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THE DISCRETE POWER HALF-NORMAL DISTRIBUTION

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SUMMARY

The discrete power half-normal distribution is introduced, as the discretization of the power half-normal distribution, based on the difference of values of the continuous survival function. The discrete distribution has a bathtub shaped failure rate or an increasing failure rate. Some statistical properties are proved. Maximum likelihood estimation is studied. A simulation study shows the good asymptotic behaviour of the maximum likelihood estimates. Applications to reliability and lifetime data are provided.

Keywords: Bathtub failure rate; Discrete power; Half-Normal distribution; Increasing failure rate; Maximum likelihood estimation.

1. INTRODUCTION

The power half-normal distribution is an important model for describing reliability and lifetime data. This distribution is proposed in Gómez and Bolfarine (2015), applying the results in Lehmann (1953) and Durrans (1992) on power distributions. If $F(x) = P(X \leq x)$ is the cumulative distribution function of a continuous random variable X with density function $f(x)$, then $G(x) = F(x)^\alpha$ is the corresponding power cumulative distribution function, with density function $g(x) = \alpha(F(x))^{\alpha-1} f(x)$, where $\alpha > 0$. See also Nadarajah and Kotz (2006), Gupta and Gupta (2008), and Pewsey *et al.* (2012). The family of power half-normal distributions is a subfamily of the distributions considered in Pescim *et al.* (2010). See also Lawless (1982), chapters 3 to 6, and Sinha (1986), for other continuous models for reliability and lifetime data.

In applications, it may be convenient to analyze data by discrete models, since we are interested in lifetime of an on/off switch or lifetime of a device that is exposed to shocks or work in cycles. See Nakagawa and Osaki (1975), Stein and Dattero (1984), Padgett and Spurrier (1985), Khan *et al.* (1989), Kulasekera (1994), Roy (2003), Roy (2004), Kemp (2008), Krishna and Pundir (2009), Nooghabi *et al.* (2011), Al-Huniti and Al-Dayan (2012), Bebbington *et al.* (2012), Chakraborty and Chakravarty (2012),

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Almalki and Nadarajah (2014), Gómez-Déniz *et al.* (2014), Abouammoh and Alhazzani (2015), Chakraborty (2015), Sangpoom and Bodhisuwan (2016), Nekoukouh and Bidram (2017), Jayakumar and Babu (2018), Jayakumar and Sankaran (2018), Jayakumar and Sankaran (2019), and Vila *et al.* (2019).

In this paper, the discrete power half-normal distribution is proposed. The discretization of the continuous model is based on the difference of values of the continuous survival function. More precisely, in Section 2 the discrete model is introduced. In Section 3, some statistical properties are shown. In Section 4 maximum likelihood estimation is studied, and a Monte Carlo experiment is performed. Finally, in Section 5, the model is applied to published data sets.

2. The model

Following Gómez and Bolfarine (2015), the cumulative distribution function of the power half-normal phn(σ,α) random variable (r.v.) is

$$G(x;\sigma,\alpha)=\left(2\Phi\left(\frac{x}{\sigma}\right)-1\right)^{\alpha}, \tag{1}$$

where Φ is the standard normal cumulative distribution function, $\sigma > 0$, $\alpha > 0$, and $x \geq 0$.

The density function of Eq. (1) is

$$g(x;\sigma,\alpha)=\frac{2\alpha}{\sigma}\phi\left(\frac{x}{\sigma}\right)\left(2\Phi\left(\frac{x}{\sigma}\right)-1\right)^{\alpha-1}, \tag{2}$$

where ϕ is the standard normal density function, $\sigma > 0$, $\alpha > 0$, and $x \geq 0$.

The survival function of Eq. (1) is

$$S(x;\sigma,\alpha)=1-\left(2\Phi\left(\frac{x}{\sigma}\right)-1\right)^{\alpha}, \tag{3}$$

where $\sigma > 0$, $\alpha > 0$, and $x \geq 0$.

The discrete power half-normal distribution dphn(σ,α) has probability mass defined as

$$p(x;\sigma,\alpha)=S(x;\sigma,\alpha)-S(x+1;\sigma,\alpha), \tag{4}$$

where $x=0,1,2,\ldots$, $\sigma > 0$, and $\alpha > 0$. See Figures 1 and 2.

The failure rate of Eq. (4) is

$$h(x;\sigma,\alpha)=\frac{p(x;\sigma,\alpha)}{S(x;\sigma,\alpha)}, \tag{5}$$

where $x=0,\ldots,m$ and $m=0,1,2,\ldots$. The failure rate is bathtub shaped for $0<\alpha<1$ and increasing for $\alpha \geq 1$. See Figures 1 and 2.

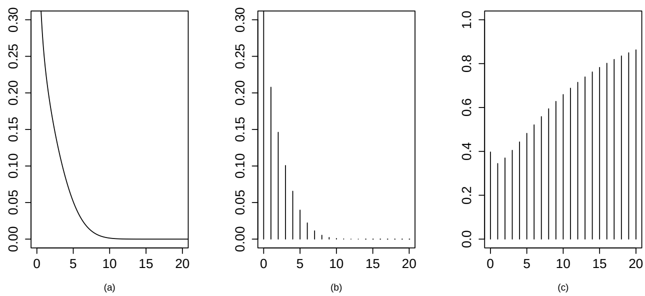

Figure 1 – Panel (a): density function $g(x;\sigma,\alpha)$, $x \geq 0$. Panel (b): discrete power half-normal distribution $p(x;\sigma,\alpha)$. Panel (c): failure rate $h(x;\sigma,\alpha)$, $x = 0,1,2,\ldots$, where $\sigma = 3.25$ and $\alpha = 0.65$.

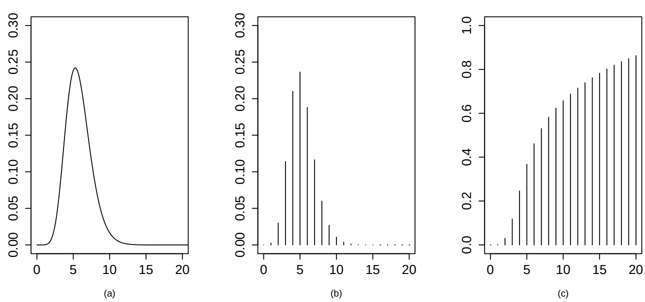

Figure 2 – Panel (a): density function $g(x;\sigma,\alpha)$, $x \geq 0$. Panel (b): discrete power half-normal distribution $p(x;\sigma,\alpha)$. Panel (c): failure rate $h(x;\sigma,\alpha)$, $x = 0,1,2,\ldots$, where $\sigma = 3.25$ and $\alpha = 7.8$.

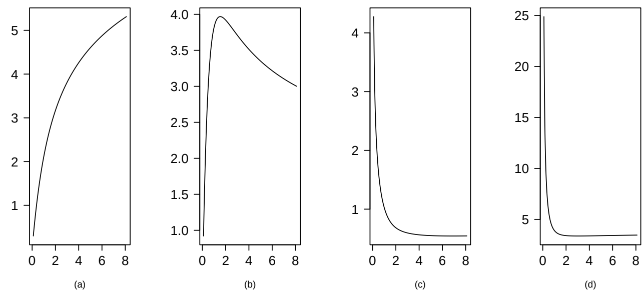

Figure 3 – Mean panel (a), variance panel (b), skewness panel (c) and kurtosis panel (d) of the discrete power half-normal distribution $\text{dphn}(\sigma,\alpha)$, for $\sigma = 3.25$ and $\alpha > 0$.

The moments,

$$\mu_r = E(X^r) = \sum_x x^r\, p(x;\sigma,\alpha),$$

cannot be expressed in closed form. Figure 3 shows the behaviour of the discrete power half-normal distribution $\text{dphn}(\sigma,\alpha)$, in terms of mean, variance, skewness, and kurtosis.

3. Some properties

THEOREM 1. *Let X_i, $i = 1,2,\ldots,n$, be non-negative independent identically distributed (i.i.d.) integer valued r.v's and $Y = \max_{1\le i\le n} X_i$. Then Y has a $dphn(\sigma,\alpha^n)$ distribution, if and only if X_i has a $dphn(\sigma,\alpha)$ distribution.*

PROOF. (If part). Let X_i, $i = 1,2,\ldots,n$, with a $\text{dphn}(\sigma,\alpha)$ distribution. Then

$$S(x) = 1 - \left(2\Phi\left(\frac{x}{\sigma}\right) - 1\right)^{\alpha}, \quad x = 0,1,2,\ldots.$$

For every $y = 0,1,2,\ldots$,

$$\begin{aligned} S(y) &= P(Y > y) = 1 - P(\text{all}\, X_i \le y) = 1 - F(y)^n \\ &= 1 - \left(2\Phi\left(\frac{y}{\sigma}\right) - 1\right)^{\alpha n}. \end{aligned}$$

(Only if part). Let $S(Y) = 1 - \left(2\Phi\left(\frac{y}{\sigma}\right) - 1\right)^{\alpha n}$, for $y = 0,1,2,\ldots$.
It is known that

$$S(x) = P(X_1 > x) = 1 - (P(\text{all}\, X_i \le x))^{1/n} = 1 - \left(2\Phi\left(\frac{x}{\sigma}\right) - 1\right)^{\alpha},$$

for $x = 0, 1, 2, \ldots$. □

THEOREM 2. *Let X_i, $i = 1, 2, \ldots$, be non-negative independent integer valued r.v.'s and $Y = \max_{1 \le i \le n} X_i$. Then, Y has a dphn(σ, α) distribution, if X_i has a dphn(σ, α_i) distribution, where $\alpha = \sum_{i=1}^{n} \alpha_i$.*

PROOF. We have that

$$S(y) = 1 - P(\text{all } X_i \le x) = 1 - \prod_{i=1}^{n} \left(2\Phi\left(\frac{x}{\sigma}\right) - 1\right)^{\alpha_i}$$
$$= 1 - \left(2\Phi\left(\frac{x}{\sigma}\right) - 1\right)^{\sum_{i=1}^{n} \alpha_i}.$$

□

Let $[\cdot]$ be the greatest integer function.

THEOREM 3. *If X has a phn(σ, α) distribution, then $Y = [X]$ has a dphn(σ, α) distribution.*

PROOF. We have that $[X] > Y \Leftrightarrow X > Y$. In fact, $([X] > Y) \subseteq (X > Y) \subseteq ([X] > [Y]) = ([X] > Y)$, where the last equality holds since Y is integer valued. Hence, $(X > Y) = ([X] > Y)$.

We also have that

$$P(Y > y) = P([X] > y) = P(X > y) = 1 - \left(2\Phi\left(\frac{y}{\sigma}\right) - 1\right)^{\alpha}.$$

□

THEOREM 4. *Let X be a non-negative r.v., and t be a positive number. Then $Y_t = [X/t]$ has a dphn(σ, α) distribution for every $t > 0$, if and only if X has a phn(σ, α) distribution.*

PROOF. (If part). We have that

$$P(Y_t = y) = P(y \le Y/t < y + 1) = S(yt) - S((y+1)t), \quad y = 0, 1, 2, \ldots.$$

Thus,

$$S_t(y) = P(Y_t > y) = S(yt),$$

where y is an integer. Moreover,

$$S_t(x) = 1 - \left(2\Phi\left(\frac{y}{\sigma t}\right) - 1\right)^{\alpha},$$

for every $t > 0$, and we have a dphn(σ, α) distribution.

(Only if part). Given that X_1 has a dphn(σ,α) distribution with survival function $S_t(y)=1-\left(2\Phi\left(\frac{y}{\sigma t}\right)-1\right)^{\alpha}$, $t>0$, $y=0,1,\ldots$. We have that $S(yt)=1-\left(2\Phi\left(\frac{y}{\sigma_t}\right)-1\right)^{\alpha}$, $t>0$, $y=0,1,2,\ldots$. Writing $yt=x$, where $0<x<+\infty$, we have

$$S(y)=1-\left(2\Phi\left(\frac{x}{\sigma_t t}\right)-1\right)^{\alpha}. \tag{6}$$

The left-hand side of Eq. (6) does not depend on t, and hence we must have on the right-hand side of Eq. (6) a positive constant $\sigma_t t=\sigma$. Thus, $S(x)=1-\left(2\Phi\left(\frac{x}{\sigma}\right)-1\right)^{\alpha}$, $0<x<+\infty$, implying that X has a phn(σ,α) distribution.

□

4. Maximum likelihood estimation

Let $\{z_1,\ldots,z_n\}$ be a sample of n i.i.d. observations from the dphn(σ,α) distribution, given by Eq. (4), on the values $x_i=0,1,2,\ldots$. The log-likelihood $l(\sigma,\alpha)=\log L(\sigma,\alpha)$ then is

$$l(\sigma,\alpha)=\sum_{i=1}^{n}\log p(z_i;\sigma,\alpha). \tag{7}$$

The score function $S(\sigma,\alpha)$ is the gradient vector $S(\sigma,\alpha)=(S(\sigma\alpha)_1,S(\sigma,\alpha)_2)$, with components $S(\sigma,\alpha)_1=(\partial/\partial\sigma)l(\sigma,\alpha)$ and $S(\sigma,\alpha)_2=(\partial/\partial\alpha)l(\sigma,\alpha)$. In particular, we have that

$$S(\sigma,\alpha)_1=\sum_{i=1}^{n}\left(\frac{1}{p(z_i;\sigma,\alpha)}\frac{\partial p(z_i;\sigma,\alpha)}{\partial\sigma}\right),$$
$$S(\sigma,\alpha)_2=\sum_{i=1}^{n}\left(\frac{1}{p(z_i;\sigma,\alpha)}\frac{\partial p(z_i;\sigma,\alpha)}{\partial\alpha}\right).$$

We have that

$$E_{(\sigma,\alpha)}(S(\sigma,\alpha)_1)=n\sum_{x}\left(\frac{\partial p(x;\sigma,\alpha)}{\partial\sigma}\right),$$
$$E_{(\sigma,\alpha)}(S(\sigma,\alpha)_2)=n\sum_{x}\left(\frac{\partial p(x;\sigma,\alpha)}{\partial\alpha}\right),$$

and $E_{(\sigma,\alpha)}(S(\sigma,\alpha))=(0,0)$.

The numerical solutions $(\hat{\sigma},\hat{\alpha})$ of the score equations $S(\sigma,\alpha)=(0,0)$ are the maximum likelihood estimates (m.l.e's) of (σ,α) in the dphn(σ,α) distribution of Eq. (4).

4.1. Information

For the log-likelihood $l(\sigma,\alpha)$, given by Eq. (7), the expected information matrix $\mathscr{I}(\sigma\alpha)$ can be obtained from minus the Hessian of $L(\sigma,\alpha)$ as

$$\mathscr{I}(\sigma,\alpha)=\begin{pmatrix}\mathscr{I}(\sigma,\alpha)_{11} & \mathscr{I}(\sigma,\alpha)_{12}\\ \mathscr{I}(\sigma,\alpha)_{21} & \mathscr{I}(\sigma,\alpha)_{22}\end{pmatrix},$$

where

$$\begin{aligned}\mathscr{I}(\sigma,\alpha)_{11}&=E_{(\sigma,\alpha)}\Big(-\frac{\partial S(\sigma,\alpha)_1}{\partial\sigma}\Big)\\ &=-n\sum_x\Big(\frac{1}{p(x;\sigma,\alpha)}\Big(\frac{\partial p(x;\sigma,\alpha)}{\partial\sigma}\Big)^2-\frac{\partial^2 p(x;\sigma,\alpha)}{\partial\sigma^2}\Big), \qquad (8)\\ \mathscr{I}(\sigma,\alpha)_{22}&=E_{(\sigma,\alpha)}\Big(-\frac{\partial S(\sigma,\alpha)_2}{\partial\alpha}\Big)\\ &=-n\sum_x\Big(\frac{1}{p(x;\sigma,\alpha)}\Big(\frac{\partial p(x;\sigma,\alpha)}{\partial\alpha}\Big)^2-\frac{\partial^2 p(x;\sigma,\alpha)}{\partial\alpha^2}\Big), \qquad (9)\\ \mathscr{I}(\sigma,\alpha)_{21}&=E_{(\sigma,\alpha)}\Big(-\frac{\partial S(\sigma,\alpha)_2}{\partial\sigma}\Big)\\ &=-n\sum_x\Big(\frac{1}{p(x;\sigma,\alpha)}\frac{\partial p(x;\sigma,\alpha)}{\partial\sigma}\frac{\partial p(x;\sigma,\alpha)}{\partial\alpha}-\frac{\partial^2 p(x;\sigma,\alpha)}{\partial\sigma\partial\alpha}\Big), \qquad (10)\\ \mathscr{I}(\sigma,\alpha)_{12}&=E_{(\sigma,\alpha)}\Big(-\frac{\partial S(\sigma,\alpha)_1}{\partial\alpha}\Big)\\ &=\mathscr{I}(\sigma,\alpha)_{21}. \qquad (11)\end{aligned}$$

4.2. Asymptotics

We assume that any closed and bounded subset of Ξ of the parameter (σ,α) is compact, and that the true parameter (σ_0,α_0) is an interior point of an open set in Ξ.

Applying Wald (1949), under some regularity conditions, and by using the strong law of large numbers, the strong convergence of the m.l.e.'s $(\hat{\sigma},\hat{\alpha})$ to (σ_0,α_0), as $n\to\infty$, can be shown.

Following Lehmann and Casella (1998), chapter 6, we can see that the third derivatives of the log-likelihood $l(\sigma,\alpha)$, given by Eq. (7), exists and can be bounded, in absolute value, by specific functions with finite expected values. The information matrix $\mathscr{I}(\sigma,\alpha)$, for the log-likelihood in Eq. (7), has finite elements in Equations (8), (9), (10), and (11), and is positive definite. We also have that $n^{1/2}((\hat{\sigma},\hat{\alpha})-(\sigma_0,\alpha_0))$ is asymptotically normal with mean $(0,0)$ and covariance matrices $\mathscr{I}(\sigma_0,\alpha_0)^{-1}$, as $n\to\infty$. Furthermore, we have that $\hat{\alpha}$ and $\hat{\sigma}$ in $(\hat{\sigma},\hat{\alpha})$ are asymptotically efficient, in the sense that

$n^{1/2}(\hat{\sigma}-\sigma_0)$ and $n^{1/2}(\hat{\alpha}-\alpha_0)$ have asymptotic variances $\mathscr{I}(\sigma_0,\alpha_0)^{-1}_{11}$ and $\mathscr{I}(\sigma_0,\alpha_0)^{-1}_{22}$, respectively, as $n \to \infty$.

4.3. Monte Carlo experiment

We performed a simulation experiment to study the bias and the mean square error of the m.l.e.'s $(\hat{\sigma},\hat{\alpha})$ in the dphn(σ,α) distribution, given by Eq. (4). We always simulated 10000 replications of the same experiment that consists in drawing a sample of n i.i.d. observations, from a dphn(σ,α) distribution, where $n = 15, 20, 50, 100, 200$ and $\sigma > 0$, and $\alpha > 0$.

We used the computational environment for statistics R, by R Core Team (2020). In particular, in the function `optim` of R, we considered the algorithm of Nelder and Mead (1965), for the optimization problems $\min_{(\sigma,\alpha)}(-l(\sigma,\alpha))$, with a log-likelihood of the form $l(\sigma,\alpha)$, given by Eq. (7).

In Tables 1 to 4 we provide the simulation results regarding the m.l.e's $(\hat{\sigma},\hat{\alpha})$ of (σ,α) for the dphn(σ,α) distribution, given by Eq. (4), with $\sigma = 32.5$, $\alpha = 0.65$, and $\alpha = 7.8$. The performance of $(\hat{\sigma},\hat{\alpha})$ improves, as n increases, but the convergence is faster for $0 < \alpha < 1$.

TABLE 1

Bias and mean square error of the m.l.e.'s $(\hat{\sigma},\hat{\alpha})$, for the discrete power half-normal distribution $p(x;\sigma,\alpha)$, where $\sigma = 3.25$ and $\alpha = 0.65$.

n	$b(\hat{\sigma})$	mse$(\hat{\sigma})$	$b(\hat{\alpha})$	mse$(\hat{\alpha})$
15	-0.14	0.81	0.16	2.15
20	-0.11	0.61	0.10	0.10
50	-0.04	0.25	0.04	0.03
100	-0.02	0.13	0.02	0.01
200	-0.01	0.06	0.01	0.01

TABLE 2

Bias and mean square error of the m.l.e.'s $(\hat{\sigma},\hat{\alpha})$, for the discrete power half-normal distribution $p(x;\sigma,\alpha)$, where $\sigma = 5.77$ and $\alpha = 0.65$.

n	$b(\hat{\sigma})$	mse$(\hat{\sigma})$	$b(\hat{\alpha})$	mse$(\hat{\alpha})$
15	-0.02	2.34	0.11	0.11
20	-0.19	1.76	0.08	0.07
50	-0.08	0.72	0.04	0.02
100	-0.03	0.36	0.01	0.01
200	-0.02	0.18	0.01	0.01

TABLE 3
Bias and mean square error of the m.l.e.'s $(\hat{\sigma}, \hat{\alpha})$, *for the discrete power half-normal distribution* $p(x;\sigma,\alpha)$, *where* $\sigma = 3.25$ *and* $\alpha = 7.8$.

n	$b(\hat{\sigma})$	mse($\hat{\sigma}$)	$b(\hat{\alpha})$	mse($\hat{\alpha}$)
15	-0.11	0.22	4.43	214.85
20	-0.08	0.17	2.76	64.13
50	–0.05	0.06	0.86	7.53
100	-0.01	0.03	0.37	2.72
200	-0.09	0.02	0.18	0.19

TABLE 4
Bias and mean square error of the m.l.e.'s $(\hat{\sigma}, \hat{\alpha})$, *for the discrete power half-normal distribution* $p(x;\sigma,\alpha)$, *where* $\sigma = 5.77$ *and* $\alpha = 7.8$.

n	$b(\hat{\sigma})$	mse($\hat{\sigma}$)	$b(\hat{\alpha})$	mse($\hat{\alpha}$)
15	-0.19	0.67	3.92	129.58
20	-0.14	0.50	2.55	52.06
50	–0.06	0.19	0.82	6.89
100	-0.02	0.09	0.35	2.51
200	-0.01	0.05	0.17	1.10

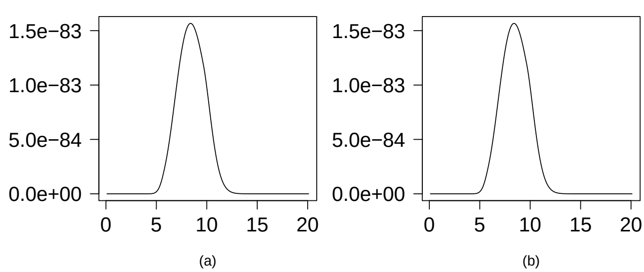

Figure 4 – Profile likelihoods $L(\alpha)$, $n = 100$, for the discrete power half-normal distribution dphn(σ,α), where $\sigma = 3.25$ and $\alpha = 0.65$ (panel (a)), and $\sigma = 3.25$ and $\alpha = 7.8$ (panel (b)).

4.4. Profile likelihood

Given a likelihood $L(\sigma,\alpha)$, the profile likelihood of α is

$$L(\alpha) = \max_{\sigma} L(\sigma,\alpha), \tag{12}$$

where $\alpha > 0$. More precisely, Eq. (12) means that $L(\alpha) = L(\hat{\sigma}_\alpha, \alpha)$, where $\hat{\sigma}_\alpha$ is the m.l.e. of θ for a given α, where $\alpha > 0$.

To calculate the profile likelihood $L(\alpha)$, defined in Eq. (12), we have used the minimization algorithm of Brent (1973), chapter 5, which is suitable for the univariate optimization problems $\min_\sigma(-L(\sigma,\alpha))$, where $\alpha > 0$. See also the function `optim` in R Core Team (2020) for further details and an implementation of the algorithm.

Figure 4 shows the good behaviour of the profile likelihood $L(\alpha)$, given by Eq. (12), for the dphn(σ,α) distribution.

5. Applications

5.1. Data set 1

We consider a data set from Birnbaum and Saunders (1969) of $n = 101$ observations, referring to maximum stress per cycle 31,000 psi (Tabel 5).

TABLE 5
Data set 1: maximum stress per cycle 31,000 psi.

70,90,96,97,99,100,103,104,104,105,107,108,108,108,109,109,112,112,113,114,
114,114,116,119,120,120,120,121,121,123,124,124,124,124,124,128,128,129,
129,130,130,130,131,131,131,131,131,132,132,132,133,134,134,134,134,134,
136,136,137,138,138,138,139,139,141,141,142,142,142,142,142,142,144,144,
145,146,148,148,149,151,151,152,155,156,157,157,157,157,158,159,162,163,
163,164,166,166,168,170,174,196,212

The m.l.e.'s for the dphn(σ,α) distribution were $\hat{\sigma} = 55.1689$ and $\hat{\alpha} = 41.2697$. The Akaike (1974) information criterion was $AIC = 861.8349$. In Figure 5, the failure rate is shown.

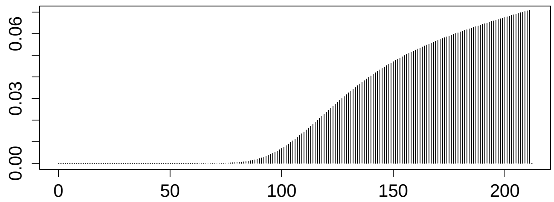

Figure 5 – Failure rate for the data set 1.

5.2. *Data set 2*

We considered a data set from Aarset (1987) of $n = 50$ observations, referring to lifetimes of devices (Table 6).

TABLE 6
Data set 2: lifetimes of devices.

0,0,1,1,1,1,1,2,3,6,7,11,12,18,18,18,18,18,21,32,36,40,45, 46,47,50,55,
60,63,63,67,67,67,67,72,75,79,82,82,83,84,84,84,85,85,85,85,85,86,86

The m.le.'s for the dphn(σ, α) distribution were $\hat{\sigma} = 62.8226$ and $\hat{\alpha} = 0.7538$. The Akaike (1974) information criterion was $AIC = 436.4083$. In Figure 6, the failure rate, having a bathtub shape, is shown.

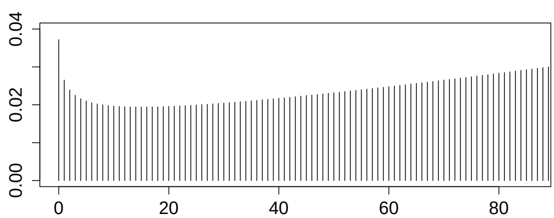

Figure 6 – Failure rate for the data set 2.

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