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CLASSIFICATION RULES FOR TWO EXPONENTIAL POPULATIONS WITH A COMMON LOCATION USING CENSORED SAMPLES

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1. INTRODUCTION

The problem of classification into several populations or groups has applications in almost all branches of science and engineering. In a general framework, the problem of classification consists in classifying a new observation, say x into one of the several populations/groups, say $\Pi_1, \Pi_2, \ldots, \Pi_k$, using the information available for x and certain methodologies. For example, let's consider a newly launched electronic/electrical/mechanical product in the market that needs to be classified into several categories in terms of its performance or quality: poor, average, good or excellent. Another example lays in medical diagnosis, where blood samples are collected to classify them into several groups: A, B, AB^+, A^+, etc. Moreover, in military surveillance, the aircrafts are recognized and identified basing on flight characteristics and moving patterns. The classification technique is also used in intelligent video surveillance applications, in pattern recognition, in the study of psychopathology, in the analysis of lifetime data, etc. One may refer to Webb (2003) for some applications of classification problems in pattern recognition for real-life situations.

In this article, we consider the classification rules for classifying a new observation or a group of observations into one of the two exponential populations $\text{Exp}(\mu, \sigma_1)$ and $\text{Exp}(\mu, \sigma_2)$ when the samples are type-II right censored, with a common location parameter μ and possibly different scale parameters σ_1 and σ_2, respectively. Here, $\text{Exp}(\mu, \sigma_i)$ denotes the exponential population Π_i, $i = 1,2$ with a probability density function

$$f_i(x|\mu, \sigma_i) = \frac{1}{\sigma_i} e^{-\left(\frac{x-\mu}{\sigma_i}\right)}, \quad x > \mu, \quad \mu \in \mathbb{R}, \quad \sigma_i > 0, \quad i = 1,2. \tag{1}$$

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The problem of classification using a complete sample has been studied under various statistical models. The exponential distribution is very useful in the study of survival analysis, life testing experiments, queuing theory, reliability theory, and related areas. Suppose two brands of electronic devices are put for a life testing experiment. Due to some constraints (maybe time or cost), the experimenter could observe only the first few observations from each of the two groups or populations. It is assumed that the lifetimes of each unit are random and follow exponential distributions with the same minimum guarantee time (location parameter). Based on the observed lifetimes of the new devices from one of these two brands, we want to classify this new device into one of these brands using classification rules. For more details on exponential distribution and classification using exponential distributions one may refer to Jana and Kumar (2016), Lawless (2003) and Basu and Gupta (1976).

The problem of classification and related parameter estimation using normal populations has been well investigated by many researchers in the past. We refer to Anderson (1951), Rueda *et al.* (1997), Long and Gupta (1998), Fernández *et al.* (2006), Conde *et al.* (2012), Jana and Kumar (2017) and to the references cited therein for details of classification and related estimation problems using several normal populations.

The classification problem using two or more exponential populations was first considered by Basu and Gupta (1971), then by Basu and Gupta (1976). The authors presented the classification rules by plugging the maximum likelihood estimators of the unknown parameters in the classification function and proved the consistency of the proposed rule. They also introduced the predictive classification rules by considering the conjugate prior distribution of the unknown parameters of the concerned populations. Furthermore, they proposed the classification rules based on the type-II censored samples. Adegboye (1993) discussed the optimal classification rule for one-parameter exponential populations. They obtained the expressions for the probabilities of misclassification and proved that it depends upon the ratio of the scale parameters. Conde *et al.* (2005) studied the classification problem for one-parameter exponential distribution with a restriction that the mean of the second population is greater than the mean of the first population and also obtained the improved classification rules. Recently, Jana and Kumar (2016) studied the classification problem for two two-parameter exponential distributions based on the mixed estimators of the associated parameters. However, the problem of classification based on censored samples has not gained much attention from researchers.

Interestingly, the classification rules obtained in this article are based on some estimators of the common location parameter under the type-II censoring scheme, which has been studied fairly in the literature of statistical inference. Chiou and Cohen (1984) dealt with the problem of estimating the common location parameter of two exponential populations using type-II censored samples. They proposed the Maximum Likelihood Estimator (MLE) and the Uniformly Minimum Variance Unbiased Estimator (UMVUE) for μ, when the scale parameters are unknown and different. Elfessi and Pal (1991) considered the estimation of common scale and different location parameters of several exponential populations using type-II censored samples. Recently, Tripathy

(2016) dealt with the problem of estimating the common location parameter of two exponential populations from a decision-theoretic viewpoint. He considered the affine equivariant class of estimators, which contains the MLE, a modification to the MLE, and the UMVUE of μ. He further obtained a sufficient condition for improving estimators in this class and proposed the improved estimators of MLE and UMVUE. It will be interesting to construct classification rules using these nice estimators under the type-II censoring scheme and compare their performances in terms of probabilities of correct classification.

When complete samples are available from two exponential populations, the problem of estimating common location parameter μ has been well investigated by several researchers in the recent past. For a detailed review and some recent updates on estimating μ using complete sample, we refer to Tripathy *et al.* (2014) and to the references cited therein. Jana and Kumar (2016) considered the classification problem for two exponential populations. The authors constructed several classification rules using some of the estimators of μ and studied their performances numerically. The problem of classification under complete sample from two or more exponential populations has been investigated by Basu and Gupta (1976), Adegboye (1993) and Conde *et al.* (2005).

The rest of our work is organized as follows. In Section 2, we discuss various estimators of the associated model parameters. In particular, we derive the UMVUEs of the scale parameters and sufficient conditions for improving equivariant estimators of the scale parameters. Section 3 discusses the classification rule for the two exponential populations with a common location parameter and different scale parameters using type-II censored samples and shows these rules are consistent. In Section 4, utilizing the MLE, the modified MLE, the UMVUE, and the improved estimators of the common location parameter proposed by Tripathy (2016), several classification rules are constructed. In Section 5, we numerically compare the performances of all the proposed classification rules in terms of probabilities of correct classification and expected probability of correct classification using the Monte-Carlo simulation method. In Section 6, we discuss a real-life example to show the potential application of our model problem.

2. Certain results on estimating the associated parameters

We note that Tripathy (2016) considered the same model problem and derived certain estimators of μ (when the scale parameters are unknown) that improve upon the MLE and UMVUE. In this Section, we consider a certain class of equivariant estimators for the two scale parameters σ_1 and σ_2, and derive sufficient conditions for improving estimators in the class using the orbit-by-orbit improvement technique of Brewster and Zidek (1974).

Suppose Π_i is the i^{th} shifted exponential population $\text{Exp}(\mu, \sigma_i)$ having probability density function

$$f_i(x|\mu,\sigma_i) = \frac{1}{\sigma_i} e^{-\left(\frac{x-\mu}{\sigma_i}\right)}, \quad x > \mu, \quad \mu \in \mathbb{R}, \quad \sigma_i > 0, \quad i = 1,2, \tag{2}$$

where μ is known as the common minimum guarantee time and σ_i the residual life after minimum guarantee period.

Let $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(r)}$, $2 \leq r \leq m$, be the type-II right censored samples taken from a random sample of size m having the probability density function f_1 (that is population Π_1). Similarly, let $Y_{(1)} \leq Y_{(2)} \leq \cdots \leq Y_{(s)}$, $2 \leq s \leq n$ be the type-II right censored sample taken from a random sample of size n having the probability density function f_2 (that is population Π_2). Here, r and s are known prefixed numbers. Note that, a complete and sufficient statistic for this model is (Z, V_1, V_2). The random variables are defined as

$$Z = \min(X_{(1)}, Y_{(1)}), \quad V_1 = U_1 - Z, \quad V_2 = U_2 - Z, \tag{3}$$

where

$$U_1 = \frac{1}{m}\left[\sum_{i=1}^{r} X_{(i)} + (m-r)X_{(r)}\right], \quad U_2 = \frac{1}{n}\left[\sum_{i=1}^{s} Y_{(i)} + (n-s)Y_{(s)}\right]. \tag{4}$$

Based on the type-II right censored samples, the MLEs of μ, σ_1 and σ_2 are, respectively, given by

$$\mu_{\text{ML}} = \min\left(X_{(1)}, Y_{(1)}\right) = Z, \quad \sigma_{1\text{ML}} = \frac{m}{r} V_1 \quad \text{and} \quad \sigma_{2\text{ML}} = \frac{n}{s} V_2. \tag{5}$$

Furthermore, a modification to the MLE of μ is obtained as

$$\mu_{\text{MM}} = Z - \frac{1}{\hat{p}}, \tag{6}$$

where $\hat{p} = m/\sigma_{1\text{ML}} + n/\sigma_{2\text{ML}}$ (see Tripathy, 2016).

The UMVUE of μ was obtained by Chiou and Cohen (1984) for equal sample sizes, however for unequal sample sizes, it was obtained by Tripathy (2016) under type-II right censoring scheme. The UMVUE of μ is given by

$$\mu_{\text{MV}} = Z - \frac{V_1 V_2}{(r-1)V_2 + (s-1)V_1}. \tag{7}$$

In the following Theorem, the UMVUEs of σ_1 and σ_2 are derived.

THEOREM 1. *The UMVUEs of σ_1 and σ_2 are given respectively as*

$$\sigma_{1MV} = \frac{m}{r(r-1)}\left[r V_1 - \frac{(s-1)V_1^2}{(s-1)V_1 + (r-1)V_2}\right] \tag{8}$$

and

$$\sigma_{2MV} = \frac{n}{s(s-1)}\left[s V_2 - \frac{(r-1)V_2^2}{(s-1)V_1 + (r-1)V_2}\right]. \tag{9}$$

PROOF. In order to prove the result, recall that the statistic (U_1, U_2, Z) is sufficient. Further the statistic (Z, V_1, V_2) is complete and sufficient. Let us define the new random variables T_1 and T_2 respectively as

$$T_1 = \sum_{i=1}^{r}(X_{(i)} - X_{(1)}) + (m-r)(X_{(r)} - X_{(1)}), \tag{10}$$

and

$$T_2 = \sum_{j=1}^{s}(Y_{(j)} - Y_{(1)}) + (n-s)(Y_{(s)} - Y_{(1)}). \tag{11}$$

Note that the the random variables $2T_1/\sigma_1$ and $2T_2/\sigma_2$ are independent and follow χ^2 distributions with degrees of freedom $2(r-1)$ and $2(s-1)$, respectively. The random variables $X_{(1)}$ and $Y_{(1)}$ follow exponential distributions $\text{Exp}(\mu, \sigma_1/m)$ and $\text{Exp}(\mu, \sigma_2/n)$, respectively, and are also independent. The random variables T_1 and T_2 follow gamma distributions with shape parameters $r-1$ and $s-1$ and scale parameters σ_1 and σ_2, respectively. Thus, an unbiased estimator of σ_1 is given by $T_1/(r-1)$.

The UMVUE of σ_1 is thus given by

$$\begin{aligned}\frac{1}{r-1}E\{T_1|(U_1-Z, U_2-Z, Z)\} &= \frac{m}{r-1}E\left[(U_1 - X_{(1)})|(U_1-Z, U_2-Z, Z)\right] \\ &= \frac{m}{r-1}E\left[(V_1 + Z - X_{(1)})|(U_1, U_2, Z)\right] \\ &= \frac{m}{r-1}\left[V_1 + Z - E\left\{X_{(1)}|(U_1, U_2, Z)\right\}\right]. \end{aligned}\tag{12}$$

The conditional expectation $E\left\{X_{(1)}|(U_1, U_2, Z)\right\}$ has been evaluated in Chiou and Cohen (1984), which is given by

$$E\left\{X_{(1)}|(U_1, U_2, Z)\right\} = Z + \frac{V_1^2}{r(r-1)W}, \tag{13}$$

where $W = \frac{V_1}{r-1} + \frac{V_2}{s-1}$.

Substituting Eq. (13) in Eq. (12), we get

$$\begin{aligned}\frac{1}{r-1}E\{T_1|(U_1-Z, U_2-Z, Z)\} &= \frac{m}{r-1}\left[V_1 - \frac{V_1^2}{r(r-1)W}\right] \\ &= \frac{m}{r(r-1)}\left[rV_1 - \frac{(s-1)V_1^2}{(s-1)V_1 + (r-1)V_2}\right] \end{aligned}\tag{14}$$

In a similar manner, the UMVUE of σ_2 can be derived, by considering $T_2/(s-1)$ as an unbiased estimator. Thus, we have

$$\frac{1}{s-1}E\{T_2|(U_1-Z, U_2-Z, Z)\} = \frac{n}{s-1}\left[V_2 + Z - E\left\{Y_{(1)}|(U_1, U_2, Z)\right\}\right]. \tag{15}$$

The conditional expectation is

$$E\left\{Y_{(1)}|(U_1,U_2,Z)\right\}=Z+\frac{V_2^2}{s(s-1)W}. \tag{16}$$

Substituting Eq. (16) in Eq. (15), we get

$$\begin{aligned}\frac{1}{s-1}E\left\{T_2|(U_1-Z,U_2-Z,Z)\right\} &= \frac{n}{s-1}\left[V_2-\frac{V_2^2}{s(s-1)W}\right]\\ &= \frac{n}{s(s-1)}\left[sV_2-\frac{(r-1)V_2^2}{(s-1)V_1+(r-1)V_2}\right]. \end{aligned}\tag{17}$$

This completes the proof of the Theorem. □

Tripathy (2016) proposed some estimators for μ under type-II censoring. These estimators, which improve upon the MLE and the UMVUE, are respectively given by

$$\hat{\mu}_{\rm ML}=\begin{cases} Z-\frac{V_2}{r+s}, & \text{if } V_1>V_2,\\ Z-\frac{V_1}{r+s}, & \text{if } V_1\le V_2,\end{cases} \tag{18}$$

and

$$\hat{\mu}_{\rm MV}=\begin{cases} Z-\frac{V_1}{r+s}\max(V,1), & \text{if } -\frac{V}{(r-1)V+s}<-\frac{1}{r+s}\max(V,1),\\ \mu_{\rm MV}, & \text{otherwise,}\end{cases} \tag{19}$$

where $V=V_2/V_1$.

Next, we propose a class of equivariant estimators for σ_i, $i=1,2$, and derive sufficient conditions for improving estimators in this class. In order to estimate the scale parameters, we use the loss function

$$L(d_i,\sigma_i)=\left(\frac{d_i-\sigma_i}{\sigma_i}\right)^2, \tag{20}$$

where d_i is an estimate for σ_i, $i=1,2$.

Consider the affine group of transformations $G_A=\{g_{a,b}:g_{a,b}(x)=ax+b, a>0$ and $b\in\mathbb{R}\}$. Under the group of transformations $g_{a,b}$, $\sigma_i\to a\sigma_i$, $\mu\to a\mu+b$. The sufficient statistics are transfered as $Z\to aZ+b$, $V_i\to aV_i$, $i=1,2$. Under this transformation, the problem remains invariant if we take the loss function as given in Eq. (20). Based on the complete and sufficient statistic (Z,V_1,V_2), the form of an affine equivariant estimator for σ_i is obtained as

$$d_{\phi_i}=V_i\phi_i(V), \tag{21}$$

where $V=V_2/V_1$ and ϕ_i is any function of V such that $\phi_i:(0,\infty)\to(0,\infty)$. Let us define the following functions $\phi_i^0(v)$ for the equivariant estimators d_{ϕ_i}, $i=1,2$, as

$$\phi_1^0(v)=\begin{cases}\frac{m}{r+s}, & \text{if } \phi_1(v)<\frac{m}{r+s}\\ \phi_1(v), & \text{otherwise}\end{cases} \tag{22}$$

and

$$\phi_2^0(v) = \begin{cases} \frac{nv}{r+s}, & \text{if } \phi_2(v) < \frac{nv}{r+s} \\ \phi_2(v), & \text{otherwise.} \end{cases} \tag{23}$$

The following Theorem is immediate and gives a sufficient condition for improving the class of estimators d_{ϕ_1} for estimating the scale parameter σ_1.

THEOREM 2. *Let d_{ϕ_1} be the class of equivariant estimators of the form in Eq. (21) for estimating the scale parameter σ_1. Let the loss function be the one given in Eq. (20). The equivariant estimator d_{ϕ_1} is inadmissible and is improved by $d_{\phi_1^0}$, if there exists some values of parameters $(\mu, \sigma_1, \sigma_2)$, such that $P\left(\phi_1(V) \neq \phi_1^0(V)\right) > 0$.*

PROOF. The Theorem can be proved by applying the orbit-by-orbit improvement technique of Brewster and Zidek (1974). Let us consider the conditional risk function of d_{ϕ_1} given V

$$R\left((d_{\phi_1}, \sigma_1)|V\right) = \frac{1}{\sigma_1^2} E\left[(V_1\phi_1(V) - \sigma_1)^2 | V = v\right]. \tag{24}$$

Observe that the above risk function is convex in ϕ_1, hence its minimizing choice is obtained as

$$\phi_1(v, \tau) = \frac{\sigma_1 E(V_1|V = v)}{E(V_1^2|V = v)}, \tag{25}$$

where $\tau = \sigma_2/\sigma_1 > 0$. The conditional expectations $E(V_1|V)$ and $E(V_1^2|V)$ have been evaluated in Tripathy (2016). Utilizing those values and simplifying, we obtain the minimizing choice of ϕ_1 as

$$\phi_1(v, \tau) = \frac{1}{r+s}[m + n\tau v]. \tag{26}$$

In order to apply the Brewster and Zidek (1974) technique, we need the supremum and infimum of ϕ_1 with respect to τ for fixed $V = v$. It is easy to observe that the function ϕ_1 is increasing for $\tau \in (0, \infty)$. Hence, we have

$$\inf_{\tau \in (0,\infty)} \phi_1(\tau, v) = \frac{m}{r+s} \text{ and } \sup_{\tau \in (0,\infty)} \phi_1(\tau, v) = \infty. \tag{27}$$

Utilizing these values, we define the function ϕ_1^0 as given in Eq. (22). As an application of Theorem 3.1.1 of Brewster and Zidek (1974), we have $R(d_{\phi_1}, \sigma_1) \geq R(d_{\phi_1^0}, \sigma_1)$, hence the Theorem is proved. □

The following Theorem gives the sufficient condition for improving upon an equivariant estimator of σ_2.

THEOREM 3. *Let d_{ϕ_2} be an equivariant estimator of the form in Eq. (21) for estimating the scale parameter σ_2. Let the loss function be the one given in Eq. (20). The equivariant estimator d_{ϕ_2} is inadmissible and is improved by $d_{\phi_2^0}$, if there exist some values of parameters $(\mu, \sigma_1, \sigma_2)$, such that $P\left(\phi_2(V) \neq \phi_2^0(V)\right) > 0$.*

PROOF. The proof is similar to the proof of the Theorem 2 and hence is omitted. □

REMARK 4. *Note that, in order to improve the MLEs and UMVUEs of σ_1 and σ_2 using Theorems 2 and 3, the sufficient conditions must hold for some choices of parameters. It is easy to see that, the MLEs of σ_1 and σ_2 are in the form of Eq. (21), that is $d_{\phi_{1ML}} = V_1\phi_{1ML}$, where $\phi_{1ML} = m/r$ and $d_{\phi_{2ML}} = V_1\phi_{2ML}$, where $\phi_{2ML} = nV/s$. The conditions for improving the MLEs do not hold true, hence could not be improved by using Theorem 2. Similarly, one can check that, the sufficient condition for improving the MLE of σ_2 does not satisfy, hence could not be improved by applying Theorem 3. Further, observe that, the UMVUE of σ_i, $i = 1, 2$, is in the form of Eq. (21), that is we can write $\sigma_{1MV} = V_1\phi_{1MV}(V)$, where $\phi_{1MV}(V) = \frac{m}{r(r-1)}[r - \frac{s-1}{(s-1)+(r-1)V}]$. In order to improve this estimator, the condition $\phi_{1MV}(V) < \frac{m}{r+s}$ must hold true. Similarly, the UMVUE $\sigma_{2MV} = V_1\phi_{2MV}(V)$, where $\phi_{2MV}(V) = \frac{n}{s(s-1)}[sV - \frac{(r-1)V^2}{(s-1)+(r-1)V}]$. In order to improve the estimator σ_{2MV}, the condition $\phi_{2MV}(V) < \frac{nV}{r+s}$ must hold true. It has been seen, from our simulation study, that the conditions hold true for a small range of parameters and v. The amount of risk improvement over the UMVUEs of σ_1 and σ_2 is insignificant. However, the Theorems 2 and 3 give two complete class results for estimating the scale parameters using censored samples.*

3. CLASSIFICATION RULE BASED ON CENSORED SAMPLES

In this Section, we will discuss a classification rule to classify a new observation, say t, into one of the populations Π_1 and Π_2, when the rule has been constructed using the type-II right censored samples from these two populations.

Suppose we have type-II right censored samples from the two populations Π_1 and Π_2, as discussed in Section 2. A new observation t is classified into the population Π_1 if

$$\log\left(\frac{f_1(t)}{f_2(t)}\right) \geq \log\left(\frac{C(2|1)q_2}{C(1|2)q_1}\right) \tag{28}$$

and into the population Π_2 if

$$\log\left(\frac{f_1(t)}{f_2(t)}\right) < \log\left(\frac{C(2|1)q_2}{C(1|2)q_1}\right), \tag{29}$$

where $C(i|j)$ is the cost of misclassification when an observation is from the population Π_j and is misclassified into the population Π_i, $i \neq j$, $i, j = 1, 2$; q_i is the prior probability that the observation belongs to the population Π_i. We assume that the populations are equally likely and the costs of misclassification remain the same, that is $q_1 = q_2$ and $C(1|2) = C(2|1)$. The details of this classification functions as well as the rules can be seen in Anderson (1951).

Using the results from Eq. (28) to Eq. (29), we obtain the classification function in our model to classify the new observation t as

$$W(t) = (t-\mu)\left(\frac{1}{\sigma_1} - \frac{1}{\sigma_2}\right) - \log\left(\frac{\sigma_1}{\sigma_2}\right). \tag{30}$$

Using this classification function W, the classification rule, say R, is given by:

$$\begin{aligned} R\text{: classify } t \text{ into } \Pi_1 \text{ if } W(t) \leq 0, \text{ and} \\ \text{classify } t \text{ into } \Pi_2 \text{ if } W(t) > 0. \end{aligned} \tag{31}$$

When the training samples are incomplete or censored (in our case it is type-II right censored), it is important to classify a group of observations rather a single observation, as it gives more information about the population. Here, we discuss a rule for classifying the group of observations which is also censored. Suppose we have to classify the ordered sample $t_{(1)} \leq t_{(2)} \leq \cdots \leq t_{(l)}$ taken from a random sample of size k, $2 \leq l \leq k$, from the population, say Π_0 (Exp(μ_0, σ_0)), having density f_0, into one of the populations Π_1 or Π_2. It is known that $(\mu_0, \sigma_0) = (\mu, \sigma_i)$, for exactly one i, with $i = 1$ or 2. In order to classify the population Π_0 into one of the populations Π_1 or Π_2, we follow the method proposed by Basu and Gupta (1976).

Let us define the statistics U_{ik} as

$$U_{ik} = (k-i+1)(t_{(i)} - t_{(i-1)}), \quad i = 1, 2, \ldots l \text{ and } t_{(0)} = \mu_0. \tag{32}$$

These statistics are independent and identically distributed and have the same probability density function f_0 with $\mu_0 = 0$. Thus, the classification rule to classify Π_0 $\left(t_{(1)} \leq t_{(2)} \cdots \leq t_{(l)}\right)$ into Π_1 or Π_2 is based on a random sample $(U_{1k}, U_{2k}, \ldots, U_{lk})$ of size l. Thus, we define the modified classification function as

$$W(\underline{U}) = (\bar{U} - \mu)\left(\frac{1}{\sigma_1} - \frac{1}{\sigma_2}\right) - \log\left(\frac{\sigma_1}{\sigma_2}\right), \tag{33}$$

where $\bar{U} = \frac{1}{l}\sum_{i=1}^{l} U_{ik}$. Utilizing this classification function, we propose the classification rule, say R_1, for our model as

$$\begin{aligned} R_1\text{:classify } \Pi_0 \text{ into } \Pi_1 \text{ if } W(\underline{U}) \leq 0,\ t_{(1)} > \mu, \text{ and} \\ \text{classify } \Pi_0 \text{ into } \Pi_2 \text{ if } W(\underline{U}) > 0,\ t_{(1)} > \mu. \end{aligned} \tag{34}$$

Let us denote $P(i|j)$ as the probability that the observation actually from the population Π_j but is misclassified into the population Π_i, using the rule R_1, $i \neq j$. In this case, the Expected Probability of Misclassification (EPM) of the rule R_1 is given by $\text{EPM}(R_1) = (P(1|2,R_1) + P(2|1,R_1))/2$. The unknown parameters $(\mu, \sigma_1, \sigma_2)$, involved in the classification statistic, will be replaced by their corresponding estimators, which are obtained in Section 2.

THEOREM 5. *The classification rule R_1 is consistent, that is EPM(R_1) tends to zero as the sample size k tends to ∞.*

PROOF. The proof is along the same line as of Theorem 2 of Basu and Gupta (1976) and hence is omitted. □

4. Classification rules using estimators of associated parameters

In this Section, we propose certain classification rules for classifying a single observation and a group of observations into one of the exponential populations. These classification rules will be formed using some popular estimators of μ as well as the scale parameters σ_1 and σ_2 under the type-II right censoring scheme.

4.1. *Classification rule based on the MLE*

Recall that the MLEs of the parameters μ, σ_1 and σ_2 have been obtained as μ_{ML}, $\sigma_{1\text{ML}}$ and $\sigma_{2\text{ML}}$, respectively. Utilizing these MLEs of the parameters, we define the classification function for classifying an observation t as

$$W_{\text{ML}}(t) = (t - \mu_{\text{ML}})\left(\frac{1}{\sigma_{1\text{ML}}} - \frac{1}{\sigma_{2\text{ML}}}\right) - \log\left(\frac{\sigma_{2\text{ML}}}{\sigma_{1\text{ML}}}\right). \tag{35}$$

Utilizing this classification function, we define the rule R_{ML} as: classify t into Π_1 if $W_{\text{ML}}(t) \leq 0$, else classify t into the population Π_2. Similarly, a group of new observations, say $(t_{(1)} \leq t_{(2)} \leq \cdots \leq t_{(l)})$ from Π_0 will be classified using the classification function

$$W_{\text{ML}}(\bar{U}) = (\bar{U} - \mu_{\text{ML}})\left(\frac{1}{\sigma_{1\text{ML}}} - \frac{1}{\sigma_{2\text{ML}}}\right) - \log\left(\frac{\sigma_{2\text{ML}}}{\sigma_{1\text{ML}}}\right). \tag{36}$$

Utilizing this function, we define the classification rule R_{ML} as: classify Π_0 into Π_1 if $W_{\text{ML}}(\bar{U}) \leq 0$, $t_{(1)} > \mu_{\text{ML}}$, and classify Π_0 into the population Π_2 if $W_{\text{ML}}(\bar{U}) > 0$, $t_{(1)} > \mu_{\text{ML}}$.

Utilizing the modified MLE μ_{MM} for μ, we construct the classification function for classifying a single observation t, as

$$W_{\text{MM}}(t) = (t - \mu_{\text{MM}})\left(\frac{1}{\sigma_{1\text{ML}}} - \frac{1}{\sigma_{2\text{ML}}}\right) - \log\left(\frac{\sigma_{2\text{ML}}}{\sigma_{1\text{ML}}}\right). \tag{37}$$

Using this function, we define a classification rule, say R_{MM} as: classify t into Π_1 if $W_{\text{MM}}(t) \leq 0$, else classify t into the population Π_2. Similarly, for classifying the sample $(t_{(1)} \leq t_{(2)} \leq \cdots \leq t_{(l)})$, we define the classification function

$$W_{\text{MM}}(\bar{U}) = (\bar{U} - \mu_{\text{MM}})\left(\frac{1}{\sigma_{1\text{ML}}} - \frac{1}{\sigma_{2\text{ML}}}\right) - \log\left(\frac{\sigma_{2\text{ML}}}{\sigma_{1\text{ML}}}\right). \tag{38}$$

Using this classification function, we define the rule R_{MM} as: classify Π_0 into Π_1 if $W_{\text{MM}}(\bar{U}) \leq 0$, $t_{(1)} > \mu_{\text{MM}}$, and classify Π_0 into the population Π_2 if $W_{\text{MM}}(\bar{U}) > 0$, $t_{(1)} > \mu_{\text{MM}}$.

Next, using the improved estimator $\hat{\mu}_{\text{ML}}$ for μ along with MLEs of σ_1 and σ_2, we construct the classification function in order to classify an observation t as

$$\hat{W}_{\text{ML}}(t) = (t - \hat{\mu}_{\text{ML}})\left(\frac{1}{\sigma_{1\text{ML}}} - \frac{1}{\sigma_{2\text{ML}}}\right) - \log\left(\frac{\sigma_{2\text{ML}}}{\sigma_{1\text{ML}}}\right). \tag{39}$$

Using this classification function, we define a classification rule, say $\hat{R}_{\text{ML}}$ as: classify t into Π_1 if $\hat{W}_{\text{ML}}(t) \leq 0$, else classify t into the population Π_2. Similarly, for classifying the sample $(t_{(1)} \leq t_{(2)} \leq \cdots \leq t_{(l)})$ from Π_0, we define the classification function as

$$\hat{W}_{\text{ML}}(\bar{U}) = (\bar{U} - \hat{\mu}_{\text{ML}})\left(\frac{1}{\sigma_{1\text{ML}}} - \frac{1}{\sigma_{2\text{ML}}}\right) - \log\left(\frac{\sigma_{2\text{ML}}}{\sigma_{1\text{ML}}}\right). \tag{40}$$

Using this classification function, we define the rule $\hat{R}_{\text{ML}}$ as: classify Π_0 into Π_1 if $\hat{W}_{\text{ML}}(\bar{U}) \leq 0$, $t_{(1)} > \hat{\mu}_{\text{ML}}$, and classify Π_0 into the population Π_2 if $\hat{W}_{\text{ML}}(\bar{U}) > 0$, $t_{(1)} > \hat{\mu}_{\text{ML}}$.

4.2. *Classification rule based on the UMVUE*

In this Section, we propose the classification rules for classifying censored samples from Π_0 into either Π_1 or Π_2, using the UMVUEs of the model parameters derived in Section 2.

Using the UMVUEs of the model parameters, we define a classification function, say $W_{\text{MV}}(t)$, for classifying a single observation t from Π_0 as

$$W_{\text{MV}}(t) = (t - \mu_{\text{MV}})\left(\frac{1}{\sigma_{1\text{MV}}} - \frac{1}{\sigma_{2\text{MV}}}\right) - \log\left(\frac{\sigma_{2\text{MV}}}{\sigma_{1\text{MV}}}\right). \tag{41}$$

Utilizing this function, we define a rule, say R_{MV}, for classifying the single observation t as: classify t into Π_1 if $W_{\text{MV}}(t) \leq 0$, else classify t into the population Π_2. Similarly, we define the classification function for classifying a sample $(t_{(1)} \leq t_{(2)} \leq \cdots \leq t_{(l)})$ from Π_0 as

$$W_{\text{MV}}(\bar{U}) = (\bar{U} - \mu_{\text{MV}})\left(\frac{1}{\sigma_{1\text{MV}}} - \frac{1}{\sigma_{2\text{MV}}}\right) - \log\left(\frac{\sigma_{2\text{MV}}}{\sigma_{1\text{MV}}}\right). \tag{42}$$

Using this classification function, we define the rule R_{MV} as: classify Π_0 into Π_1 if $W_{MV}(\bar{U}) \leq 0$, $t_{(1)} > \mu_{MV}$, and classify Π_0 into the population Π_2 if $W_{MV}(\bar{U}) > 0$, $t_{(1)} > \mu_{MV}$.

Using the improved estimator $\hat{\mu}_{MV}$ for the UMVUE of μ proposed by Tripathy (2016) along with the UMVUEs of σ_1 and σ_2, we define a new classification function for classifying a single observation t as

$$\hat{W}_{MV}(t) = (t - \hat{\mu}_{MV})\left(\frac{1}{\sigma_{1MV}} - \frac{1}{\sigma_{2MV}}\right) - \log\left(\frac{\sigma_{2MV}}{\sigma_{1MV}}\right). \tag{43}$$

Using this classification function we define the classification rule $\hat{R}_{MV}$ as: classify t into Π_1 if $\hat{W}_{MV}(t) \leq 0$, else classify t into Π_2. Similarly, we define a classification function for classifying a sample $(t_{(1)} \leq t_{(2)} \leq \cdots \leq t_{(l)})$ as

$$\hat{W}_{MV}(\bar{U}) = (\bar{U} - \hat{\mu}_{MV})\left(\frac{1}{\sigma_{1MV}} - \frac{1}{\sigma_{2MV}}\right) - \log\left(\frac{\sigma_{2MV}}{\sigma_{1MV}}\right). \tag{44}$$

Utilizing this classification function, we define the rule $\hat{R}_{MV}$ as: classify Π_0 into Π_1 if $\hat{W}_{MV} \leq 0$, $t_{(1)} > \hat{\mu}_{MV}$, and classify Π_0 into Π_2 if $\hat{W}_{MV} > 0$, $t_{(1)} > \hat{\mu}_{MV}$.

5. A SIMULATION STUDY FOR COMPARING THE CLASSIFICATION RULES

In this Section, we compare all the proposed classification rules, such as R_{ML}, R_{MM}, R_{MV}, $\hat{R}_{ML}$ and $\hat{R}_{MV}$ as given in the Section 4. Though we have derived the classification rules for classifying a single observation t and a group of observations, we only present the simulation results for classifying the latter one. The simulation results for classifying a single observation can also be obtained in a very similar manner.

To compare the classification rules using type-II censored samples from the two exponential populations, we consider the following steps.

1. Generate training sample, that is, a type-II censored sample of size r from the exponential population Π_1 with scale parameter σ_1 and location parameter μ. Here, we first generate a random sample of size m from the population Π_1, then, after sorting in increasing order, we collect the first r observations, which is the type-II right censored sample. In a very similar manner, we also generate a sample of size n from the population Π_2 with scale parameter σ_2 and location parameter μ, then, after ordering, we collect type-II right censored sample of size s.

2. Using these training samples, the unknown parameters σ_1, σ_2 and μ, which are involved in the classification rules, are estimated and consequently used in the classification rules.

3. Next, we generate a type-II censored sample from the population Π_1 with same r and m and estimate the value of the statistic $\bar{U}$; then, using it in the classification rules we check whether it belongs to the population Π_1.

4. Similarly, we generate a type-II censored sample from the population Π_2 with same s and n and estimate the value of the statistic $\bar{U}$; then, using it in the classification rules we check whether it belongs to the population Π_2.

The above procedure is carried out using the well known Monte-Carlo simulation method to compare the probabilities of correct classification $P(1|1)$, $P(2|2)$ and the expected probability of correct classification (EPC) for all the proposed classification rules. The number of replications is 20,000. We assume that the costs of misclassification and the prior probabilities are equal, that is $C(1|2) = C(2|1)$ and $q_1 = q_2 = 0.5$ for the simulation study. A high level of accuracy is achieved and the standard error is checked (it is of the order 10^{-3}). Here, the accuracy of simulation means the sample generation, construction of estimators and consequently the construction of classification rules are accurate with an error of the order 10^{-3}. We also note that the probabilities of misclassification and correct classification are a function of $\tau = \sigma_2/\sigma_1 > 0$ only.

The expected probability of correct classification (EPC) for a given rule, say R is calculated as

$$EP(R) = \frac{1}{2}(P(1|1,R) + P(2|2,R)). \tag{45}$$

We further compute the percentage of relative improvements in EPC values for all the proposed rules with respect to the benchmark rule R_{ML} as

$$\begin{aligned} EP1 &= \left(\frac{EP(\hat{R}_{\text{ML}})}{EP(R_{\text{ML}})} - 1\right) \times 100, \quad EP2 = \left(\frac{EP(R_{\text{MV}})}{EP(R_{\text{ML}})} - 1\right) \times 100 \\ EP3 &= \left(\frac{EP(\hat{R}_{\text{MV}})}{EP(R_{\text{ML}})} - 1\right) \times 100, \quad EP4 = \left(\frac{EP(R_{\text{MM}})}{EP(R_{\text{ML}})} - 1\right) \times 100. \end{aligned} \tag{46}$$

Further, we define the censoring factors CF1 and CF2 for both the populations as the ratio of number of observed samples to total number of samples. It is $\text{CF1} = r/m$ for first population and $\text{CF2} = s/n$ for the second population. Note that the values of CF1 and CF2 are between 0 and 1.

A comprehensive simulation study is carried out by considering various combinations of sample sizes, censoring factors and τ. However, for illustration purpose, we present in Tables 1 to 3 the percentage of relative improvements in expected probability of correct classification of rules $\hat{R}_{\text{ML}}$, R_{MV}, $\hat{R}_{\text{MV}}$ and R_{MM} over R_{ML} for some specific choices of sample sizes, censoring factors and τ. Specifically, in Table 1, we present the percentage of relative improvements in EPC for sample sizes $(16,16)$ and $(24,24)$. The first column gives the values of τ. Each value of τ corresponds to four values of the relative percentage of improvement. These four values are obtained for

CF1 = CF2 = 0.25, 0.50, 0.75, 1.0, respectively. Similarly, in Tables 2 and 3 the percentage of relative improvement in EPC are presented for unequal sample sizes.

We compute the probabilities of correct classification $P(1|1)$ and $P(2|2)$ for all the proposed classification rules with various censoring factors and parameter choices. For illustration purposes, we plot the graphs of $P(1|1)$ and $P(2|2)$ for all the classification rules with sample sizes $(16,16)$ and censoring factors CF1 = CF2 = 0.25, 0.50, 0.75, 1 in Figures 1 and 2. Specifically, in Figure 1 the values of $P(1|1)$, (a) and (b), are given for censoring factors 0.25 and 0.75, respectively. In Figure 1 the values of $P(2|2)$, (c) and (d), are presented for all the rules with censoring factors 0.25 and 0.75, respectively. In a similar manner, we plot in Figure 2 the values of $P(1|1)$ and $P(2|2)$, (a) to (d), against τ for all the rules with censoring factors 0.50 and 1. The notations ML, MM, MLI, MV and MVI are used for the probabilities of correct classifications $P(1|1)$ and $P(2|2)$ for the classification rules R_{ML}, R_{MM}, $\hat{R}_{\text{ML}}$, R_{MV} and $\hat{R}_{\text{MV}}$, respectively.

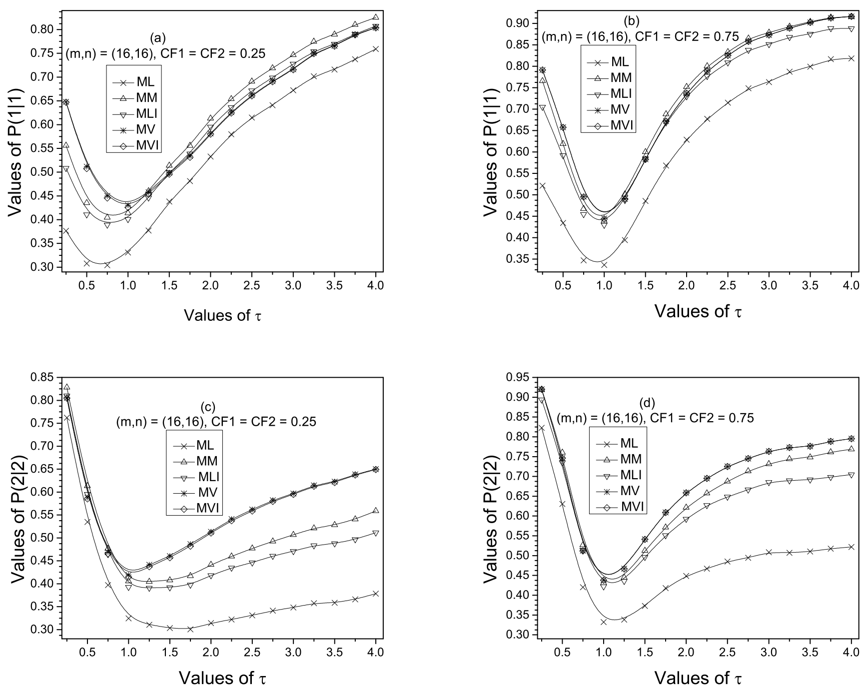

Figure 1 – Comparison of probability of correct classification for several classification rules: CF1 = CF2 = 0.25, 0.75.

TABLE 1
Percentage of relative improvement in EPC for all the proposed rules:
$CF1 = CF2 = 0.25, 0.50, 0.75, 1.00.$

τ	(m,n)=(16,16)				(m,n)=(24,24)			
	$\hat{R}_{ML}$	R_{MM}	R_{MV}	$\hat{R}_{MV}$	$\hat{R}_{ML}$	R_{MM}	R_{MV}	$\hat{R}_{MV}$
0.25	16.34	22.11	**28.63**	28.45	18.07	24.24	**28.62**	28.60
	18.48	25.02	**28.24**	**28.24**	19.02	25.85	**27.92**	**27.92**
	19.19	25.88	**27.82**	**27.82**	19.55	26.73	**27.79**	**27.79**
	19.58	26.34	**27.68**	**27.68**	19.88	27.00	**27.82**	**27.82**
0.50	19.73	24.44	**30.54**	29.86	21.87	26.57	**30.44**	30.18
	22.09	26.87	**29.94**	29.84	23.46	28.37	**30.56**	30.53
	23.42	28.12	**30.29**	30.27	23.98	28.56	**30.13**	30.12
	23.95	28.43	**29.99**	29.97	24.05	28.75	**29.79**	29.78
0.75	21.15	25.09	**29.69**	28.70	23.80	27.43	**31.45**	30.87
	24.84	28.66	**31.01**	30.69	25.83	28.95	**31.00**	30.83
	25.94	29.34	**30.98**	30.86	26.38	29.10	**30.69**	30.60
	26.99	30.06	**30.79**	30.73	27.18	29.72	**30.75**	30.71
1.00	21.71	25.51	**30.15**	29.02	24.41	27.81	**30.45**	29.78
	24.53	27.29	**29.64**	29.26	26.58	29.03	**31.35**	31.18
	26.25	28.92	**30.90**	30.68	27.94	30.08	**31.45**	31.30
	28.06	30.50	**31.26**	31.13	28.84	31.02	**31.59**	31.50
1.25	21.09	25.03	**29.84**	28.55	23.62	27.04	**29.69**	29.13
	25.77	29.13	**31.86**	31.48	26.48	29.44	**31.05**	30.88
	25.76	28.70	**30.40**	30.21	28.04	30.87	**32.09**	32.00
	26.45	29.02	**30.48**	30.33	29.01	31.26	**32.01**	31.95
1.50	20.24	24.74	**30.02**	29.03	23.63	27.50	**31.20**	30.72
	24.83	28.70	**31.38**	31.14	25.55	29.26	**31.34**	31.15
	25.11	28.62	**30.55**	30.38	26.51	29.94	**31.42**	31.40
	25.47	28.96	**30.60**	30.52	27.35	30.66	**31.70**	31.68
1.75	20.30	24.69	**29.92**	29.13	22.43	26.75	**30.84**	30.49
	23.39	27.62	**30.52**	30.32	23.82	28.13	**30.55**	30.50
	24.42	28.67	**31.19**	31.11	25.60	29.90	**31.25**	31.23
	24.86	29.04	**30.87**	30.83	26.17	30.33	**31.34**	31.32
2.00	19.19	23.83	**29.48**	28.88	21.49	26.43	**30.68**	30.32
	22.08	26.81	**29.85**	29.75	24.06	28.91	**31.21**	31.14
	23.46	27.91	**30.29**	30.26	23.71	28.34	**29.82**	29.81
	24.22	28.72	**30.20**	**30.20**	24.13	28.84	**29.99**	**29.99**
2.25	18.70	23.66	**29.15**	28.68	20.96	26.15	**29.97**	29.77
	22.23	27.67	**31.14**	31.04	23.22	28.25	**30.30**	30.25
	22.52	27.70	**29.91**	29.88	23.85	29.14	**30.76**	30.75
	23.32	28.58	**30.19**	30.18	24.05	29.24	**30.52**	30.52
2.50	18.31	23.52	**28.82**	28.46	20.78	25.88	**30.27**	30.13
	20.69	25.99	**29.19**	29.10	22.16	27.46	**29.81**	29.80
	21.70	27.10	**29.48**	29.47	22.45	28.14	**29.43**	29.42
	22.45	27.83	**29.52**	**29.52**	22.54	28.16	**29.25**	**29.25**
2.75	18.03	23.62	**29.25**	28.89	19.70	25.30	**29.64**	29.52
	20.63	26.25	**29.61**	29.53	20.47	25.97	**28.39**	28.38
	21.09	26.94	**28.97**	28.96	21.84	27.77	**29.14**	**29.14**
	21.51	27.28	**28.99**	**28.99**	22.03	28.14	**29.13**	**29.13**
3.00	17.58	23.13	**29.33**	28.98	19.60	25.48	**29.70**	29.59
	20.32	26.05	**29.26**	29.22	20.84	27.24	**29.59**	**29.59**
	20.87	26.86	**28.93**	28.92	21.26	27.61	**29.09**	**29.09**
	21.37	27.36	**29.00**	**29.00**	21.50	28.05	**28.93**	**28.93**

TABLE 2
Percentage of relative improvement in EPC for all the proposed rules:
$CF1 = CF2 = 0.25, 0.50, 0.75, 1.00.$

$\tau\downarrow$	(m,n)=(12,16)				(m,n)=(16,24)			
	$\hat{R}_{ML}$	R_{MM}	R_{MV}	$\hat{R}_{MV}$	$\hat{R}_{ML}$	R_{MM}	R_{MV}	$\hat{R}_{MV}$
0.25	15.07	22.09	**30.63**	30.37	16.20	24.20	**31.06**	30.95
	17.37	24.66	**29.12**	29.09	17.49	25.80	**28.96**	28.95
	18.13	26.06	**28.87**	**28.87**	19.51	27.44	**29.51**	**29.51**
	18.56	26.81	**28.72**	**28.72**	18.84	27.97	**29.29**	**29.29**
0.50	18.92	24.61	**31.66**	30.74	19.83	26.12	**31.01**	30.46
	21.73	27.39	**31.49**	31.24	22.42	28.48	**31.42**	31.31
	22.55	28.17	**31.24**	31.14	24.60	29.70	**32.04**	31.98
	23.74	28.94	**31.09**	31.08	23.78	29.76	**31.51**	31.49
0.75	20.11	24.93	**31.56**	30.24	21.92	26.82	**31.53**	30.67
	23.66	27.50	**30.82**	30.25	25.25	29.37	**31.09**	30.90
	26.25	30.10	**31.96**	31.69	27.41	30.35	**31.73**	31.54
	27.13	30.62	**32.19**	32.06	27.22	30.65	**31.83**	31.77
1.00	20.41	24.12	**29.88**	28.48	23.12	26.78	**30.34**	29.48
	24.30	27.52	**31.17**	30.61	26.40	29.43	**31.23**	30.90
	26.17	29.12	**30.59**	30.28	28.48	30.86	**32.12**	31.86
	26.78	29.41	**30.66**	30.48	29.21	31.26	**32.43**	32.30
1.25	19.93	23.93	**29.03**	27.79	22.27	25.75	**29.94**	29.28
	24.58	27.76	**30.58**	30.17	26.44	29.28	**30.73**	30.52
	25.51	28.40	**30.15**	29.86	26.36	29.27	**30.42**	30.32
	26.82	29.47	**30.96**	30.78	27.91	30.15	**31.30**	31.24
1.50	20.17	23.80	**29.22**	28.13	22.43	26.01	**29.90**	29.36
	23.35	26.36	**29.59**	29.26	24.98	27.95	**29.79**	29.66
	25.33	28.41	**30.48**	30.32	24.53	28.01	**29.30**	29.29
	26.48	29.37	**30.83**	30.75	25.89	28.40	**29.59**	29.55
1.75	19.21	22.97	**28.70**	27.77	20.69	24.10	**27.79**	27.41
	23.12	26.82	**29.77**	29.46	24.00	27.07	**29.18**	29.05
	23.75	27.17	**29.03**	28.94	24.17	28.06	29.28	**29.29**
	23.90	27.35	**29.23**	29.18	25.67	28.71	**29.73**	29.73
2.00	19.37	23.58	**29.32**	28.61	20.60	24.24	**27.76**	27.55
	22.10	25.92	**28.86**	28.67	22.58	26.48	**28.52**	28.47
	23.68	27.47	**29.57**	29.51	23.02	27.60	**29.09**	29.08
	23.81	27.70	**29.21**	29.18	24.34	28.05	**29.21**	29.20
2.25	18.16	22.28	**27.98**	27.32	20.43	24.32	**28.09**	27.72
	21.57	25.80	**29.12**	28.97	21.90	25.35	**27.61**	27.57
	22.24	26.18	**28.59**	28.54	22.08	26.65	**28.16**	28.15
	22.93	26.97	**28.65**	28.64	23.43	27.19	**28.43**	28.42
2.50	18.06	22.40	**27.76**	27.30	20.22	23.88	**27.78**	27.53
	21.52	25.63	**28.84**	28.77	21.68	25.65	**27.73**	27.71
	21.83	26.15	**28.38**	28.37	22.24	26.99	**28.45**	**28.45**
	22.17	26.81	**28.30**	28.29	23.31	27.45	**28.46**	**28.46**
2.75	17.17	21.10	**26.65**	26.31	19.55	23.40	**27.87**	27.70
	20.75	25.22	**28.57**	28.51	21.35	25.63	**27.68**	27.66
	21.73	26.16	**28.44**	**28.44**	21.07	26.08	**27.49**	**27.49**
	22.04	26.81	**28.36**	28.35	22.07	26.56	**27.43**	**27.43**
3.00	17.65	22.00	**27.80**	27.40	19.23	23.35	**27.36**	27.24
	20.19	25.05	**28.10**	28.05	20.85	25.09	**27.10**	27.09
	20.51	25.18	**27.30**	27.25	20.08	25.37	**26.84**	**26.84**
	21.07	25.73	**27.26**	**27.26**	21.31	26.12	**26.91**	**26.91**

TABLE 3

Percentage of relative improvement in EPC for all the proposed rules:
$CF1 = CF2 = 0.25, 0.50, 0.75, 1.00.$

$\tau\downarrow$	(m,n)=(16,12)				(m,n)=(24,16)			
	$\hat{R}_{ML}$	R_{MM}	R_{MV}	$\hat{R}_{MV}$	$\hat{R}_{ML}$	R_{MM}	R_{MV}	$\hat{R}_{MV}$
0.25	16.27	20.88	**26.74**	26.51	17.94	22.36	**26.32**	26.27
	18.53	23.71	**26.83**	26.82	19.42	24.07	**25.87**	**25.87**
	19.32	24.83	**26.83**	**26.83**	19.60	24.59	**25.73**	**25.73**
	19.53	25.12	**26.55**	26.54	20.14	25.26	**26.06**	**26.06**
0.50	19.26	23.23	29.28	**28.58**	21.07	24.52	**28.80**	28.45
	22.90	26.64	**29.46**	29.24	23.33	27.00	**29.31**	29.27
	23.31	26.64	**28.91**	28.86	23.77	27.23	**28.38**	28.36
	23.35	26.98	**28.76**	28.75	24.34	27.87	**28.95**	**28.95**
0.75	19.77	23.43	**28.71**	27.57	22.75	26.23	**30.05**	29.30
	23.72	26.92	**29.07**	28.66	25.74	28.58	**30.40**	30.12
	25.49	28.22	**29.63**	29.36	26.79	29.08	**30.32**	30.21
	25.87	28.15	**29.59**	29.49	27.59	29.84	**30.70**	30.66
1.00	20.48	24.91	**30.04**	28.69	22.78	26.65	**31.27**	30.36
	24.32	27.63	**30.42**	29.94	26.83	29.91	**31.84**	31.49
	26.35	29.18	**31.05**	30.83	26.96	29.69	**31.45**	31.29
	27.84	30.31	**31.10**	30.81	28.89	31.08	**32.07**	31.90
1.25	20.03	24.50	**30.85**	29.48	23.35	28.15	**32.51**	31.46
	24.52	28.54	**31.77**	31.34	26.54	30.46	**32.71**	32.49
	25.31	28.76	**30.83**	30.64	26.58	29.97	**31.72**	31.61
	27.36	30.62	**32.19**	32.01	28.82	32.11	**33.35**	33.29
1.50	19.34	24.08	**30.49**	29.14	21.81	26.99	**32.38**	31.47
	22.75	27.38	**30.90**	30.61	24.75	29.62	**31.95**	31.67
	23.90	28.14	**31.17**	30.95	26.15	30.92	**32.38**	32.29
	24.67	28.94	**30.89**	30.74	26.26	30.54	**32.11**	32.06
1.75	18.61	23.87	**31.01**	29.93	21.34	27.25	**32.73**	32.18
	22.42	27.84	**31.40**	31.09	24.48	30.22	**33.27**	33.09
	23.37	28.63	**31.10**	31.00	24.81	30.17	**32.28**	32.22
	23.98	28.79	**31.05**	30.99	25.83	31.36	**32.95**	32.92
2.00	18.24	23.77	**31.31**	30.25	20.56	26.87	**32.63**	32.27
	21.60	27.06	**31.54**	31.29	23.08	29.20	**32.48**	32.39
	22.57	28.47	**31.00**	30.93	23.76	30.09	**32.27**	32.24
	22.99	28.54	**30.74**	30.70	24.29	30.19	**32.09**	32.07
2.25	17.49	23.17	**30.88**	30.09	19.61	25.94	**31.80**	31.30
	20.97	26.84	**30.55**	30.43	21.72	28.42	**31.66**	31.60
	21.85	27.81	**30.69**	30.65	22.37	28.86	**31.14**	31.13
	22.15	28.32	**30.81**	30.79	22.73	29.43	**31.17**	**31.17**
2.50	17.25	23.12	**31.23**	30.56	19.50	26.37	**32.49**	32.16
	20.61	26.95	**31.42**	31.29	21.48	28.58	**32.03**	32.00
	21.80	28.38	**31.30**	31.28	21.50	28.23	**30.46**	30.44
	22.39	28.84	**31.06**	31.05	22.48	29.63	**31.32**	**31.32**
2.75	16.59	22.66	**31.15**	30.59	18.27	25.25	**31.48**	31.23
	19.56	26.21	**30.72**	30.64	19.97	27.36	**30.58**	30.56
	20.43	27.44	**30.29**	30.28	20.99	28.78	**31.15**	31.14
	20.61	27.70	**29.89**	**29.89**	20.67	28.39	**29.96**	**29.96**
3.00	16.50	22.93	**31.43**	30.84	18.16	25.83	**32.66**	32.47
	19.24	26.44	**31.08**	30.98	20.39	28.12	**31.67**	31.65
	20.36	27.52	**30.60**	30.58	20.42	28.41	**30.65**	**30.65**
	20.48	27.90	**30.25**	30.24	20.93	29.14	**30.70**	**30.70**

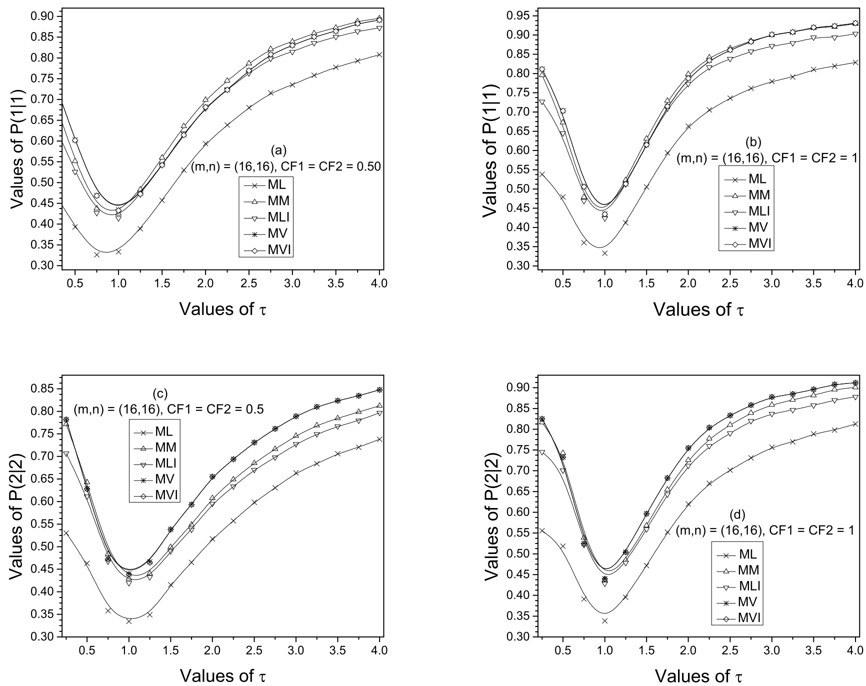

Figure 2 – Comparison of probability of correct classification for several classification rules: CF1 = CF2 = 0.50, 1.

The following observations are made from our simulation study as well as Tables 1 to 3 and Figures 1 to 2.

- As the values of τ increase, the probabilities of correct classification for all the rules first decrease and attain their minimum, then increase and finally converge to some values between 0 and 1 (see Figures 1 and 2). The maximum probability of correct classification $P(1|1)$ is obtained for the rules R_{MM} and R_{MV}, whereas the maximum probability of correct classification $P(2|2)$ is noticed for the rules R_{MV} and R_{MM}.

- The expected probabilities of correct classification for the rules $\hat{R}_{\mathrm{ML}}$, R_{MM}, R_{MV} and $\hat{R}_{\mathrm{MV}}$ are always higher than the expected probability of correct classification of the rule R_{ML}.

- The relative percentage of improvement in EPC values for all the rules with respect to R_{ML} varies between 16% and 33%. The maximum percentage of improvement in EPC values is observed in the case of R_{MV}.

- A similar pattern is noticed in terms of probabilities of correct classification and expected probabilities of correct classification for other combinations of sample sizes, censoring factors, and parameters for all the proposed rules.

- Based on our computational results, it is recommended to use the rule R_{MV} for classifying a type-II censored sample $(t_{(1)} \leq t_{(2)} \leq \cdots \leq t_{(l)})$ into one of the populations Π_1 or Π_2.

REMARK 6. *We present the simulation results by considering the test sample size and the training sample size as equal; however, the overall conclusion regarding the performances of the various proposed rules remains the same if one chooses the test sample size different from the training sample size. This is verified by the simulation study.*

6. A REAL LIFE EXAMPLE

In this Section, we consider a real-life situation that can be modeled using the two-parameter exponential distributions and satisfy the equality of location parameters. Using these data sets, we compute the various classification rules and illustrate the methodologies proposed in the article.

Lawless (2003) considered survival data on 40 advanced lung cancer patients. This data set was previously examined by Prentice (1973). Their purpose was to compare the effects of two chemotherapy treatments, namely standard and test, in prolonging the survival times of the patients, who can have four different types of tumors, namely Squamous, Small, Adeno, and Large. For our purpose, we consider the data on large tumors of two different types of test, such as standard and test. The data sets are given as:

Standard, Large: 177, 50, 66, 16, 12, 40, 68, 12, 200, 80, 41, 12, 250, 70, 53, 8, 100, 60, 37, 13;

Test, Large: 164, 70, 68, 15, 19, 30, 39, 4, 43, 60, 49, 11, 340, 80, 64, 10, 231, 70, 67, 18.

Using the goodness of fit chi-square test, we find that the two-parameter exponential distribution fits these two data sets well with p-values 0.25 and 0.30, respectively. Further, we perform the test proposed by Hsieh (1986) to check the equality of the location parameters. The equality of the location parameters can not be rejected with the level of significance 0.05. This is a situation where our model fits well.

In order to illustrate the classification rules, we focus on the following cases. Let us first consider the test observation $t = 49$ from the training population Π_2, i.e., Test, Large. Utilizing the proposed classification rules, we check which rules classify the test observation correctly. For this purpose, we collect the type-II censored data from these two training data with $r = 5$ and $s = 5$ and, then, we compute the values of classification statistics. The various values of classification statistics are obtained as $W_{\text{ML}} = -0.04522$, $\hat{W}_{\text{ML}} = -0.03639$, $W_{\text{MV}} = 0.000962$, $\hat{W}_{\text{MV}} = 0.00086$, $W_{\text{MM}} = -0.037991$. This shows that the classification rules R_{ML}, $\hat{R}_{\text{ML}}$ and R_{MM} classify $t = 49$ incorrectly into the first population (Standard, Large), whereas the rules R_{MV} and $\hat{R}_{\text{MV}}$ classify it into the second population correctly. Similarly, we consider the other data points and classify using the proposed rules. In this case, the percentage of correct classification for the rules R_{MV} and $\hat{R}_{\text{MV}}$ is 52.5%. The percentage of correct classification for the rest of the rules is 50%. We also compute the percentages of correct classification for all the rules by considering some other choices of censoring factors. For example, when $r = 18$ and $s = 12$, the percentage of correct classification for the rules R_{MV} and $\hat{R}_{\text{MV}}$ is 50%, whereas for the rules R_{ML}, $\hat{R}_{\text{ML}}$ and R_{MM} it is 47%. In another case, when $r = 16$ and $s = 4$, the percentage of correct classification for the rules R_{MV} and $\hat{R}_{\text{MV}}$ is 55%, whereas the rules R_{ML}, $\hat{R}_{\text{ML}}$ and R_{MM} have percentage of correct classification equal to 45%.

Next, we classify a group of observations into one of the two data sets. Let us consider that the test observation is $\Pi_0 = (4, 10, 11, 15, 18)$ from the training population Π_2. Using the proposed classification rules we will classify this data set into one of the populations and check which rules identify the Π_0 correctly. The values of classification statistics are computed as $W_{\text{ML}} = -0.04522$, $\hat{W}_{\text{ML}} = -0.000007$, $W_{\text{MV}} = 0.0004022$, $\hat{W}_{\text{MV}} = 0.000384$, $W_{\text{MM}} = -0.0000085$. This shows that the classification rules R_{ML}, $\hat{R}_{\text{ML}}$ and R_{MM} classify Π_0 incorrectly into the first population (Standard, Large), whereas the rule R_{MV} and $\hat{R}_{\text{MV}}$ classify Π_0 into the second population correctly.

Finally, we use the proposed classification rules to classify a new observation into one of the two data sets. Suppose, a new observation, say $t = 65$, is given and we want to classify it into one of the two populations. Assuming the original data as training sample, we collect the type-II right censored samples with $r = 14$ and $s = 14$ and we use them compute the classification statistics as $W_{\text{ML}} = -0.000367$, $\hat{W}_{\text{ML}} = -0.00011$, $W_{\text{MV}} = 0.0002981$, $\hat{W}_{\text{MV}} = 0.00028$, $W_{\text{MM}} = -0.00011$. Thus, the classification rules R_{ML}, $\hat{R}_{\text{ML}}$ and R_{MM} classify $t = 65$ into the first population (Standard, Large), whereas the rules R_{MV} and $\hat{R}_{\text{MV}}$ classify $t = 65$ into the second population (Test, Large).

7. Discussion and conclusions

It is worth mentioning that a fair amount of research work has been done on classification under the same model set-up using the whole samples from two or more shifted exponential populations. However, when censored samples are available, not much attention has been paid in this direction to the best of our knowledge. This article consider the problem of classification into one of the two exponential populations with a common location parameter and different scale parameters using type-II right censored samples. Tripathy (2016) considered the same model set-up and estimated the common location parameter using the decision-theoretic approach. The author notably proposed improved estimators for the MLE and the UMVUE and a modification to the MLE. Moreover, we derive the MLEs and the UMVUEs for the associated scale parameters and obtain sufficient conditions for improving these estimators. Utilizing all these estimators for the associated model parameters, we construct several classification rules to classify a single observation and a group of observations into one of the two exponential populations. Performances of all the classification rules are evaluated through probabilities of correct classification and the EPC numerically. Our simulation study establish that the rules based on the UMVUE and its improved version for the common location parameter have the best performance in terms of EPC values.

The problem we consider in this article can be generalized to the case of $k(\geq 2)$ exponential populations. Suppose we have type-II right censored samples from k exponential populations $\Pi_1, \Pi_2, \ldots, \Pi_k$ having density functions $f_1, f_2, \ldots, f_k$, respectively, where $f_i \sim \text{Exp}(\mu, \sigma_i)$, $i = 1, 2, \ldots, k$. Utilizing the estimators of the associated parameters, we can construct classification rules to classify an observation t or a group of observations $(t_{(1)} \leq t_{(2)} \leq \ldots \leq t_{(r_i)})$ (type-II right censored sample) from the population Π_i into one of the k populations as follows: classify t into the population Π_i if $f_i/f_j \geq 0$, $j = 1, 2, \ldots, k$, $i \neq j$. The details of the classification problem for k populations will be considered separately. In this article we only consider the case for $k = 2$ populations and derive all the results related to $k = 2$ only. Moreover, the case of classification problem using multivariate exponential distribution will be more challenging and interesting. We hope that the present study will shed some light on the classification problems using certain censoring schemes from other probabilistic models, that may arise in practice.

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Summary

The problem of classification into two exponential populations with a common location parameter and different scale parameters under the type-II censoring scheme is considered. First, we consider classes of equivariant estimators for the scale parameters and derive sufficient conditions for improving estimators in these classes. Utilizing the maximum likelihood estimators (MLEs) and the uniformly minimum variance unbiased estimators (UMVUEs) for the associated parameters, various classification rules are constructed for classifying an observation and a group of observations into one of the two exponential populations. More importantly, a detailed and in-depth simulation study has been done to numerically compare the probabilities of correct classification and the expected probability of correct classification for all the proposed classification rules. Finally, a real-life example has been presented to illustrate the applicability of the proposed classification rules under the type-II censoring scheme.

Keywords: Classification using censored sample; Equivariant estimators; Maximum likelihood estimator (MLE); Probability of correct classification; Simulation study; Type-II censoring; Uniformly minimum variance unbiased estimator (UMVUE).