed results to simulated and real-life data. The performance of the proposed estimators based on simulated data from the CTBE$(\alpha_2, \alpha_3, \theta_1, \theta_2)$ distribution under the RSS, LRSS, URSS, and ERSS$_1$ schemes is demonstrated. Further, the RSS sampling schemes are used to estimate the mean of minimum mercury concentration in sampled fish using mercury concentration data under the assumption that $(X, Y) \sim$ CTBE$(\alpha_2, \alpha_3, \theta_1, \theta_2)$ distribution.

7.1. Estimation based on simulated data

To compare the performance of estimators under various RSS schemes, we perform a simulation study. The following steps are followed to get the results.

- Simulate 5000 pairs of observations from the distribution CTBE$(\alpha_2 = \alpha_2^0, \alpha_3 = \alpha_3^0, \theta_1 = \theta_1^0, \theta_2 = \theta_2^0)$ using the `rctbe()` function developed in `Matlab` and treat it as a population.
- From this populations of size 5000, generate RSS, LRSS, URSS, and ERSS samples of size n each, using the `RSSampling` package (Sevinc *et al.*, 2019) in `R`.

- Estimate θ_2 and its variance for each of the RSS, LRSS, URSS, and ERSS samples under the assumption that $(X,Y) \sim \text{CTBE}(\alpha_2^0, \alpha_3^0, \theta_1^0, \theta_2)$.

The experiments are performed for $(\alpha_2^0, \alpha_3^0, \theta_1^0, \theta_2^0) = (-0.1, 0.9, 1, 1), (-0.1, -0.9, 1, 1)$ and $(-0.1, 0.9, 1, 2)$ with $n = 10$. Results are reported in Table 7.

TABLE 7
The Estimates of θ_2 under different RSS schemes for simulated data.

$(X,Y) \sim \text{CTBE}(\alpha_2, \alpha_3, \theta_1, \theta_2)$	Scheme	$\hat{\theta}_2$	$\text{Var}(\hat{\theta}_2)$
(-0.1,0.9,1,1)	RSS	1.62	0.24
	ERSS_1	1.30	0.35
	LRSS	1.36	0.23
	URSS	1.19	0.09
(-0.1,-0.9,1,1)	RSS	1.16	0.12
	ERSS_1	1.25	0.19
	LRSS	0.98	0.06
	URSS	0.93	0.11
(-0.1,0.9,1,2)	RSS	1.79	0.29
	ERSS_1	1.25	0.33
	LRSS	1.38	0.24
	URSS	1.80	0.20

The results shows that for $\alpha_3 > 0$, $\hat{\theta}_{2,\text{URSS}}$ has the smallest variance and for $\alpha_3 < 0$, $\hat{\theta}_{2,\text{LRSS}}$ has the smallest variance. This confirms the outcomes of the efficiency performance study.

7.2. *Estimation of mean minimum mercury concentration in sampled fish*

We consider a data set used by Lange *et al.* (1993) who studied the influence of water chemistry on mercury concentration in largemouth bass from 53 different Florida lakes. The data consist of amount of alkalinity (mg/l), calcium (mg/l), chlorophyll (mg/l) etc. in each of the water samples. Then the sample of fishes was taken from each lake to measure the minimum mercury concentration (μg/g). Lange *et al.* (1993) observed that the bio-accumulation of mercury in the largemouth bass was strongly influenced by the chemical characteristics of the lakes. In the present study we consider bivariate data, namely the amount of alkalinity (mg/l) in water sample as variable X and the minimum mercury concentration in the sampled fish as variable Y. These data are also used by Mohsin *et al.* (2014) and Chacko (2017). Mohsin *et al.* (2014) fitted a bivariate exponential distribution to the data while Chacko (2017) assumed (X,Y) to have an MTBE distribution. We assume (X,Y) to follow the $\text{CTBE}(\alpha_2, \alpha_3, \theta_1, \theta_2)$ distribution with the objective to estimate the scale parameter θ_2 associated with Y and hence to estimate the mean minimum mercury concentration in largemouth bass from different Florida lakes. We draw random samples of size 6 from the data set using `RSSampling` package in R using the RSS, LRSS, and ERSS_1 schemes. The samples under the RSS schemes are given in Table 8.

TABLE 8
Samples under various RSS schemes.

Scheme	Sample values for y-variable					
RSS	0.31	0.37	0.26	0.04	0.31	0.07
LRSS	0.31	0.31	0.31	0.36	0.25	0.30
ERSS_1	0.31	0.04	0.31	0.04	0.25	0.07

The estimator of θ_2 under different RSS schemes is a function of α_2 and α_3, which are unknown in this situation. Hence, we obtain moment estimates of α_2 and α_3 based on all available data as proposed in Section 4. These give $\hat{\alpha}_2 = -0.7759$ and $\hat{\alpha}_3 = -0.2241$.

We also wish to compare the estimators of θ_2 under different RSS schemes when $(X,Y) \sim \text{MTBE}(\alpha_3, \theta_1, \theta_2)$ (i.e. $\text{CTBE}(0, \alpha_3, \theta_1, \theta_2)$). Under this assumption we need an estimator of α_3. We use the estimator of α_3 given by Chacko and Thomas (2008). Accordingly, $\hat{\alpha}_3 = -1$ as $\text{corr}(X,Y) = -0.5254$. Table 9 shows the estimates of θ_2, when $(X,Y) \sim \text{CTBE}(\alpha_2, \alpha_3, \theta_1, \theta_2)$ and $(X,Y) \sim \text{MTBE}(\alpha_3, \theta_1, \theta_2)$ under the RSS, LRSS and ERSS_1 schemes.

TABLE 9
Estimate of θ_2 under different RSS schemes for bivariate data.

Assumption about distribution	Scheme	Estimator of θ_2	Estimate of θ_2	Variance $/\theta_2^2$	Estimated mean min. mercury conc.
$(X,Y) \sim \text{CTBE}(\alpha_2, \alpha_3, \theta_1, \theta_2)$	RSS	$\hat{\theta}_{2,\text{RSS}}$	0.17	0.11	0.23
	LRSS	$\hat{\theta}_{2,\text{LRSS}}$	0.21	0.10	0.29
	ERSS_1	$\hat{\theta}_{2,\text{ERSS}_1}$	0.13	0.17	0.18
$(X,Y) \sim \text{MTBE}(\alpha_3, \theta_1, \theta_2)$ (i.e. $\text{CTBE}(0, \alpha_3, \theta_1, \theta_2)$)	RSS	$\hat{\theta}_{2,\text{RSS}}$	0.23	0.16	0.23
	LRSS	$\hat{\theta}_{2,\text{LRSS}}$	0.23	0.11	0.23
	ERSS_1	$\hat{\theta}_{2,\text{ERSS}_1}$	0.17	0.20	0.17

The results in Table 9 are consistent with those for simulated data. That is, among the RSS, LRSS and ERSS_1 schemes, $\hat{\theta}_{2,\text{LRSS}}$ has the smallest variance. This confirms the conclusion that for even sample sizes, the LRSS scheme gives the most efficient estimator of θ_2 when $\alpha_3 < 0$. Further, observe that the estimates of θ_2 under respective RSS schemes are better in terms of variances when the variables (X,Y) are assumed to have the CTBE distribution than that of when they are assumed to have the MTBE distribution.

8. Conclusions

This paper considers the problem of estimation of scale parameter associated with one variate of CTBE distribution under various RSS schemes. Some important aspects of CTBE distribution are obtained, namely its COS, and moment estimators of all parameters. The core contribution of the paper is estimation of the scale parameter associated with a study variable under different RSS schemes, namely RSS, LRSS, URSS, and ERSS. The efficiency performance of the proposed estimators is presented numerically and variation in efficiencies with respect to association parameters and sample size is presented graphically. The proposed estimators' performance is also studied through a simulation study and implemented to real-life data. The other important contribution of the paper is an algorithm for simulation from the CTBE distribution and its implementation in software. The simulation is extensively used to verify the results developed and to confirm the feasibility of the proposed moment estimators.

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