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THE MARSHALL-OLKIN EXTENDED UNIT-GOMPERTZ DISTRIBUTION: ITS PROPERTIES, REGRESSION MODEL AND APPLICATIONS

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SUMMARY

In this paper, a new bounded generalization of the unit-Gompertz distribution called the Marshall-Olkin extended unit-Gompertz distribution (MOEUGD) is introduced. The mathematical properties and an associated quantile regression model of the proposed distribution are derived. The maximum likelihood estimation method is employed for estimating the parameters of the proposed distribution, and a Monte Carlo simulation study is carried out to investigate the asymptotic behaviour of the parameter estimates of the proposed distribution. Finally, the applicability of the proposed distribution is illustrated by means of two real data sets defined on a unit-interval andan application of the regression model to a real data set.

Keywords: Unit-Gompertz distribution; Marshall-Olkin; Quantile regression model.

1. INTRODUCTION

Lifetime distributions are applied to vast areas of real life phenomena such as in reliability theory and survival analysis. For example, the exponential distribution has been widely applied to model survival data sets and has also been generalized by many researchers due to the disadvantage of exhibiting only a constant hazard rate property. Some of these generalizations include the Weibull distribution, gamma distribution, exponentiated exponential distribution, and Gompertz distribution. Increasing the flexibility of the classical lifetime distributions in analyzing real life phenomena has remained a strong reason for the generalization of classical distributions.

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Several unit distributions have been introduced for modelling datasets in the field of Biological sciences, Engineering, Actuarial sciences, Economics and Financial Risk Management, etc. Examples of such distributions include the unit-Weibull distribution due to Mazucheli *et al.* (2020), the unit-Bur XII distribution due to Korkmaz and Chesneau (2021), the log-weighted exponential distribution due to Altun (2021), the Marshall-Olkin extended Topp-Leone distribution due to Opone and Iwerumor (2021), the logit slash distribution due to Korkmaz (2020), the unit-Johnson SU distribution due to Gunduz and Korkmaz (2020), the unit-Gamma distribution due to Mazucheli *et al.* (2018a),the unit-Birnbaum-Saunders distribution due to Mazucheli *et al.* (2018b), the bounded weighted exponential distribution due to Mallick *et al.* (2021), the bounded M-O extended exponential distribution due to Ghosh *et al.* (2019), the transmuted Marshall-Olkin Topp-Leone distribution developed by Opone and Osemwenkhae (2022), the transmuted continuous Bernoulli distribution introduced by Chesneau *et al.* (2022), the power continuous Bernoulli distribution proposed by Chesneau and Opone (2022), etc.

Mazucheli *et al.* (2019) introduced a new generalization of the Gompertz distribution with bounded support using the logarithm transformation for which $X = e^{-Y}$ in the Gompertz distribution. The density function of the unit-Gompertz distribution is defined by

$$f(x) = \lambda\beta x^{-(\beta+1)} \exp\{-\lambda(x^{-\beta} - 1)\}, \qquad x, \lambda, \beta > 0, \tag{1}$$

and the corresponding cumulative distribution function is obtained as

$$F(x) = \exp\{-\lambda(x^{-\beta} - 1)\}, \qquad x, \lambda, \beta > 0. \tag{2}$$

In this paper, we extend the unit-Gompertz distribution by using the method of generalization in Marshall and Olkin (1997).

Suppose the survival function of a known probability distribution is defined by $\bar{F}(x)$. Marshall and Olkin (1997) defined the survival function of the Marshall-Olkin extended family of distributions as

$$\bar{G}(x,\alpha) = \frac{\alpha\bar{F}(x)}{1-\bar{\alpha}\bar{F}(x)} = \frac{\alpha\bar{F}(x)}{F(x)+\alpha\bar{F}(x)}, \qquad -\infty < x < \infty, \quad 0 < \alpha < \infty. \tag{3}$$

If F is a cumulative distribution function with a density function f, then G has a density function given as

$$g(x,\alpha) = \frac{\alpha f(x)}{\{1-\bar{\alpha}\bar{F}(x)\}^2}, \qquad -\infty < x < \infty, \quad 0 < \alpha < \infty, \tag{4}$$

where $\bar{\alpha} = 1 - \alpha$ is called a "tilt parameter", since the hazard $h(x)$ of the transformed distribution is shifted below ($\alpha \geq 1$) or above ($0 < \alpha \leq 1$) from the hazard $r(x)$ of the baseline distribution. In fact, for all $x \geq 0, h(x) \leq r(x)$ when $\alpha \geq 1$, and $h(x) \geq r(x)$ when $0 < \alpha \leq 1$.

Considering the unit-Gompertz dsitribution as the baseline distribution in Eq. (3) and Eq. (4), the resulting distribution is what we call the Marshall-Olkin extended unit-Gompertz distribution (MOEUGD). The remaining Sections of this paper are organized as follows: Section 2 presents some mathematical properties of the proposed distribution, which include the survival function, cumulative distribution function, probability density function, hazard rate function, quantile function, median, moments, moment generating function, and Renyi entropy. The parameter estimates of the proposed distribution using the maximum likelihood method and a Monte Carlo simulation study to investigate the performance of the maximum likelihood estimators are given in Section 3. The quantile regression model of the proposed distribution and its validation is introduced in Section 4. Section 5 illustrates the applicability of the proposed distribution in analyzing two real data sets, while Section 6 presents the application of the quantile regression model of the proposed distribution to a real data set. Finally, we give a concluding remark in Section 7.

2. Mathematical properties of the proposed MOEUG distribution

2.1. *Probability density and cumulative distribution functions of the MOEUGD*

The density function of a random variable X following the proposed Marshall-Olkin extended unit-Gompertz distribution is defined by

$$g(x) = \frac{\alpha\beta\lambda x^{-(\beta+1)}\exp\{-\lambda(x^{-\beta}-1)\}}{\{1-\bar{\alpha}\left[1-\exp\{-\lambda(x^{-\beta}-1)\}\right]\}^2}, \qquad 0 < x < 1, \quad \alpha,\beta,\lambda > 0. \tag{5}$$

The corresponding cumulative distribution function of the MOEUGD is obtained as

$$G(x) = \frac{\exp\{-\lambda(x^{-\beta}-1)\}}{1-\bar{\alpha}\left[1-\exp\{-\lambda(x^{-\beta}-1)\}\right]}, \qquad 0 < x < 1, \quad \alpha,\beta,\lambda > 0. \tag{6}$$

The density function defined in Eq. (5) can further be expressed in a series representation using the generalized binomial expansion for any positive real number and $|z| < 1$, reported in George and Thobias (2017) as

$$(1-z)^{-s} = \sum_{k=0}^{\infty} \binom{s+k-1}{k} z^k. \tag{7}$$

Thus, we have that

$$\left(1-\bar{\alpha}\left[1-e^{-\lambda(x^{-\beta}-1)}\right]\right)^{-2} = \sum_{i=0}^{\infty} \binom{i+1}{i} \bar{\alpha}^i \left[1-e^{-\lambda(x^{-\beta}-1)}\right]^i, \tag{8}$$

$$\left[1-e^{-\lambda(x^{-\beta}-1)}\right]^i = \sum_{i=0}^{\infty} \binom{i}{j} (-1)^j e^{-\lambda(x^{-\beta}-1)j}, \tag{9}$$

$$e^{-\lambda(x^{-\beta}-1)(j+1)} \approx e^{-\lambda(j+1)x^{-\beta}+\lambda(j+1)}. \tag{10}$$

Recall that

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}, \tag{11}$$

so that

$$e^{-\lambda(j+1)x^{-\beta}} = \sum_{k=0}^{\infty} \frac{(-\lambda(j+1))^k x^{-\beta k}}{k!}. \tag{12}$$

Hence, Eq. (5) can be expressed as

$$g(x) = \alpha\beta\lambda \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \binom{i+1}{i}\binom{i}{j} \frac{\bar{\alpha}^i(-1)^{j+k}}{k!} e^{\lambda(j+1)}[\lambda(j+1)]^k x^{-\beta(k+1)-1}. \tag{13}$$

The graphical representation of the density function of the MOEUG distribution is given in Figure 1.

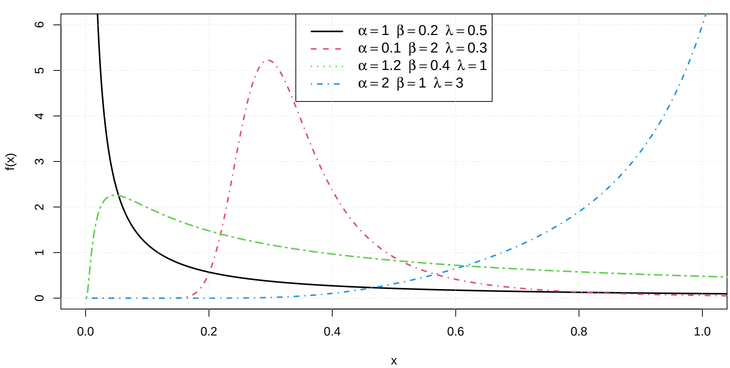


Figure 1 – Density function of the MOEUGD for different values of the parameters.

Figure 1 shows that the density function of the MOEUG distribution accomodates decreasing, increasing, and right-skewed unimodal shapes for different choices of the parameters of the distribution.

2.2. *Survival and hazard rate functions of the MOEUGD*

The survival and hazard rate functions of the proposed MOEUG distribution are, respectively, defined by

$$\begin{aligned} S(x) &= 1-G(x)\\ &= 1-\frac{\exp\{-\lambda(x^{-\beta}-1)\}}{1-\bar{\alpha}\left[1-\exp\{-\lambda(x^{-\beta}-1)\}\right]},\\ &= \frac{\alpha\left[1-\exp\{-\lambda(x^{-\beta}-1)\}\right]}{1-\bar{\alpha}\left[1-\exp\{-\lambda(x^{-\beta}-1)\}\right]}, \qquad 0<x<1, \quad \alpha,\beta,\lambda>0, \end{aligned} \tag{14}$$

and

$$\begin{aligned} H(x) &= \frac{g(x)}{1-G(x)}\\ &= \frac{\alpha\beta\lambda x^{-(\beta+1)}\exp\{-\lambda(x^{-\beta}-1)\}}{\{1-\bar{\alpha}\left[1-\exp\{-\lambda(x^{-\beta}-1)\}\right]\}^2} \times \frac{1-\bar{\alpha}\left[1-\exp\{-\lambda(x^{-\beta}-1)\}\right]}{\alpha\left[1-\exp\{-\lambda(x^{-\beta}-1)\}\right]}\\ &= \frac{\alpha\beta\lambda x^{-(\beta+1)}\exp\{-\lambda(x^{-\beta}-1)\}}{1-\bar{\alpha}\left[1-\exp\{-\lambda(x^{-\beta}-1)\}\right]\alpha\left[1-\exp\{-\lambda(x^{-\beta}-1)\}\right]}. \end{aligned} \tag{15}$$

The plot of the hazard rate function of the MOEUG distribution is shown in Figure 2.

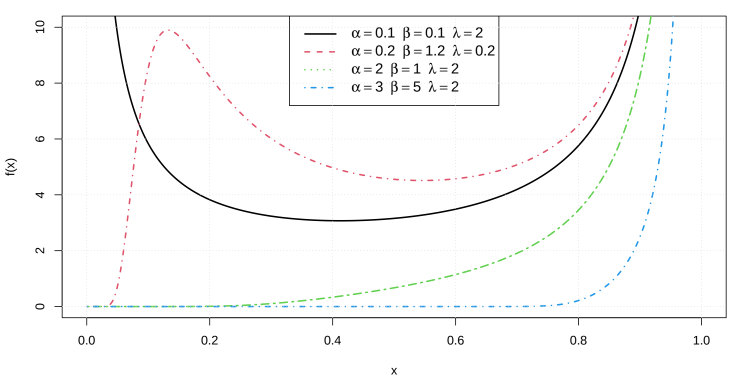


Figure 2 – Hazard rate function of the MOEUGD for different values of the parameters.

The plot indicates that the hazard rate function of the MOEUG distribution accomodates an increasing, bathtub, and upsidedown bathtub shaped property.

2.3. The quantile function of the MOEUGD

Suppose that $G(x)$ is the cumulative distribution function of a continous random variable X, then the quantile function of X, $(Q_X(u))$ is derived by solving the system of equation $G(x) = u$ to obtain $Q_X(u) = G^{-1}(u)$, where u is uniformly distributed. The quantile function of a random variable X, following the Marshall-Olkin extended unit-Gompertz distribution, is therefore defined as

$$\begin{aligned}
\frac{\exp\{-\lambda(x^{-\beta}-1)\}}{1-\bar{\alpha}\left[1-\exp\{-\lambda(x^{-\beta}-1)\}\right]} &= u \\
\exp\{-\lambda(x^{-\beta}-1)\} &= u - u\bar{\alpha}\left[1-\exp\{-\lambda(x^{-\beta}-1)\}\right] \\
\exp\{-\lambda(x^{-\beta}-1)\}(1-u\bar{\alpha}) &= u(1-\bar{\alpha}) \\
\exp\{-\lambda(x^{-\beta}-1)\} &= \frac{\alpha u}{(1-u\bar{\alpha})} \\
x^{-\beta} &= 1-\frac{1}{\lambda}\ln\left(\frac{\alpha u}{(1-u\bar{\alpha})}\right) \\
Q_X(u) &= \left[1-\frac{1}{\lambda}\ln\left(\frac{\alpha u}{(1-u\bar{\alpha})}\right)\right]^{-\frac{1}{\beta}}. \qquad (16)
\end{aligned}$$

The median of the MOEUG distribution is obtained by assuming $u = \frac{1}{2}$ in Eq. (16), which yields

$$\text{Median} = Q_2\left(\frac{1}{2}\right) = \left[1-\frac{1}{\lambda}\ln\left(\frac{\alpha}{2\left(1-\frac{\bar{\alpha}}{2}\right)}\right)\right]^{-\frac{1}{\beta}}. \qquad (17)$$

Some numerical computation of quantiles from the MOEUG distribution for different values of the parameters are given in Table 1.

TABLE 1
Some quantiles from the MOEUG distribution ($\alpha = 2$).

u	$(\beta=3,\lambda=2)$	$(\beta=1,\lambda=1)$	$(\beta=2,\lambda=0.5)$	$(\beta=1,\lambda=3)$
0.1	0.8142	0.3697	0.4762	0.6377
0.2	0.8642	0.4765	0.5593	0.7320
0.3	0.8967	0.5640	0.6267	0.7951
0.4	0.9211	0.6412	0.6869	0.8428
0.5	0.9403	0.7115	0.7431	0.8809
0.6	0.9562	0.7766	0.7967	0.9125
0.7	0.9695	0.8374	0.8487	0.9392
0.8	0.9811	0.8946	0.8996	0.9622
0.9	0.9911	0.9487	0.9500	0.9823

Table 1 shows some quantiles from the MOEUG distribution, which can also be seen as random samples from the distribution. We clearly observe that for varying values of the parameters of the distribution, the random variates fall within the unit-interval, which conforms with the support of the proposed distribution.

2.4. The r^{th} moment and moment generating function of the MOEUGD

Let X be a continuous random variable with probability density function $g(x)$, then the r^{th} moment about the origin of X is defined by

$$\mu_r^{'} = E(X^r) = \int_{-\infty}^{\infty} x^r g(x) \mathrm{d}x. \tag{18}$$

Substituting the series representation of the density function of the MOEUG distribution into Eq. (18), the r^{th} moment of the MOEUG distribution is obtained as

$$\begin{aligned}
\mu_r^{'} &= \alpha\beta\lambda\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{i+1}{i}\binom{i}{j}\frac{\bar{\alpha}^i(-1)^{j+k}}{k!}e^{\lambda(j+1)}[\lambda(j+1)]^k\int_0^1 x^{r-\beta(k+1)-1}\mathrm{d}x \\
&= \alpha\beta\lambda\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{i+1}{i}\binom{i}{j}\frac{\bar{\alpha}^i(-1)^{j+k}}{k![r-\beta(k+1)]}e^{\lambda(j+1)}[\lambda(j+1)]^k. \quad (19)
\end{aligned}$$

The first four r^{th} moments of the MOEUG distribution in terms of infinite series are obtained from Eq. (19) as

$$\begin{aligned}
\mu_1^{'} &= \alpha\beta\lambda\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{i+1}{i}\binom{i}{j}\frac{\bar{\alpha}^i(-1)^{j+k}}{k![1-\beta(k+1)]}e^{\lambda(j+1)}[\lambda(j+1)]^k, \\
\mu_2^{'} &= \alpha\beta\lambda\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{i+1}{i}\binom{i}{j}\frac{\bar{\alpha}^i(-1)^{j+k}}{k![2-\beta(k+1)]}e^{\lambda(j+1)}[\lambda(j+1)]^k, \\
\mu_3^{'} &= \alpha\beta\lambda\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{i+1}{i}\binom{i}{j}\frac{\bar{\alpha}^i(-1)^{j+k}}{k![3-\beta(k+1)]}e^{\lambda(j+1)}[\lambda(j+1)]^k, \\
\mu_4^{'} &= \alpha\beta\lambda\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{i+1}{i}\binom{i}{j}\frac{\bar{\alpha}^i(-1)^{j+k}}{k![4-\beta(k+1)]}e^{\lambda(j+1)}[\lambda(j+1)]^k. \quad (20)
\end{aligned}$$

The variance (σ^2), measures of skewness (S_k), and kurtosis (K_s) of the MOEUG distribution can be derived by substituting the values of the r^{th} moments into the expressions

$$\sigma^2 = \mu_2^{'} - \mu^2,$$
$$S_k = \frac{\mu_3^{'} - 3\mu_2^{'}\mu + 2\mu^3}{\left(\mu_2^{'} - \mu^2\right)^{\frac{3}{2}}},$$
$$K_s = \frac{\mu_4^{'} - 4\mu_3^{'}\mu + 6\mu_2^{'}\mu^2 - 3\mu^4}{\left(\mu_2^{'} - \mu^2\right)^2}. \tag{21}$$

The generating function of a continuous random variable X with density function $g(x)$ is defined by

$$M_X(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} g(x) dx, \tag{22}$$

thus, the generating function of the MOEUG distribution is given by

$$M_X(t) = \alpha\beta\lambda \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \binom{i+1}{i}\binom{i}{j} \frac{\bar{\alpha}^i (-1)^{j+k} t^m e^{\lambda(j+1)} [\lambda(j+1)]^k}{k! m! [m - \beta(k+1)]}, \tag{23}$$

since $e^{tx} = \sum_{m=0}^{\infty} \frac{t^m x^m}{m!}$.

Numerical computation of the theoretical moments of the MOEUG distribution for selected values of the parameters is shown in Table 2.

TABLE 2
Theoretical Moments of the MOEUG distribution (β=1).

$\mu_r^{'}$	$(\alpha=2, \lambda=3)$	$(\alpha=1, \lambda=1)$	$(\alpha=3, \lambda=1)$	$(\alpha=0.2, \lambda=0.5)$
$\mu_1^{'}$	0.840	0.5985	0.7311	0.2764
$\mu_2^{'}$	0.7254	0.4065	0.5767	0.109
$\mu_3^{'}$	0.6388	0.301	0.4772	0.0578
$\mu_4^{'}$	0.5712	0.234	0.4078	0.0374
σ^2	0.019	0.0481	0.0422	0.0326
S_k	-1.1692	0.0074	-0.7059	1.6394
K_s	4.2006	1.9908	2.6304	5.599

Table 2 displays the theoretical moments of the MOEUG distribution for different values of the parameters of the distribution. The Table indicates that the MOEUG distribution is positively (right)-skewed ($S_k > 0$), negatively (left)-skewed ($S_k < 0$), and approximately symmetric ($S_k \approx 0$). The MOEUG distribution can also be leptokurtic ($K_s > 3$), platykurtic ($K_s < 3$) and mesokurtic ($K_s \approx 3$). This claim supports the graphical illustration of the density function of the MOEUG distribution in Figure 1.

2.5. *The Renyi entropy of the MOEUGD*

An entropy of a random variable X is a measure of variation of uncertainty associated with the ramdom variable X. Rényi (1961) defined the Renyi entropy of X with density function $g(x)$ as

$$\tau_R(\gamma)=\frac{1}{1-\gamma}\log\left[\int g^{\gamma}(x)\mathrm{d}x\right], \qquad \gamma>0,\gamma\neq 1. \tag{24}$$

Substituting Eq. (5) in Eq. (24), we have

$$\begin{aligned}\tau_R(\gamma) &= \frac{1}{1-\gamma}\log\int_0^1(\alpha\beta\lambda)^{\gamma}x^{-\gamma(\beta+1)}e^{-\lambda(x^{-\beta}-1)} \\ &\times \left(1-\bar{\alpha}\left[1-e^{-\lambda(x^{-\beta}-1)}\right]\right)^{-2\gamma}\mathrm{d}x. \end{aligned} \tag{25}$$

Using the generalized binomial expansion defined in Eq. (7),

$$\begin{aligned}\left(1-\bar{\alpha}\left[1-e^{-\lambda(x^{-\beta}-1)}\right]\right)^{-2\gamma} &= \sum_{i=0}^{\infty}\binom{i+\gamma}{i}\bar{\alpha}^{\gamma i}\left(1-e^{-\lambda(x^{-\beta}-1)}\right)^{\gamma i}, \\ \left(1-e^{-\lambda(x^{-\beta}-1)}\right)^{\gamma i} &= \sum_{j=0}^{\infty}(-1)^j\binom{\gamma i}{j}e^{-\lambda(x^{-\beta}-1)\gamma j}, \\ e^{-\lambda\gamma(j+1)(x^{-\beta}-1)} &\approx e^{-\lambda\gamma(j+1)-\lambda\gamma(j+1)x^{-\beta}}, \\ e^{-\lambda\gamma(j+1)x^{-\beta}} &= \sum_{k=0}^{\infty}\frac{(-1)^k[\lambda\gamma(j+1)]^k x^{-\beta k}}{k!}, \end{aligned} \tag{26}$$

then Eq. (25) becomes

$$\tau_R(\gamma) = \frac{1}{1-\gamma}\log(\alpha\beta\lambda)^{\gamma}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{i+\gamma}{i}\binom{\gamma i}{j}(-1)^{j+k}\bar{\alpha}^{\gamma i}e^{-\lambda\gamma(j+1)} \tag{27}$$

$$\times \; [\lambda\gamma(j+1)]^k\int_0^1 x^{-\beta(\gamma+k)-\gamma}\mathrm{d}x. \tag{28}$$

Evaluating the integral part of Eq. (27) yields

$$\begin{aligned}\tau_R(\gamma) &= \frac{1}{1-\gamma}\log(\alpha\beta\lambda)^{\gamma}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{i+\gamma}{i}\binom{\gamma i}{j}(-1)^{j+k}\bar{\alpha}^{\gamma i} \\ &\times \frac{e^{-\lambda\gamma(j+1)}[\lambda\gamma(j+1)]^k}{1-\beta(\gamma+k)-\gamma}. \end{aligned} \tag{29}$$

Golshani and Pasha (2010) provided some important properties of the measure given in Eq. (24):

(i) The Renyi entropy can be negative;

(ii) For any $\gamma_1 < \gamma_2$, $R_{\gamma_2} \leq R_{\gamma_1}$, and equality holds if and only if X is a uniform random variable.

According to Kayal and Kumar (2017), the Renyi entropy is more or less sensitive to the shape of the probability distribution due to the parameter γ. For large values of γ, the measure given in Eq. (24) is more sensitive to events that occur often, while for small values of γ, it is more sensitive to event that occur rarely.

Numerical computation of the Renyi entropy of the MOEUGD for varying values of parameter γ is shown in Table 3.

TABLE 3
Numerical Computation of the Renyi Entropy of the MOEUGD (λ =1).

γ	$(\alpha = 4, \beta = .3)$	$(\alpha = 2, \beta = .2)$	$(\alpha = 4, \beta = 2)$	$(\alpha = 3, \beta = 1)$
0.01	-0.0001	-0.0030	-0.1545	-0.0271
0.03	-0.0003	-0.0090	-0.2488	-0.0565
0.5	-0.0049	-0.01792	-0.7750	-0.2717
0.7	-0.0069	-0.2718	-0.8843	-0.3198
2	-0.0206	-1.1815	-1.2921	-0.5142
4	-0.0460	-2.2117	-1.5512	-0.6681
7	-0.0947	-2.7264	-1.7135	-0.7844
9	-0.1320	-2.8805	-1.7716	-0.8300

From Table 3, we clearly observe that for any two consecutive values of parameter γ_i, say, γ_1 and γ_2, the Renyi entropy R_{γ_i}, say, R_{γ_1} and R_{γ_2}, satisfies the condition $\gamma_1 < \gamma_2$, $R_{\gamma_2} \leq R_{\gamma_1}$ as suggested by Golshani and Pasha (2010).

3. Parameter estimation

3.1. *Maximum likelihood estimation*

Here, we present the maximum likelihood estimates(MLEs) of the parameters of MOEUGD(α, β, λ). Let $x_1, x_2, \cdots, x_n$ be random samples from the MOEUGD with density function defined in Eq. (5), then the log-likelihood function is given by

$$\begin{aligned}
\ell(x,\varphi) &= \sum_{i=1}^{n} \ln[g(x)], \\
\ell(x,\varphi) &= \sum_{i=1}^{n} \ln\left(\frac{\alpha\beta\lambda x^{-(\beta+1)}\exp\{-\lambda(x^{-\beta}-1)\}}{\{1-\bar{\alpha}\left[1-\exp\{-\lambda(x^{-\beta}-1)\}\right]\}^2}\right), \qquad \varphi=(\alpha,\beta,\lambda), \\
&= n\ln(\alpha,\beta,\lambda)-(\beta+1)\sum_{i=1}^{n} ln(x_i)-\lambda\sum_{i=1}^{n}(x_i^{-\beta}-1) \\
&- 2\sum_{i=1}^{n}\ln\left[1-\bar{\alpha}\left(1-e^{-\lambda(x_i^{-\beta}-1)}\right)\right]. \qquad (30)
\end{aligned}$$

The first partial derivatives of the log-likelihood function with respect to the different parameters are given by

$$\frac{\partial\ell(x,\varphi)}{\partial\alpha} = \frac{n}{\alpha}-2\sum_{i=1}^{n}\frac{1-e^{-\lambda(x_i^{-\beta}-1)}}{1-\bar{\alpha}\left[1-e^{-\lambda(x^{-\beta}-1)}\right]}, \qquad (31)$$

$$\frac{\partial\ell(x,\varphi)}{\partial\beta} = \frac{n}{\beta}-\sum_{i=1}^{n}\ln x_i+\lambda\sum_{i=1}^{n}x_i^{-\beta}\ln x_i-2\lambda\bar{\alpha}\sum_{i=1}^{n}\frac{e^{-\lambda(x_i^{-\beta}-1)}x_i^{-\beta}\ln x_i}{1-\bar{\alpha}\left[1-e^{-\lambda(x^{-\beta}-1)}\right]}, \qquad (32)$$

$$\frac{\partial\ell(x,\varphi)}{\partial\lambda} = \frac{n}{\lambda}-\sum_{i=1}^{n}\left(x^{-\beta}-1\right)+2\bar{\alpha}\sum_{i=1}^{n}\frac{e^{-\lambda(x_i^{-\beta}-1)}x_i^{-\beta}-e^{-\lambda(x_i^{-\beta}-1)}}{1-\bar{\alpha}\left[1-e^{-\lambda(x^{-\beta}-1)}\right]}. \qquad (33)$$

The maximum likelihood estimate $\hat{\varphi}$ of the parameters φ is obtained by solving the system of non-linear equations $\frac{\partial\ell(x,\varphi)}{\partial\varphi}=0$. These equations can be solved using a numerical method known as New Raphson iterative scheme given by

$$\hat{\varphi}=\varphi_k-H^{-1}(\varphi_k)U(\varphi_k), \quad \hat{\varphi}=(\hat{\alpha},\hat{\beta},\hat{\lambda})^T, \qquad (34)$$

where $U(\varphi_k)$ is the score function and $H^{-1}(\varphi_k)$ is the Hessian matrix, which is the second partial derivative of the log-likelihood function. The `bbmle` package in the `R` statistical software package is used to evaluate the maximum likelihood estimates of the parameters of the MOEUG distribution.

3.2. *Simulation study*

Here, we investigate the performance and the accuracy of the maximum likelihood estimate of the parameters of the Marshall-Olkin extended unit-Gompertz distribution through a simulation study. Using a similar simulation algorithm to that reported in the works of Opone and Ekhosuehi (2018) and Tuoyo *et al.* (2021), a Monte Carlo simulation of pseudo-random samples from the MOEUG distribution is repeated 1000 times for different sample sizes $n = (20, 50, 100, 200)$ and fixed parameter values of $(\alpha = 0.1, \beta = 0.5, \lambda = 0.2)$, $(\alpha = 0.3, \beta = 0.8, \lambda = 0.1)$, and $(\alpha = 0.1, \beta = 2.0, \lambda = 0.1)$. The asymptotic behaviour of the parameter estimates is examined through the computation of three quantities, namely the Bias, Root Mean-Squared Error (RMSE), and Coverage Probability (CP) of the 95% confidence interval of the parameter estimates. Table 4 displays the Monte Carlo simulation results for the Bias, RMSE, and the coverage probability (CP) of the 95% confidence interval of the parameter estimates.

From Table 4 we observe that the bias and root mean-squared error of the parameter estimates of the MOEUG distribution decreases (tends to zero) as the sample size n increases, which holds in case of the consistency property of an estimator. Also, we observe that the coverage probabilities of the CIs of the parameter estimates are close to the nominal level of 95%.

4. QMOEUG REGRESSION MODEL

In this Section, we introduce an alternative quantile regression model based on the reparameterization of the MOEUG distribution. Let

$$\alpha = \frac{\exp(\lambda)(u-1)}{u\left(\exp(\lambda) - \exp\left(\lambda\mu^{-\beta}\right)\right)}. \tag{35}$$

Inserting Eq. (35) in the MOEUG density, we have the probability density function of the quantile MOEUG (QMOEUG) distribution:

$$\begin{aligned} f(y) &= \frac{\exp(\lambda)(u-1)\beta\lambda y^{-(\beta+1)}}{u\left(\exp(\lambda) - \exp\left(\lambda\mu^{-\beta}\right)\right)} \exp\left(-\lambda\left(y^{-\beta}-1\right)\right) \\ &\times \left\{ \begin{array}{l} 1-\left(1-\frac{\exp(\lambda)(u-1)}{u\left(\exp(\lambda)-\exp\left(\lambda\mu^{-\beta}\right)\right)}\right) \\ \times\left[1-\exp\left\{-\lambda\left(y^{-\beta}-1\right)\right\}\right] \end{array} \right\}^{-2}, \end{aligned} \tag{36}$$

where $\mu \in (0,1)$ is the quantile parameter, u is a predefined value, $\beta, \lambda > 0$ are the shape parameters. Hereafter, the density in Eq. (36) is denoted as $Y \sim \text{QMOEUG}(\mu, \beta, \lambda; u)$.

Using the density function in Eq. (36), we define the QMOEUG regression model.

Let $Y_1, Y_2, \ldots, Y_n$ be random variables from the QMOEUG distribution with unknown μ_i, β, and λ parameters, and known u. We use the link function to link the covariates to the quantile parameter of the QMOEUG distribution. The appropriate link function should be chosen based on the domain of the random variable Y. The QMOEUG distribution is defined on $(0,1)$, so the logit-link function is chosen. The logit-link function is defined by

$$g(\mu_i) = \log\left(\frac{\mu_i}{1-\mu_i}\right) = \mathbf{x}_{\mathrm{i}}\gamma^{\mathbf{T}}, \tag{37}$$

TABLE 4
Monte Carlo Simulation Results for Bias, RMSE and CP of the Parameters of the MOEUGD.

parameters	n	Bias (α)	Bias (β)	Bias (λ)	RMSE (α)	RMSE (β)	RMSE (λ)	CP (α)	CP (β)	CP (λ)
$\alpha=0.3$	20	0.4284	0.1300	0.2482	0.9466	0.3652	0.9636	0.865	0.982	0.748
$\beta=0.8$	50	0.3201	0.0716	0.2418	0.7639	0.3312	0.8097	0.849	0.943	0.760
$\lambda=0.1$	100	0.3103	0.0582	0.2025	0.7481	0.3079	0.6589	0.821	0.855	0.749
	200	0.2306	0.0485	0.1307	0.5692	0.2687	0.4465	0.847	0.832	0.753
$\alpha=0.1$	20	0.2642	0.1500	0.2452	0.5305	0.3105	0.9944	0.888	0.999	0.668
$\beta=0.5$	50	0.2035	0.1054	0.2453	0.4320	0.2791	0.8317	0.857	0.960	0.676
$\lambda=0.2$	100	0.1817	0.0846	0.1980	0.3984	0.2476	0.6933	0.848	0.891	0.704
	200	0.1473	0.0687	0.1083	0.3556	0.2171	0.4173	0.856	0.849	0.732
$\alpha=0.1$	20	0.1840	0.3523	0.1832	0.4026	0.9344	0.6229	0.865	0.910	0.723
$\beta=2.0$	50	0.1423	0.2204	0.1730	0.3275	0.8543	0.5151	0.843	0.975	0.717
$\lambda=0.1$	100	0.1286	0.2011	0.1571	0.2967	0.8168	0.4699	0.821	0.907	0.729
	200	0.0749	0.0853	0.1203	0.2139	0.6909	0.3251	0.800	0.877	0.772

where $\gamma = \left(\gamma_0, \gamma_1, \ldots, \gamma_p\right)^{\mathrm{T}}$ represents the unknown regression parameters, $x_i = \left(1, x_{i1}, x_{i2}, \ldots, x_{ip}\right)^{\mathrm{T}}$ represents realizations of the covariates for the i-th individual.

The MLE method is preferred to estimate the unknown parameters of the QMOEUG regression model. Isolating μ_i from Eq. (37), we have

$$\mu_i = \frac{\exp\left(x_i \gamma^T\right)}{1 + \exp\left(x_i \gamma^T\right)}, \tag{38}$$

where $i = 1, 2, \ldots, n$. Let $\phi = \left(\gamma^{\mathrm{T}}, \beta, \lambda\right)$ be the unknown parameter vector, and u is a pre-defined value. The log-likelihood function of the QMOEUG regression model can be written as follows

$$\begin{aligned} \ell(\phi) &= n\lambda + n\log(u-1) + n\log(\beta\lambda) - (\beta+1)\sum_{i=1}^{n}\log(y_i) \\ &- \lambda\sum_{i=1}^{n}\left(y_i^{-\beta} - 1\right) - \sum_{i=1}^{n}\log\left\{u\left(\exp(\lambda) - \exp\left(\lambda\mu_i^{-\beta}\right)\right)\right\} \\ &- 2\sum_{i=1}^{n}\log\left\{\begin{array}{l} 1 - \left(1 - \frac{\exp(\lambda)(u-1)}{u\left(\exp(\lambda) - \exp\left(\lambda\mu_i^{-\beta}\right)\right)}\right) \\ \times\left[1 - \exp\left\{-\lambda\left(y_i^{-\beta} - 1\right)\right\}\right] \end{array}\right\}, \end{aligned} \tag{39}$$

where μ_i is defined in Eq. (37). The MLE of the unknown parameter vector ϕ is obtained by direct maximization of the log-likelihood function, given in Eq. (39). Some statistical software can be used for this purpose. The `optim` function of the `R` software is used in this study. This function requires good initial parameter vector to converge to the global maximum value of the log-likelihood function. The initial parameter vector is obtained following the idea of Mazucheli *et al.* (2020).

4.1. Validation

Assessment of the model fitting is a critical part of any statistical model. The accuracy of the model is discussed with the randomized quantile residuals of Dunn and Smyth (1996), which is defined by

$$r_{q,i} = \Phi\left\{F\left(y_i; \hat{\mu}_i, \hat{\beta}, \hat{\lambda}\right)\right\}^{-1}, \tag{40}$$

where Φ is the cumulative distribution function of the standard normal distribution, $F\left(y_i; \hat{\mu}_i, \hat{\beta}, \hat{\lambda}\right)$ is the cumulative distribution function of the QMOEUG distribution. According to Dunn and Smyth (1996), randomized quantile residuals are distributed as $N(0, 1)$ once the fitted model is valid.

5. Application of the MOEUGD to lifetime data sets

Here, the applicability of the MOEUG distribution is illustrated using two real data sets defined on a unit interval. Some well-known bounded distributions are also used to fit the real data sets along with the proposed MOEUG distribution. These distributions with their corresponding density function include:

(i) unit-Gompertz Distribution (UGD)

$$f(x) = \lambda\beta x^{-(\beta+1)} \exp\{-\lambda(x^{-\beta} - 1)\}, \tag{41}$$

(ii) Kumaraswamy Distribution (KD)

$$f(x) = \alpha\beta x^{\alpha-1}(1 - x^{\alpha})^{\beta-1}, \tag{42}$$

(iii) Beta Distribution

$$f(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}, \qquad B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}. \tag{43}$$

TABLE 5
Flood Level and Rock Sample Data Sets.

Data set I	0.26, 0.27, 0.30, 0.32, 0.32, 0.34, 0.38, 0.38, 0.39, 0.40, 0.41, 0.42, 0.42, 0.42, 0.45,0.48, 0.49, 0.61, 0.65, 0.74
Data Set II	0.0903296, 0.2036540, 0.2043140, 0.2808870, 0.1976530, 0.3286410, 0.1486220,0.1623940, 0.2627270, 0.1794550, 0.3266350, 0.2300810, 0.1833120, 0.1509440, 0.2000710, 0.1918020, 0.1541920, 0.4641250, 0.1170630, 0.1481410, 0.1448100, 0.1330830, 0.2760160, 0.4204770, 0.1224170, 0.2285950, 0.1138520, 0.2252140, 0.1769690, 0.2007440, 0.1670450, 0.2316230, 0.2910290, 0.3412730, 0.4387120, 0.2626510, 0.1896510, 0.1725670, 0.2400770, 0.3116460, 0.1635860, 0.1824530, 0.1641270,0.1534810, 0.1618650, 0.2760160, 0.2538320, 0.2004470.

The data sets are displayed in Table 5, where:

Data set I The first data set represents 20 observations of the maximum flood level (in millions of cubic feet per second) for the Susquehanna River at Harrisburg, Pennsylvania. The data set was first reported in Dumonceaux and Antle (1973), and recently was used in Mazucheli *et al.* (2019) to illustrate the potentiality of the unit-Gompertz distribution. The data set is right skewed with skewness value $S_k = 0.9939$ and leptokurtic with kurtosis value $K_s = 3.3053$.

Data set II The second data set consists of 48 rock samples from a petroleum reservoir reported in Cordeiro and Brito (2012). The data set is also right skewed with skewness value $S_k = 1.1330$ and leptokurtic with kurtosis value K_s=3.9404.

The parameter estimates of the distributions, the Akaike Information Criterion (AIC), the Bayesian Information Criterion (BIC), the Kolmogorov-Smirnov test statistic (K-S), the Anderson Darling test statistic (A^*), and the Cramer-von Mises test statistic (W^*) with their respective p-values are employed to compare the fitness obtained from the distributions for the two data sets.

TABLE 6
Summary Statistics for the Flood Level Data Set, n=20.

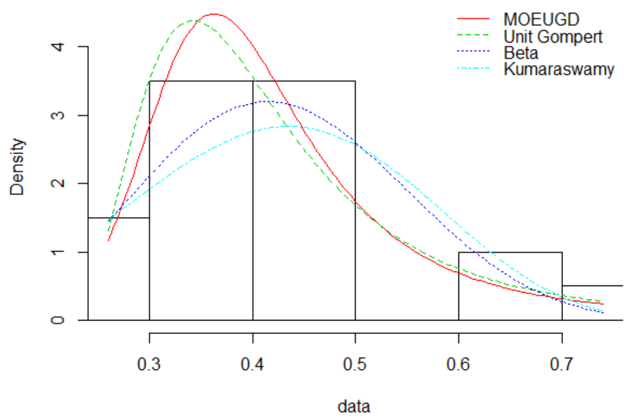
Distributions	Parameter Estimates	AIC	BIC	K-S (p-value)	A^* (p-value)	W^* (p-value)
MOEUGD	$\alpha = 0.029$ $\beta = 1.162$ $\lambda = 1.854$	-26.664	-23.676	0.131 (0.885)	0.247 (0.972)	0.042 (0.927)
UGD	$\alpha = 0.015$ $\beta = 4.115$	-28.740	-26.741	0.152 (0.743)	0.293 (0.943)	0.053 (0.862)
Beta	α=6.832 β=9.238	-24.367	-22.376	0.206 (0.363)	0.730 (0.532)	0.124 (0.482)
Kumaraswamy	α=3.378 β=12.006	-21.947	-19.955	0.218 (0.300)	0.937 (0.391)	0.165 (0.348)

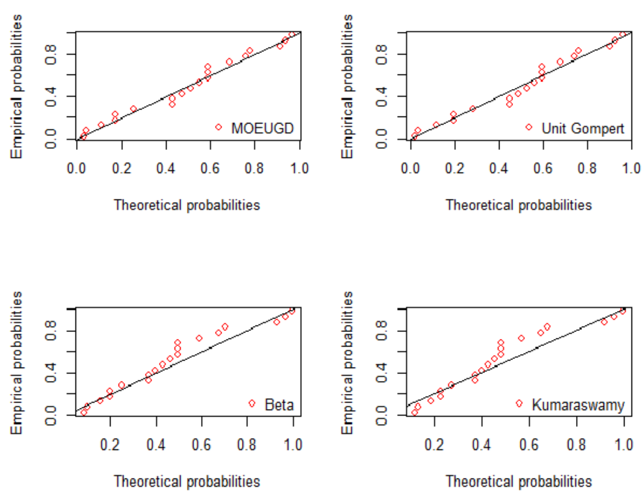
Tables 6 and 7 respectively show the parameter estimates, the Akaike Information Criterion (AIC), the Bayesian Information Criterion (BIC), the Kolmogorov-Smirnov test statistic (K-S), the Anderson Darling test statistic (A^*), and the Cramer-von Mises test statistic (W^*) with their respective p-values of the distributions for the flood level and rock sample data sets. The Tables reveal the superiority of the proposed MOEUG distribution over the unit-Gompertz, beta and Kumaraswamy distributions in analyzing the two dats sets, since the proposed MOEUG distribution has the smallest values in terms of the goodness of fit test statistics.

The graphical illustration of goodness of fit in terms of the density fit and the probability-probability (P-P) plots of the distributions for the two data sets are displayed in Figures 3 and 4 to further support the superiority of the proposed MOEUG distribution over the existing distributions.

TABLE 7
Summary Statistics for the Rock Samples from a Petroleum Reservoir Data Set, n=48.

Distributions	Parameter Estimates	AIC	BIC	K-S (p-value)	A^* (p-value)	W^* (p-value)
MOEUGD	$\alpha = 0.081$ $\beta = 1.513$ $\lambda = 0.240$	-107.609	-101.995	0.056 (0.998)	0.268 (0.960)	0.025 (0.990)
UGD	$\alpha = 0.005$ $\beta = 2.989$	-109.287	-105.545	0.081 (0.913)	0.357 (0.889)	0.043 (0.918)
Beta	α=6.832 β=9.238	-24.367	-22.376	0.718 (2.2e-16)	88.403 (1.2 e-05)	10.690 (2.2 e-16)
Kumaraswamy	α=2.719 β=44.670	-100.983	-97.241	0.153 (0.209)	1.289 (0.236)	0.201 (0.257)





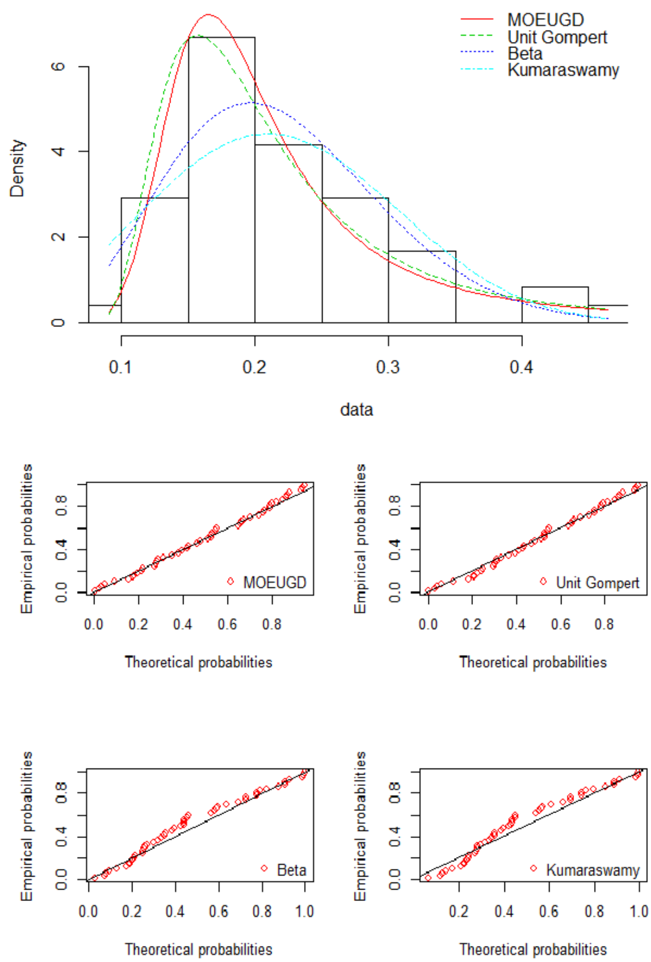


Figure 4 – Density fit and P-P plots of the distributions for the Rock Sample Data.

6. Application of the QMOEUGD regression model

In this Section, an application of the QMOEUG regression model is presented. We use similar data to the work of Korkmaz *et al.* (2022). They use the unit-Chen (UC) quantile regression model to analyze the recovery rates for viable $CD34+$ cells with the covariates such as gender (1: male, 0: female), history of chemotherapy (0: 1-day chemotherapy, 1: 3-day chemotherapy), and ages of the individuals. Detailed information for the data can be found in the `simplexreg` package of the `R` software. Here, the recovery rate of the $CD34+$ cells is considered as response variable. QMOEUGD, unit-Chen and Kumaraswamy regression models are used to model the data set. The fitted regression model is

$$\text{logit}(\mu_i) = \gamma_0 + \gamma_1 \text{GENDER}_i + \gamma_2 \text{CHEMO}_i + \gamma_3 \text{AGE}_i, \tag{44}$$

where $i = 1, 2, 3, \ldots, 239$. The pre-defined value u is selected as $u = 0.5$ to model the conditional median. The estimated parameters of the QMOEUGD regression model are given in Table 8. The estimated parameters of the UC and Kumaraswamy regression models are omitted, except $-\ell$ and AIC values, since these are reported in Korkmaz *et al.* (2022). From Tables 8 and 9, we conclude that the QMOEUGD regression model produce better result than the UC and Kumaraswamy regression models, because it has lowest values of $-\ell$ and AIC values.

TABLE 8
Estimated parameters of the QMOEUGD regression model.

Parameters	Estimates	Standard errors	p values
Intercept	0.957	0.142	< 0.001
Gender	0.068	0.104	0.256
Chemo	0.246	0.118	0.019
Age	0.019	0.006	< 0.001
β	650.893	20.639	-
λ	0.015	0.019	-

TABLE 9
Model selection criteria for QMOEUGD, UC and Kumaraswamy regression models.

Models	$-\ell$	AIC
QMOEUGD	-197.491	-382.981
UC	-195.826	-381.651
Kumaraswamy	-192.830	-375.659

According to the estimated parameters of the QMOEUGD regression model, there is no significant difference between female and male individuals for the recovery rate of the $CD34+$ cells. However, when the age increases, the recovery rate of the $CD34+$ cells increases. Additionally, individuals receiving a 3-day chemotherapy have a higher recovery rate than the individuals receiving a 1-day chemotherapy.

6.0.1. Residual analysis

The P-P plots of the randomized quantile residuals for the fitted regression models are displayed in Figure 5. From these Figures, it is obvious that the points of the QMOEUGD residuals are nearer to the diagonal line than those of the UC and Kumaraswamy regression models. So, we conclude that the QMOEUGD regression model provides a good fit for the data.

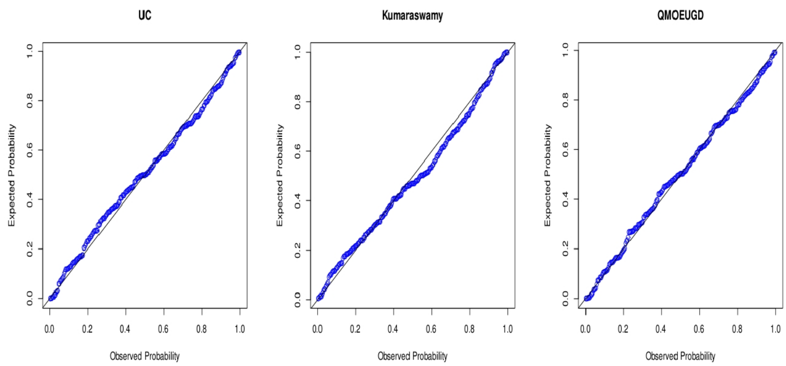


Figure 5 – PP plots of randomized quantile residuals for the fitted regression models.

7. Conclusion

This paper extends the unit-Gompertz distribution using the Marshall-Olkin method of generalization. Explicit expressions for the mathematical properties of the MOEUG distribution have been derived. The maximum likelihood estimation method has been employed to estimate the unknown parameters of the MOEUG distribution. We have provided a quantile regression model based on the MOEUG distribution and have compared it with the unit-Chen and unit-Kumaraswamy regression models. The results obtained from the two data sets under study reveal that the MOEUG distribution as well as the QMOEUG regression model provide a better fit than some existing models. We hope that in the future, the QMOEUG regression model attracts researchers when analyzing lifetime data sets.

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