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THE EXPONENTIATED GUMBEL-LOMAX DISTRIBUTION: PROPERTIES AND APPLICATIONS

Uchenna Ugwunnaya Uwadi [1]
Department of Mathematics/Computer Science/Statistics/Informatics, Alex Ekwueme Federal University Ndufu-alike, Nigeria

Elebe Emmanuel Nwezza
Department of Mathematics/Computer Science/Statistics/Informatics, Alex Ekwueme Federal University Ndufu-alike, Nigeria

Chukwuemeka Onwuzuruike Omekara
Department of Statistics, Micheal Okpara University of Agriculture, Umudike, Nigeria

SUMMARY

A new five-parameter distribution called exponentiated Gumbel Lomax (EGuL), a special model from the exponentiated Gumbel Family of distributions is proposed and studied in this paper. The proposed distribution has reverse J-shaped, inverted bathtub-shaped and J-shaped hazard rate function making it suitable for modeling survival and lifetime data. The density of the new distribution is expressed as a linear combination of the exponentiated density of the Lomax distribution. We derive the explicit expression for the quantile function, moments, incomplete moment, moment of residual life, entropy and order statistics of EGuL distribution. The estimation of the parameters of the new model is done using the method of maximum likelihood. Furthermore, a simulation study is used to ascertain the performance of the maximum likelihood estimates. Two applications are used to illustrate that the new distribution provides a better fit compared to other distributions with the same baseline.

Keywords: Exponentiated Gumbel distributionl; Lomax distribution; Moments; Maximum likelihood estimation.

1. INTRODUCTION

Probability distributions are very useful in describing and predicting real-world phenomenon. Many classical distributions have served this purpose. Recently, distributions have been modified or extended to enhance their flexibility. The modification of

[1] Corresponding Author. E-mail: uchenna.uwadi@funai.edu.ng

probability distribution can be done through the addition of parameters to the original distribution or transformation of the original distribution see (Lee *et al.*, 2013). Distributions have been proposed using numerous methods provided in the literature. There are scenarios where datasets do not follow known distributions; hence there are always rooms for developing distributions that are either more flexible or for fitting specific real-world scenario (Pathak and Chaturvedi, 2013). The Lomax distribution also known as Pareto Type II was defined byLomax (1954). It is a heavy-tailed distribution that has found application in lifetime data from many fields like Economics, Computer Science, engineering, medical and biological sciences, etc. Lomax distribution according to Alghamdi (2018) is considered as an alternative to exponential, gamma and Weibull distribution when the data are heavy-tailed. The cumulative density function (cdf) and probability density function (pdf) of the Lomax distribution for $x > 0$, $a > 0$, and $b > 0$ are respectively given as

$$F(x) = 1 - \left(1 + \frac{x}{b}\right)^{-a} \tag{1}$$

and

$$f(x) = \frac{a}{b}\left(1 + \frac{x}{b}\right)^{-(a+1)}, \tag{2}$$

where a is a shape parameter and b is the scale parameter.

Lomax distribution does not give good parametric fit for lifetime data with non-monotonic hazard rate function which is very common in survival and reliability studies. Many authors have generalized and extended Lomax distribution to provide a better fit for data with various non-monotonic failure rates. Some generalized and extended Lomax distribution in the literature include: Marshal-Olkin extended Lomax by Ghitany *et al.* (2007), beta Lomax, Kumaraswamy Lomax and McDonald Lomax by Lemonte and Cordeiro (2013), Transmuted Lomax by Ashour and Eltehiwy (2013), Exponentiated Generalized Lomax by Pathak and Chaturvedi (2013), Exponentiated Lomax by Salem (2014), Weibull Lomax by Tahir *et al.* (2015), Gumbel Lomax by Tahir *et al.* (2016), Power Lomax by Rady *et al.* (2016), Exponentiated Weibull Lomax by Hassan and Abd-Allah (2018), Alpha Power Transformed Lomax by Dey *et al.* (2019) and the Marshall-Olkin Gumbel Lomax by Nwezza and Ugwuowo (2020). A univariate family of distribution generated by exponentiated Gumbel distribution was recently introduced by Uwadi *et al.* (2019). The cdf of exponentiated Gumbel generated (EGu-G) family of distribution is given by

$$G(x) = 1 - \left[1 - \exp\left\{-B\left(\frac{F(x)}{1-F(x)}\right)^{-\frac{1}{\sigma}}\right\}\right]^{\alpha}, \quad \alpha, \sigma > 0, \quad -\infty < \mu < \infty. \tag{3}$$

The corresponding pdf to Eq. (3) is

$$g(x) = \frac{\alpha B f(x)(F(x))^{-\frac{1}{\sigma}-1}}{\sigma(1-F(x))^{-(\frac{1}{\sigma}-1)}} \exp\left\{-B\left(\frac{F(x)}{1-F(x)}\right)^{-\frac{1}{\sigma}}\right\} \times \left[1-\exp\left\{-B\left(\frac{F(x)}{1-F(x)}\right)^{-\frac{1}{\sigma}}\right\}\right]^{\alpha-1}, \quad (4)$$

where $B = \exp(\frac{\mu}{\sigma})$, α, σ are two extra shape parameters and μ is the location parameter. If $\alpha = 1$, Eq. (3) and Eq. (4) reduces to the cdf and pdf of Gumbel-X distribution defined by Al-Aqtash *et al.* (2015).

The aim this paper is to derive a new extension of the Lomax distribution using EGu-G family of distributions proposed by Uwadi *et al.* (2019), called exponentiated Gumbel Lomax (EGuL) distribution. We are motivated to introduce and study EGuL because it has various hazard rate shapes which include J-shaped, reverse J-shaped, and inverted bathtub shape thus EGuL distribution is practically applicable where Lomax distribution is not. Furthermore, empirical evidence from modelling survival times of infected guinea pigs data and bladder cancer data show that the EGuL distribution provides a better fit when compared with other competing life time models with the same baseline distribution. The rest of the paper is organized as follows. In Section 2, we define the Exponentiated Gumbel Lomax distribution. Mathematical properties including the shape of the density function, moments, re-representation of cdf and pdf, quantile function, moments, moments of residual life, entropy and order statistics of EGuL are presented in Section 3. Maximum likelihood estimates (MLE) of the parameters of the proposed distribution is discussed in Section 4. In Section 5, we investigate the performance of the MLEs through simulation studies. Analysis of two real data sets to illustrate the flexibility of the new distribution is in Section 6. The paper is concluded in Section 7.

2. The Exponentiated Gumbel Lomax Distribution

The cdf of EGuL ($G_L(x)$) distribution is obtained by substituting Eq. (1) and Eq. (2), in Eq. (3). This is given as

$$G_L(x) = 1 - \left[1-\exp\left\{-B\left[\left(1+\frac{x}{b}\right)^{a}-1\right]^{-\frac{1}{\sigma}}\right\}\right]^{\alpha}. \quad (5)$$

The corresponding pdf to Eq. (5) is

$$g_L(x) = \frac{\alpha B a}{\sigma b}\left(1+\frac{x}{b}\right)^{-(\frac{a}{\sigma}+1)}\left[1-\left(1+\frac{x}{b}\right)^{-a}\right]^{-(\frac{1}{\sigma}+1)} \times \exp\left\{-B\left[\left(1+\frac{x}{b}\right)^{a}-1\right]^{-\frac{1}{\sigma}}\right\} \times$$

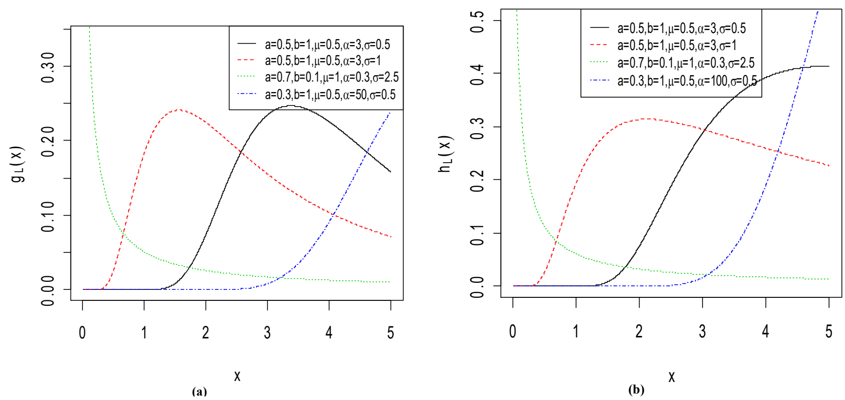

Figure 1 – Plots of pdf and hazard rate function: (a) EGuL pdf plots for selected parameter values, (b) EGuL hazard rate function plots for selected parameter value.

$$\left[1-\exp\left\{-B\left[\left(1+\frac{x}{b}\right)^{a}-1\right]^{-\frac{1}{\sigma}}\right\}\right]^{\alpha-1}. \tag{6}$$

The survival function, $S_L(x)$, and the hazard rate function, $h_L(x)$, corresponding to Eq. (5), are given by Eq. (7) and Eq. (8), respectively.

$$S_L(x)=\left[1-\exp\left\{-B\left[\left(1+\frac{x}{b}\right)^{a}-1\right]^{-\frac{1}{\sigma}}\right\}\right]^{\alpha} \tag{7}$$

and

$$h_L(x)=\frac{\alpha B a\left(1+\frac{x}{b}\right)^{-a\left(\frac{1}{\sigma}+1\right)}\exp\left\{-B\left[\left(1+\frac{x}{b}\right)^{a}-1\right]^{-\frac{1}{\sigma}}\right\}}{\sigma b\left[1-\left(1+\frac{x}{b}\right)^{-a}\right]^{\left(\frac{1}{\sigma}+1\right)}\left[1-\exp\left\{-B\left[\left(1+\frac{x}{b}\right)^{a}-1\right]^{-\frac{1}{\sigma}}\right\}\right]}. \tag{8}$$

The left side (a) of Figure 1 is the plot of EGuL distribution pdf for selected parameter values. It reveals that EGuL density can be unimodal, reverse J-shaped, and increasing, while the right side (b) of Figure 1 shows the $h_L(x)$ may be J-shaped, unimodal and decreasing.

The reverse–hazard rate function, $R_L(x)$, of EGuL distribution is given as

$$R_L(x) = \frac{\alpha B a\left(1+\frac{x}{b}\right)^{-a\left(\frac{1}{\sigma}+1\right)}\left[1-\left(1+\frac{x}{b}\right)^{-a}\right]^{-\left(\frac{1}{\sigma}+1\right)}}{\sigma b\left(1-\left[1-\exp\left\{-B\left[\left(1+\frac{x}{b}\right)^{a}-1\right]^{-\frac{1}{\sigma}}\right\}\right]^{\alpha}\right)}\times$$

$$\exp\left\{-B\left[\left(1+\frac{x}{b}\right)^{a}-1\right]^{-\frac{1}{\sigma}}\right\}\times$$
$$\left[1-\exp\left\{-B\left[\left(1+\frac{x}{b}\right)^{a}-1\right]^{-\frac{1}{\sigma}}\right\}\right]^{\alpha-1}, \quad (9)$$

while cumulative hazard rate function, $H_L(x)$, is given by

$$H_L(x)=\alpha\ln\left[1-\exp\left\{-B\left[\left(1+\frac{x}{b}\right)^{a}-1\right]^{-\frac{1}{\sigma}}\right\}\right]^{\alpha}. \quad (10)$$

3. Mathematical Properties

The mathematical properties of EGuL distribution are given in this Section.

3.1. *Shape of $g_L(x)$ and $h_L(x)$*

The shape of the $g_L(x)$ can be described analytically by taking the log of Eq. (6) differentiating with respect to x and equating to zero. Hence the critical points of EGuL are the roots of Eq. (11)

$$\begin{aligned}\frac{d\log(g_L(x))}{dx} &= \left(1+\frac{x}{b}\right)^{-1}\left[(a+1)+\frac{a\left(\frac{1}{\sigma}+1\right)}{\left(1+\frac{x}{b}\right)^{a}\left[1-\left(1+\frac{x}{b}\right)^{a}\right]}-a\left(\frac{1}{\sigma}-1\right)\right]\\ &-\frac{aB\left[1-\left(1+\frac{x}{b}\right)^{a}\right]^{-\frac{1}{\sigma}-1}}{\sigma\left(1+\frac{x}{b}\right)^{-\left(\frac{a}{\sigma}+1\right)}}\times\\ &\left[1-\frac{(\alpha-1)\exp\left\{-B\left[\left(1+\frac{x}{b}\right)^{a}-1\right]^{-\frac{1}{\sigma}}\right\}}{\left[1-\exp\left\{-B\left[\left(1+\frac{x}{b}\right)^{a}-1\right]^{-\frac{1}{\sigma}}\right\}\right]}\right]=0. \quad (11)\end{aligned}$$

There may be more than one root to Eq. (11). If $x=x_0$ is a root of Eq. (11), then it corresponds to a local maximum, local minimum or a point of inflexion depending on whether $\psi(x_0)<0$, $\psi(x_0)>0$ or $\psi(x_0)=0$, respectively; where $\psi(x_0)=\frac{d^2\log(g_L(x))}{dx^2}$. The critical points of $h_L(x)$ are obtained through equating the derivative of Eq. (8) to zero as given in Eq. (12)

$$\frac{d\log(g_L(x))}{dx} = \frac{1}{\left(1+\frac{x}{b}\right)}\left[(a+1)+\frac{a}{\left(1+\frac{x}{b}\right)^{a}}\left[\frac{\left(\frac{1}{\sigma}+1\right)}{\left[1-\left(1+\frac{x}{b}\right)^{-a}\right]}+\frac{\left(\frac{1}{\sigma}-1\right)}{\left(1+\frac{x}{b}\right)^{-a}}\right]\right]$$

$$+\frac{Ba\left[1-\left(1+\frac{x}{b}\right)^{-a}\right]^{-\left(\frac{1}{\sigma}+1\right)}}{\sigma\left(1+\frac{x}{b}\right)^{-\left[a\left(2+\frac{1}{\sigma}\right)+1\right]}}\times$$

$$\left[1+\frac{\exp\left\{-B\left[\left(1+\frac{x}{b}\right)^{a}-1\right]^{-\frac{1}{\sigma}}\right\}}{\left[1-\exp\left\{-B\left[\left(1+\frac{x}{b}\right)^{a}-1\right]^{-\frac{1}{\sigma}}\right\}\right]}\right]=0. \tag{12}$$

There may be more than one root to Eq. (12). If $x = x_0$ is a root of Eq, (12), then it corresponds to a local maximum, local minimum or a point of inflexion depending on whether $\omega(x_0) < 0$, $\omega(x_0) > 0$ or$\omega(x_0) = 0$, respectively; where $\omega(x_0) = \frac{d^2 \log[h_L(x)]}{dx^2}$.

3.2. *Useful expansion*

Here we derive a useful mixture representation of the pdf and cdf of EGuL. Applying the binomial series expansion of $(1-w)^{\beta-1} = \sum_{i=0}^{\infty}(-1)^i \binom{\beta-1}{i} w^i$; where $|w| < 1$ and power series expansion for the exponential function to Eq. (6), we can re-write Eq. (6) as

$$g_L(x) = \sum_{i,j,k=0}^{\infty} W_{i,j,k}\, h_{k-\frac{1}{\sigma}(j+1)}(x), \tag{13}$$

where

$$W_{i,j,k} = \frac{\alpha}{\sigma}\sum_{i,j,k=0}^{\infty}\frac{(-1)^{i+j+k}}{j!\left[k-\frac{1}{\sigma}(j+1)\right]}B^{j+1}(i+1)^j\binom{\alpha-1}{i}\binom{\frac{j}{\sigma}+\frac{1}{\sigma}-1}{k}$$

and $h_{p^*}(x) = p^* g(x)G(x)^{p^*-1}$. By Eq.(13), the pdf EGuL is expressed as a mixture of exponentiated Lomax distribution with parameters a, b, and $k-\frac{1}{\sigma}(j+1)$.

Furthermore, by applying the binomial series expansion twice on $(G_L(x))^{t'}$, where t' is a positive integer yields

$$\left(G_L(x)\right)^{t'} = \sum_{t'}^{p=0}\sum_{q,m,l=0}^{\infty}\vartheta_{p,q,l,m} H_{l-\frac{m}{\sigma}}(x), \tag{14}$$

where $\vartheta_{p,q,l,m} = \frac{(-1)^{p+q+l+m}}{m!}\binom{t'}{p}\binom{\alpha p}{q}\binom{\frac{m}{\sigma}}{l}(Bq)^m$ and $H_{l-\frac{m}{\sigma}}(x)$ is the cdf of exponentiated Lomax distribution with parameters a, b, and $l-\frac{m}{\sigma}$.

Eq. (13) and Eq. (14) are the important results of this section which will assist in the studying of the properties of EGuL.

3.3. Quantile function

The quantile function $x_u = Q(u) = F^{-1}(u)$ for $u \in (0,1)$ of EGuL distribution is obtained by inverting Eq. (5) and is given by

$$x_u = b\left[\left(\left\{-B^{-1}\log\left[1-(1-u)^{\frac{1}{\alpha}}\right]\right\}^{\sigma}+1\right)^{\frac{1}{a}}-1\right] = Q(u). \tag{15}$$

Eq. (15) can be used to simulate samples of random numbers from EGuL distribution. In particular, the median of EGuL can be obtained by substituting for $u = 0.5$ in Eq. (15)

$$Q(0.5) = b\left[\left(\left\{-B^{-1}\log\left[1-(0.5)^{\frac{1}{\alpha}}\right]\right\}^{\sigma}+1\right)^{\frac{1}{a}}-1\right]. \tag{16}$$

The quantile based measures of skewness S and kurtosis K can be used to examine the effect of the shape parameters on the skewness and kurtosis of a distribution. Skewness and kurtosis based on quartiles and octiles are calculated respectively using the relationships of Galton (1983) and Moor (1988). These measures of skewness and kurtosis exist for distributions without moment and are less sensitive to outliers (Alizadeh *et al.*, 2015). Using the quantile function in Eq. (15), the Galton's skewness and Moors kurtosis of the proposed family is given by

$$S = \frac{Q\left(\frac{3}{4}\right)-2Q\left(\frac{1}{2}\right)+Q\left(\frac{1}{4}\right)}{Q\left(\frac{3}{4}\right)-Q\left(\frac{1}{4}\right)} \tag{17}$$

and

$$K = \frac{Q\left(\frac{7}{8}\right)-Q\left(\frac{5}{8}\right)+Q\left(\frac{3}{8}\right)-Q\left(\frac{1}{8}\right)}{Q\left(\frac{6}{8}\right)-Q\left(\frac{2}{8}\right)}. \tag{18}$$

Measures based on quartiles and octiles as given in Eq.(17) and Eq.(18) are used to investigate the effect of the shape parameters α, a and σ on the skewness and kurtosis of the EGuL distribution. The Skewness and Kurtosis of the five- parameter EGuL distribution is as shown in Table 1 for different combinations of the parameter values. In Table 1 the values of the scale and the location parameters are respectively fixed as follows, $b = 1$, $\mu = -2$ and $\mu = 1$ while the values of the shape parameters are varied. Table 1 reveals that for fixed α and σ, skewness is directly related a, while kurtosis is inversely related to a for case 1. In case 2 skewness and kurtosis increases for increasing values of σ when a and α are fixed. For case 3 the values both skewness and kurtosis decreases as the value α increases for fixed values of a and σ.

TABLE 1
The skewness (S) and Kurtosis (K) of EGuL distribution for $b = 1$.

Cases	α	σ	a	$\mu = -1$		$\mu = 1$	
				S	K	S	K
	2.0	0.5	0.3	0.83	8.39	0.99	138099.20
	2.0	0.5	0.8	2.11	-1.42	0.98	94.01
	2.0	0.5	2.5	2.25	-1.41	1.88	-1.82
	2.0	0.5	5.0	3.52	-0.88	2.38	-1.41
	2.0	0.5	10	8.78	-0.41	4.89	-0.72
Case 1	2.0	0.5	20	47.22	-0.12	17.82	-0.28
	0.5	0.3	2.0	0.34	1.05	0.28	0.83
	0.5	0.8	2.0	0.55	2.06	0.43	1.48
	0.5	2.5	2.0	0.82	6.01	0.77	5.69
	0.5	5.0	2.0	0.96	33.99	0.96	33.94
	0.5	10	2.0	0.99	1155.89	0.99	1155.89
Case 2	0.5	20	2.0	0.99	1336096	0.9999	1336096
	0.3	2.0	0.5	0.99	10491.76	0.99	10506.03
	0.8	2.0	0.5	0.91	28.17	0.96	39.27
	2.5	2.0	0.5	0.52	2.05	0.69	3.78
	5.0	2.0	0.5	0.38	1.19	0.49	1.80
	10	2.0	0.5	0.28	0.82	0.35	1.075
Case 3	20	2.0	0.5	0.21	0.59	0.25	0.72

3.4. Ordinary and incomplete moments

If a random variable X follows EGuL distribution, then the r^{th} moment of X from the origin (Non-central moment) is derived by using Eq. (13) as shown below

$$\mu_r^{'} = \sum_{i,j,k=0}^{\infty} W_{i,j,k} \int_0^{\infty} x^r h_{k-\frac{1}{\sigma}(j+1)}(x) \mathrm{d}x$$

$$\begin{aligned}\mu_r^{'} &= \sum_{i,j,k=0}^{\infty} W_{i,j,k} \frac{a}{b} \int_0^{\infty} x^r \left[k - \frac{1}{\sigma}(j+1)\right]\left(1+\frac{x}{b}\right)^{-(a+1)} \times \\ &\quad \left[1-\left(1+\frac{x}{b}\right)^{-a}\right]^{k-\frac{1}{\sigma}(j+1)-1} \mathrm{d}x.\end{aligned}$$

Let $y\left(1+\frac{x}{b}\right)^{-a} \Rightarrow x = b\left(y^{-\frac{1}{a}} - 1\right)$, and applying the general binomial expansion. The r^{th} moment of EGuL distribution is given by

$$\mu_r^{'} = \sum_{i,j,k=0}^{\infty} \sum_{n=0}^{r} Z_{i,j,k,n} b^r \left[k - \frac{1}{\sigma}(j+1)\right] B\left[1 - \frac{1}{a}(r-n), k - \frac{1}{\sigma}(j+1)\right], \tag{19}$$

where $Z_{i,j,k,n} = (-1)^n \binom{r}{n}$ and $B[.,.]$ is a beta function. Substituting for $r = 1,2,3,4$ in Eq. (19) generates the first four moments of EGuL about the origin. The moment generating function (mgf) of EGuL distribution is given by

$$\begin{aligned}M_X(t) &= \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r^{'} \\ M_X(t) &= \sum_{r,i,j,k=0}^{\infty} \sum_{n=0}^{r} Z_{i,j,k,n} \frac{t^r}{r!} b^r \left[k - \frac{1}{\sigma}(j+1)\right] \times \\ &\quad B\left[1 - \frac{1}{a}(r-n), k - \frac{1}{\sigma}(j+1)\right].\end{aligned} \tag{20}$$

The r^{th} incomplete moment $\varphi_r(z)$ of EGuL distribution is obtained using Eq. (13) as follows

$$\varphi_r(z) = \sum_{i,j,k=0}^{\infty} W_{i,j,k} \int_0^{z} x^r h_{k-\frac{1}{\sigma}(j+1)}(x) \mathrm{d}x$$

$$\varphi_r(z) = \sum_{i,j,k=0}^{\infty} W_{i,j,k}\frac{a}{b}\int_0^z x^r\left[k-\frac{1}{\sigma}(j+1)\right]\left(1+\frac{x}{b}\right)^{-(a+1)} \times \left[1-\left(1+\frac{x}{b}\right)^{-a}\right]^{k-\frac{1}{\sigma}(j+1)-1}.$$

Simplifying further, the incomplete moment of EGuL distribution is obtained as

$$\varphi_r(z) = \sum_{n=0}^{r} b_n\left\{B(\eta_1,\eta_2)-B_{(1+\frac{z}{b})^{-a}}(\eta_1,\eta_2)\right\}, \tag{21}$$

where $b_n=(-1)^n b^r \sum_{i,j,k=0}^{\infty} W_{i,j,k}\left[k-\frac{1}{\sigma}(j+1)\right]$, $\eta_1=1-\frac{1}{a}(r-n)$, $\eta_2=k-\frac{1}{\sigma}(j+1)$, and $B_z(.,.)$ is an incomplete beta function.

One major application of incomplete moment is in the derivation of mean deviation about mean and median. The mean deviation about mean $\omega_1=E(|X-\mu|)$ and about the median $\omega_2=E(|X-M|)$ can be derived using the first incomplete moment $\varphi_1(t)$ as $\omega_1 = 2\mu G_L(\mu)-2\varphi_1(\mu)$, and $\omega_2=\mu-2\varphi_1(M)$ respectively; where $\mu=E(X)$ and $M=$ median of EGuL distribution. Another application of incomplete moments can be found in the inequality curves such as Lorenz and Bonferroni curves defined as $L_G(t)=\frac{\varphi_1(t)}{\mu}$ and $B_G(t)=\frac{L_G(t)}{G_L(t)}$ respectively.

3.5. *Probability weighted moments (PWM)*

Probability weighted moments of a random variable X from EGuL distribution is obtained using Eq. (22)

$$\tau_{r,s}=E(X^r G(x)_L^s)=\int_0^{\infty} x^r g_L(x)G_L(x)^s \mathrm{d}x. \tag{22}$$

Substituting Eq. (13) and Eq. (14) in Eq. (22) yields

$$\tau_{r,s} = \sum_{i,j,k=0}^{\infty}\sum_{p=0}^{s}\sum_{q,m,l=0}^{\infty} W_{i,j,k}\varphi_{p,q,l,m}\frac{a}{b}\left[k-\frac{1}{\sigma}(j+1)\right]\times \int_0^{\infty} x^r\left(1+\frac{x}{b}\right)^{-(a+1)}\left[1-\left(1+\frac{x}{b}\right)^{-a}\right]^{(k+l)-\frac{1}{\sigma}(j+m+1)-1}$$

$$\tau_{r,s} = \sum_{i,j,k,q,m,l=0}^{\infty}\sum_{p=0}^{s}\sum_{\ell=0}^{\infty} W_{i,j,k}\varphi_{p,q,l,m}(-1)^{\ell}\binom{r}{\ell}\left[k-\frac{1}{\sigma}(j+1)\right]b^r\times B\left[1-\left(\frac{r-\ell}{a}\right),(k+l)-\frac{1}{\sigma}(j+m+1)-1\right].$$

3.6. *Moments of residual life*

The n^{th} moment of residual life of a random variable X is defined as $m_n(z) = E\left[(X-z)^n|X>z\right]$, $n = 1,2,\ldots$ and $z > 0$. Subsequently, the n^{th} moment of residual life of EGuL distribution is given by

$$m_n(z) = \frac{1}{1-G_L(z)} \int_z^\infty (x-z)^n g_L(x)\mathrm{d}x, \tag{23}$$

substituting Eq. (13) in Eq. (23) and simplifying, we obtain Eq. (23) as

$$m_n(z) = \frac{\sum_{i,j,k=0}^\infty \sum_{r=0}^n \sum_{l_1=0}^r W^*_{i,j,k,r,l_1} b^r B_{\left(1+\frac{z}{b}\right)^a}\left[\eta^*, \eta_2\right]}{\left[1-\exp\left\{-B\left[\left(1+\frac{z}{b}\right)-1\right]^{-\frac{1}{\sigma}}\right\}\right]^\alpha}, \tag{24}$$

where $W^*_{i,j,k,r,l_1} = (-1)^{n+r+l_1} W_{i,j,k} \binom{n}{r}\binom{r}{l_1} z^{n-r}\left[k-\frac{1}{\sigma}(j+1)\right]$, $\eta^* = 1-\frac{1}{a}(r-l_1)$ and $\eta_2 = k-\frac{1}{\sigma}(j+1)$.

For $n = 1$ in Eq. (24) gives the mean residual life (MRL) function which is another important function. The MRL represents the mean additional life length for a unit that is alive at age x.

3.7. *Entropy*

The entropy of a random variable is the measure of the variation of the uncertainty. A large entropy value indicates greater uncertainty in the data. Renyi (1961) defined the entropy of a continuous random variable X with density function $f(x)$ as $I_R(x) = \frac{1}{1-\gamma}\log\left(\int_{-\infty}^\infty f^\gamma(x)\mathrm{d}x\right)$; $\gamma > 0$ and $\gamma \neq 1$.

The Renyi entropy of a random variable X that has EGuL distribution is defined as

$$I_R(x) = \frac{1}{1-\gamma}\log\left(\int_{-\infty}^\infty g_L^\gamma(x)\mathrm{d}x\right);\ \gamma > 0,\ \gamma \neq 1. \tag{25}$$

Applying the generalized binomial series expansion, we obtained $g_L^\gamma(x)$ as

$$g_L^\gamma(x) = \sum_{i,j,k=0}^\infty V_{i,j,k}\left(1+\frac{x}{b}\right)^{-\gamma(a+1)}\left[1-\left(1+\frac{x}{b}\right)^{-a}\right]^{k-\gamma\left(\frac{1}{\sigma}+1\right)-\frac{j}{\sigma}},$$

where $V_{i,j,k} = (-1)^{i+j+k}\frac{B^j}{j!}(i+j)^j\left(\frac{\alpha Ba}{\sigma b}\right)^\gamma \binom{\gamma(\alpha-1)}{i}\binom{\frac{1}{\sigma}(\gamma+j)-\gamma}{k}$.

Substituting the expression of $g_L^{\gamma}(x)$ in Eq. (25), the Renyi entropy of EGuL is given by

$$I_R(x)=\frac{1}{1-\gamma}\log\left\{\sum_{i,j,k=0}^{\infty}\frac{b}{a}B\left[\gamma+\frac{1}{a}(\gamma-1),(k+1)-\gamma\left(\frac{1}{\sigma}+j\right)+\frac{j}{\sigma}\right]\right\}.$$

3.8. Order statistics

Order statistics have many applications in statistical theory and practice. Suppose $X_1,X_2,\ldots,X_n$ be a random sample from EGuL distribution. Let X_r denotes the r^{th} order statistics. The pdf of the r^{th} order statistics is given by

$$g_{r:n}(x)=\frac{\sum_{u=0}^{n-r}(-1)^u\binom{n-r}{u}}{B(r,n-r+1)}g_L(x)G_L^{u-r+1}(x), \tag{26}$$

where $B(.,.)$ is the beta function. Considering Eq. (14) and substituting Eq.(13) in Eq.(26); the r^{th} order statistics of EGuL distribution can be expressed as

$$g_{r:n}(x)=\sum_{u=0}^{n-r}\sum_{i,j,k,q,m,l=0}^{\infty}\sum_{p=0}^{u+r-1}C_{i,j,k,m,l,u,p}\,h_{(k+l)-\frac{1}{\sigma}(j+m+1)}(x), \tag{27}$$

where

$$C_{i,j,k,m,l,u,p} = \frac{(-1)^{u+p+q+m+l}\binom{n-r}{u}\binom{u+r-1}{p}\binom{\alpha p}{q}\binom{\frac{m}{\sigma}}{l}}{m!B(r,n-r+1)\left[(k+l)-\frac{1}{\sigma}(j+m+1)\right]}\times (Bq)^m\left[k-\frac{1}{\sigma}(j+1)\right]W_{i,j,k}.$$

Hence the pdf of r^{th} order statistics of EGuL can be expressed as a mixture of exponentiated Lomax density with parameters a,b and $(k+l)-\frac{1}{\sigma}(j+m+1)$.

The pdf of the smallest order statistic for EGuL distribution is obtained for $r=1$ in Eq. (27) and is given by

$$f_{1:n}(x)=\sum_{u=0}^{n-1}\sum_{i,j,k,q,m,l=0}^{\infty}\sum_{p=0}^{u}\tau_{i,j,k,m,l,u,p}\,h_{(k+l)-\frac{1}{\sigma}(j+m+1)}(x),$$

where

$$\tau_{i,j,k,m,l,u,p} = \frac{(-1)^{u+p+q+m+l}\binom{n-1}{u}\binom{u}{p}\binom{\alpha p}{q}\binom{\frac{m}{\sigma}}{l}}{m!\left[(k+l)-\frac{1}{\sigma}(j+m+1)\right]} \times (Bq)^m\left[k-\frac{1}{\sigma}(j+1)\right] n W_{i,j,k}.$$

Furthermore, the pdf of the largest order statistics associated with EGuL is obtained by substituting $r = n$ in Eq. (27) and it is given by

$$f_{n:n}(x) = \sum_{i,j,k,q,m,l=0}^{\infty} \sum_{p=0}^{n-1} \pi_{i,j,k,m,l,u,p}\, h_{(k+l)-\frac{1}{\sigma}(j+m+1)}(x),$$

where

$$\pi_{i,j,k,m,l,p} = \frac{(-1)^{p+q+m+l}\binom{n-1}{p}\binom{\alpha p}{q}\binom{\frac{m}{\sigma}}{l}}{m!\left[(k+l)-\frac{1}{\sigma}(j+m+1)\right]} \times (Bq)^m\left[k-\frac{1}{\sigma}(j+1)\right] n W_{i,j,k}.$$

4. Estimation

Here we obtain estimates of the parameters of EGuL distribution. Maximum likelihood method of estimation is considered here because of its desirable properties. Given a random sample $X_1, X_2, \ldots, X_n$ of size n from Eq. (6), the log-likelihood function of EGuL distribution is given by

$$\begin{aligned}
\ell(\Omega) = {} & n\log(\alpha) + n\log(B) + n\log(a) - n\log(\sigma) - n\log(b) \\
& -(a+1)\sum_{i=1}^{n}\log\left(1+\frac{x_i}{b}\right) - \left(\frac{1}{\sigma}+1\right)\sum_{i=1}^{n}\left[1-\left(1+\frac{x_i}{b}\right)^{-a}\right] \\
& +\left(\frac{1}{\sigma}-1\right)\sum_{i=1}^{n}\log\left\{1-\left[1-\left(1+\frac{x_i}{b}\right)^{-a}\right]\right\} \\
& -B\sum_{i=1}^{n}\left\{\frac{1-\left(1+\frac{x_i}{b}\right)^{-a}}{1-\left[1-\left(1+\frac{x_i}{b}\right)^{-a}\right]}\right\}^{-\frac{1}{\sigma}}
\end{aligned}$$

$$+(\alpha-1)\sum_{i=1}^{n}\log\left[1-\exp\left\{-B\left[\frac{1-\left(1+\frac{x_1}{b}\right)^{-a}}{1-\left[1-\left(1+\frac{x_i}{b}\right)^{-a}\right]}\right]^{-\frac{1}{\sigma}}\right\}\right]. \quad (28)$$

The vector of score functions associated to Eq. (28) is defined as

$$Z(\Omega)=\left(\frac{\partial\ell}{\partial\alpha},\frac{\partial\ell}{\partial B},\frac{\partial\ell}{\partial\sigma},\frac{\partial\ell}{\partial a},\frac{\partial\ell}{\partial b}\right)^{T},$$

where

$$\frac{\partial\ell}{\partial\alpha}=\frac{n}{\alpha}+\sum_{i=i}^{n}\log\left[1-\exp\left\{-B\left[\frac{1-\left(1+\frac{x_i}{b}\right)^{-a}}{1-\left[1-\left(1+\frac{x_i}{b}\right)^{-a}\right]}\right]^{-\frac{1}{\sigma}}\right\}\right],$$

$$\begin{aligned}\frac{\partial\ell}{\partial B} &= \frac{n}{B}\sum_{i=n}^{n}\left[\frac{1-\left(1+\frac{x_i}{b}\right)^{-a}}{1-\left[1-\left(1+\frac{x_i}{b}\right)^{-a}\right]}\right]^{-\frac{1}{\sigma}}\\ &+(\alpha-1)\sum_{i=1}^{n}\left(\frac{\exp\left\{-B\left[\frac{1-\left(1+\frac{x_i}{b}\right)^{-a}}{1-\left[1-\left(1+\frac{x_i}{b}\right)^{-a}\right]}\right]^{-\frac{1}{\sigma}}\right\}}{\left[1-\exp\left\{-B\left[\frac{1-\left(1+\frac{x_i}{b}\right)^{-a}}{1-\left[1-\left(1+\frac{x_i}{b}\right)^{-a}\right]}\right]^{-\frac{1}{\sigma}}\right\}\right]}\right),\end{aligned}$$

$$\begin{aligned}\frac{\partial\ell}{\partial\sigma} &= -\frac{n\mu}{\sigma}-\frac{n}{\sigma}+\frac{1}{\sigma^2}\sum_{i=1}^{n}\log\left(1-\left(1+\frac{x_i}{b}\right)^{-a}\right)\\ &-\frac{1}{\sigma^2}\sum_{i=1}^{n}\log\left[1-\log\left(1-\left(1+\frac{x_i}{b}\right)^{-a}\right)\right]\\ &-\frac{B}{\sigma^2}\sum_{i=1}^{n}\left\{\left(\frac{1-\left(1+\frac{x_i}{b}\right)^{-a}}{1-\left[1-\left(1+\frac{x_i}{b}\right)^{-a}\right]}\right)^{-\frac{1}{\sigma}}\left[\log\left(\frac{1-\left(1+\frac{x_i}{b}\right)^{-a}}{1-\left[1-\left(1+\frac{x_i}{b}\right)^{-a}\right]}\right)-\mu\right]\right\}\end{aligned}$$

$$+\frac{(\alpha-1)}{\sigma^2}\sum_{i=1}^{n}\left\{\begin{array}{c}\dfrac{\left[\exp\left\{-B\left[\frac{1-\left(1+\frac{x_i}{b}\right)^{-a}}{1-\left[1-\left(1+\frac{x_i}{b}\right)^{-a}\right]}\right]^{-\frac{1}{\sigma}}\right\}+\frac{\mu}{\sigma}\right]}{\ln\left[1-\exp\left\{-B\left[\frac{1-\left(1+\frac{x_i}{b}\right)^{-a}}{1-\left[1-\left(1+\frac{x_i}{b}\right)^{-a}\right]}\right]^{-\frac{1}{\sigma}}\right\}\right]}\times\\ \left\{\left(\frac{1-\left(1+\frac{x_i}{b}\right)^{-a}}{1-\left[1-\left(1+\frac{x_i}{b}\right)^{-a}\right]}\right)^{-\frac{1}{\sigma}}\left[\ln\left(\frac{1-\left(1+\frac{x_i}{b}\right)^{-a}}{1-\left[1-\left(1+\frac{x_i}{b}\right)^{-a}\right]}\right)-\mu\right]\right\}\end{array}\right\},$$

$$\begin{aligned}\frac{\partial\ell}{\partial a} &= \frac{n}{a}-\sum_{i=1}^{n}\ln\left(1+\frac{x_i}{b}\right)-\left(\frac{1}{\sigma}+1\right)\sum_{i=1}^{n}\left(\frac{\ln\left(1+\frac{x_i}{b}\right)}{\left(1+\frac{x_i}{b}\right)^{a}\left(1-\left(1+\frac{x_i}{b}\right)^{-a}\right)}\right)\\ &-\left(\frac{1}{\sigma}-1\right)\sum_{i=1}^{n}\left(\frac{\ln\left(1+\frac{x_i}{b}\right)}{\left(1+\frac{x_i}{b}\right)^{a}\left[1-\left(1-\left(1+\frac{x_i}{b}\right)^{-a}\right)\right]}\right)\\ &+\frac{B}{\sigma}\sum_{i=1}^{n}\left(\frac{1-\left(1+\frac{x_i}{b}\right)^{-a}}{1-\left(1-\left(1+\frac{x_i}{b}\right)^{-a}\right)}\right)^{-\frac{1}{\sigma}-1}\left(\frac{\ln\left(1+\frac{x_i}{b}\right)}{\left(1+\frac{x_i}{b}\right)^{-a}}\right)\\ &+\frac{(\alpha-1)}{\sigma^2}\sum_{i=1}^{n}\left\{\begin{array}{c}\dfrac{\left[\exp\left\{-B\left[\frac{1-\left(1+\frac{x_i}{b}\right)^{-a}}{1-\left[1-\left(1+\frac{x_i}{b}\right)^{-a}\right]}\right]^{-\frac{1}{\sigma}}\right\}+\frac{\mu}{\sigma}\right]}{\ln\left[1-\exp\left\{-B\left[\frac{1-\left(1+\frac{x_i}{b}\right)^{-a}}{1-\left[1-\left(1+\frac{x_i}{b}\right)^{-a}\right]}\right]^{-\frac{1}{\sigma}}\right\}\right]}\times\\ \left(\frac{1-\left(1+\frac{x_i}{b}\right)^{-a}}{1-\left(1-\left(1+\frac{x_i}{b}\right)^{-a}\right)}\right)^{-\frac{1}{\sigma}-1}\left(\frac{\ln\left(1+\frac{x_i}{b}\right)}{\left(1+\frac{x_i}{b}\right)^{-a}}\right)\end{array}\right\},\end{aligned}$$

$$\begin{aligned}\frac{\partial\ell}{\partial b} &= \frac{n}{b}+\left(\frac{a+1}{b^2}\right)\sum_{i=1}^{n}\left(\frac{x_i}{\left(1+\frac{x_i}{b}\right)}\right)+\frac{a}{b^2}\left(\frac{1}{\sigma}+1\right)\sum_{i=1}^{n}\left[\frac{x_i\left(1+\frac{x_i}{b}\right)^{-a-1}}{1-\left(1+\frac{x_i}{b}\right)^{-a}}\right]\\ &+\frac{a}{b^2}\left(\frac{1}{\sigma}-1\right)\sum_{i=1}^{n}\left(\frac{x_i}{\left(1+\frac{x_i}{b}\right)}\right)-\frac{aB}{\sigma b^2}\sum_{i=1}^{n}x_i\left(1+\frac{x_i}{b}\right)^{-a-1}\times\\ &\left[\frac{1-\left(1+\frac{x_i}{b}\right)^{-a}}{1-\left(1-\left(1+\frac{x_i}{b}\right)^{-a}\right)}\right]^{-\frac{1}{\sigma}-1}+\frac{a(\alpha-1)B}{\sigma b^2}\times\end{aligned}$$

$$\sum_{i=1}^{n} \frac{x_i\left(1+\frac{x_i}{b}\right)^{a-1}\left[\frac{1-\left(1+\frac{x_i}{b}\right)^{-a}}{1-\left[1-\left(1+\frac{x_i}{b}\right)^{-a}\right]}\right]^{-\frac{1}{\sigma}} \exp\left\{-B\left[\frac{1-\left(1+\frac{x_i}{b}\right)^{-a}}{1-\left[1-\left(1+\frac{x_i}{b}\right)^{-a}\right]}\right]^{-\frac{1}{\sigma}}\right\}}{\left[1-\exp\left\{-B\left[\frac{1-\left(1+\frac{x_i}{b}\right)^{-a}}{1-\left[1-\left(1+\frac{x_i}{b}\right)^{-a}\right]}\right]^{-\frac{1}{\sigma}}\right\}\right]}.$$

The maximum likelihood estimates are obtained numerically by solving $Z(\Omega) = 0$ which is a system of non-linear equations. Numerical optimization methods such as Newton Raphson's algorithm are normally used in solving such systems of equations.

5. SIMULATIONS

A Monte Carlo simulation study is conducted to assess the performance of the maximum likelihood estimators of the parameters of the EGuL distribution. The performance of the maximum likelihood estimators is examined using various simulations for different sample sizes and different parameter values. The simulation is repeated for$N = 500$ times each with sample size $n = 25, 100, 200, 400, 800$ and parameter values $a = 5, \alpha = 4, b = 1.5, B = 2, \sigma = 2$. Random samples are simulated from the EGuL. Five quantities are computed in the simulations and these include: Mean estimates (M), Average bias (AVB), Root mean squared error (RMSE), Coverage probability (CP) of 95% confidence intervals and Average width (AW) of 95% confidence intervals of the maximum likelihood estimator of the parameters. Table 2 contains the result for the M, AVB, RMSE, AW and CP values of the parameters a, α, b, B and σ for different sample sizes. Results from Table 2 shows that the average biases for the parameters α and B are all negative. The average biases for the parameter a are all positive. It can also be observed that as the sample size increases, the root mean square errors of all the parameters decrease to zero underlining the consistency of the maximum likelihood estimators of the parameters for a larger sample size. Also, the results in Table 2 shows that the coverage probabilities of the confidence intervals are quite close to the nominal level of 95% and that the average confidence widths decrease as the sample size increases.

6. APPLICATIONS

Two real data application to illustrate empirically the potentiality of EGuL distribution is done in this subsection. In order to ascertain the shape of the hazard function of the datasets used in the application of EGul distribution, a graphical method based on Total Time to Test (TTT) plot (Aarset, 1987) is used. The TTT plot is derived by plotting

$$T\left(\frac{r}{n}\right) = \frac{\left[\left(\sum_{i=1}^{r} D_{i:n}\right) + (n-r)D_{r:n}\right]}{\left(\sum_{i=1}^{r} D_{i:n}\right)},$$

TABLE 2
Results of Monte Carlo simulations of EGuL.

Parameter	Sample size	M	AVB	RMSE	AW	CP
a	25	6.2060	4.7622	15.7065	172.0324	0.95
	100	5.8215	2.2223	9.2012	148.3069	0.97
	200	5.9712	5.1506	4.4352	92.2781	0.97
	400	5.5131	1.2164	4.0243	86.8572	0.99
	800	5.3522	8.7142	3.9116	59.8291	0.99
α	25	6.2452	-1.5945	7.2454	305.5785	0.96
	100	6.0833	-1.9410	6.9356	192.8970	0.99
	200	5.1264	-1.0898	5.2682	102.6727	0.99
	400	4.9056	-1.2637	5.1895	94.4513	1
	800	4.0002	-1.7267	5.0969	86.6881	1
b	25	3.4662	-0.7604	8.2652	17.3376	1
	100	3.3965	-7.2381	7.1013	10.2126	1
	200	2.7810	0.6229	6.7565	5.5381	1
	400	1.5946	2.1649	4.1720	5.1249	1
	800	1.5001	5.3713	3.1053	2.5324	1
B	25	3.1336	-9.4521	12.2604	207.3733	0.86
	100	2.9544	-3.4345	9.7181	201.2795	0.96
	200	2.1070	-4.2144	7.6636	197.2892	0.98
	400	2.0046	-7.3237	4.5533	82.34751	0.99
	800	2.0100	-8.2386	3.3072	29.6237	0.99
σ	25	3.2490	-1.5176	5.1338	91.6371	1
	100	2.8955	-0.3132	3.6862	72.9415	1
	200	2.5916	-2.2701	3.4471	39.9563	1
	400	2.0890	3.1745	2.9397	19.4261	1
	800	1.9992	-4.7258	2.1002	9.8400	1

where $r = 1, 2, \ldots, n$ and $D_{i:n} i = 1, 2, \ldots, n$, are the order statistics of the sample, against $\frac{r}{n}$ (Mudholkar *et al.*, 1996). If the plot of $\left(\frac{r}{n}, T\left(\frac{r}{n}\right)\right)$ is concave, convex concave followed by convex, it implies that the hazard function is increasing, decreasing and unimodal, respectively.

We fitted the EGuL distribution alongside other competing models with the same baseline density. The models compared with EGuL distribution are Lomax (L), Exponentiated Lomax (EL), Exponentiated Generalized Lomax distribution (EGL), Kumaraswamy Lomax(KL), Exponentiated Kumaraswamy Lomax (EKL) and Beta Lomax distribution(BL). The density functions of the competing models are given below:

$$\text{EL: } f(x) = \frac{\theta a}{b}\left(1+\frac{x}{b}\right)^{-(a+1)}\left(1-\left(1+\frac{x}{b}\right)^{-a}\right)^{\theta-1},$$

$$\text{EGL: } f(x) = \alpha\theta\left(\frac{a}{b}\right)\left(1+\frac{x}{b}\right)^{-a(\theta+1)}\left\{1-\left(1+\frac{x}{b}\right)^{-a\theta}\right\}^{\alpha-1},$$

$$\begin{aligned}\text{KL: } f(x) &= \alpha\theta\left(\frac{a}{b}\right)\left(1+\frac{x}{b}\right)^{-(a+1)}\left\{1-\left(1+\frac{x}{b}\right)^{-a}\right\}^{\theta-1}\times \\ &\left\{1-\left\{1-\left(1+\frac{x}{b}\right)^{-a}\right\}^{\theta}\right\}^{\alpha-1},\end{aligned}$$

$$\begin{aligned}\text{EKL: } f(x) &= \alpha\theta\left(\frac{a}{b}\right)\left(1+\frac{x}{b}\right)^{-(a+1)}\left\{1-\left(1+\frac{x}{b}\right)^{-a}\right\}^{\theta-1}\times \\ &\left\{1-\left\{1-\left(1+\frac{x}{b}\right)^{-a}\right\}^{\theta}\right\}^{\alpha-1}\times \\ &\left[\left\{1-\left\{1-\left(1+\frac{x}{b}\right)^{-a}\right\}^{\theta}\right\}^{\alpha}\right]^{\beta-1},\end{aligned}$$

$$\text{BL: } f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\left(\frac{a}{b}\right)\left(1+\frac{x}{b}\right)^{-(a\beta+1)}\left[1-\left(1+\frac{x}{b}\right)^{-a}\right]^{\alpha-1}.$$

The Anderson-Darling (A^*) and Cramer-von Mises (W^*) statistics are used in the comparison of EGuL with other competing models with the same baseline distribution. A^* and W^* are chosen because they are widely used to compare non-nested models and to determine how closely a specific cdf fits the empirical distribution of a given data set Corderio *et al.* (2018). Anderson-Darling (A^*) and Cramer-von Mises (W^*) statistics are respectively given by

$$A^* = \left(\frac{9}{4n^2} + \frac{3}{4n} + 1\right)\left\{n + \frac{1}{n}\sum_{j=1}^{n}(2j-1)\log\left[z_i\left(1 - z_{n-j+1}\right)\right]\right\}$$

and

$$W^* = \left(\frac{1}{2n} + 1\right)\left\{\sum_{j=1}^{n}\left(z_i - \frac{2j-1}{2n}\right)^2 + \frac{1}{12n}\right\},$$

where $z_i = F(y_i)$ and the y_i are values of the ordered observations. In general, the smaller the values of these statistics, the better the fit of the distribution to the data.

6.1. Dataset 1

Dataset 1 represents the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli obtained from Bjerkedal (1960). These data have been studied by Elgarhy *et al.* (2017) and are reported in Table 3.

TABLE 3
Data on the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli.

0.1	0.33	0.44	0.56	0.59	0.72	0.74	0.77	0.92
0.93	0.96	1	1	1.02	1.05	1.07	0.7	0.08
1.08	1.08	1.09	1.12	1.13	1.15	1.16	1.2	1.21
1.22	1.22	1.24	1.3	1.34	1.36	1.39	1.44	1.46
1.53	1.59	1.6	1.63	1.63	1.68	1.71	1.72	1.76
1.83	1.95	1.96	1.97	2.02	2.13	2.15	2.16	2.22
2.3	2.31	2.4	2.45	2.51	2.53	2.54	2.54	2.78
2.93	3.27	3.42	3.47	3.61	4.02	4.32	4.58	5.55

The TTT plot (a) in Figure 2 for dataset 1 is concave, indicating that the dataset has increasing failure rate and the EGuL distribution is very suitable in modeling dataset 1. Furthermore, we fitted EGuL and other competing models to dataset 1. The MLEs of the parameters and their standard errors (in parentheses) are listed in Table 4, while the values of the A^* and W^* are listed in Table 5. Table 5 reveals EGuL has the smallest values of A^* and W^* among the fitted models. Hence, the EGuL can be chosen as the best model for the data.

To further assess the suitability EGuL in modeling dataset 1. The histogram of the dataset and estimated pdfs are displayed in plot (b) of Figure 2, while estimated cdfs with empirical cdf are displayed in plot (c). The density plot (b) in Figure 2 compares the empirical histogram from dataset 1 with the competing fitted densities. It is obvious that EGuL fitted pdf is closer to the empirical histogram than the other competing models. Also, the estimated cdf of the EGuL is closer to the empirical cdf of dataset 1 than that of L, EL, EGL, KL, EKL, and BL.

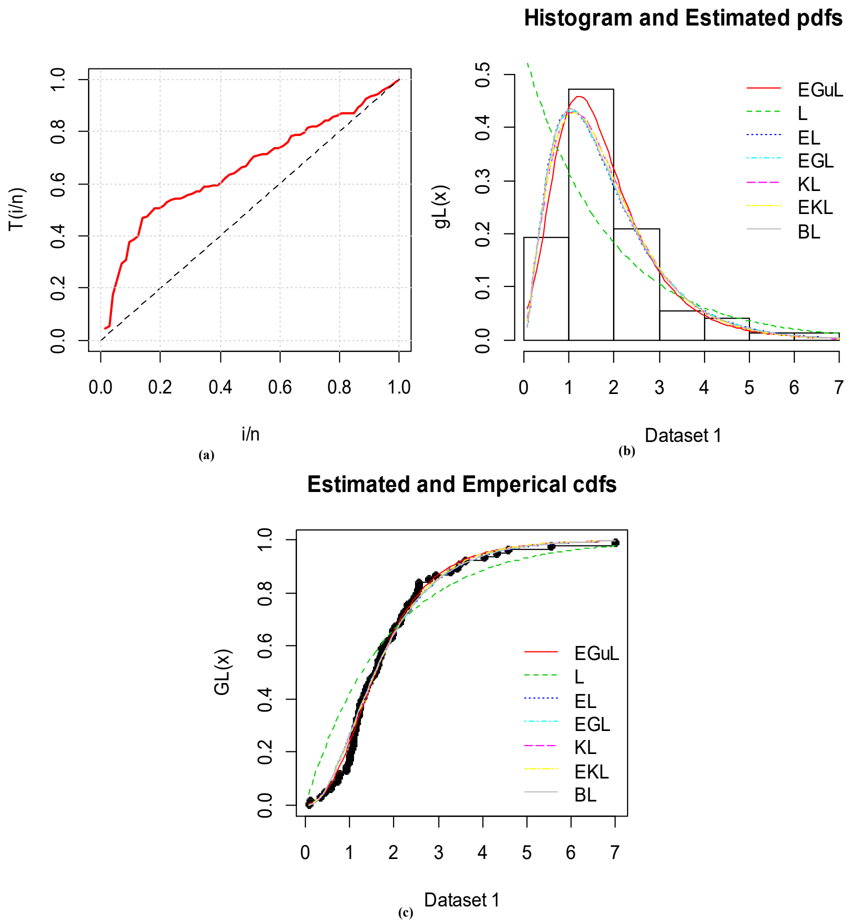

Figure 2 – Plots of TTT for the first dataset, estimated pdfs, empirical cdf with estimated cdfs: (a) TTT plot, (b) Estimated pdfs of competing distributions, (c) Estimated cdfs of competing distributions with empirical cdf.

TABLE 4
MLEs of EGuL parameters and other competing models for dataset1 (standard errors in parenthesis).

Distributions	Estimates				
EGuL (a,b,B,α,σ)	23.91 (38.89)	2.49 (4.16)	6.37 (2.01)	2.08 (3.27)	7.28 (5.36)
L(a,b)	130.59 (231.14)	238.69 (424.08)			
EL(a,b,θ)	59.26 (148.76)	60.67 (157.22)	2.76 (0.55)		
EGL(a,b,θ,α)	10.60 (45.27)	41773.77 (255.10)	3731.24 (15937.65)	2.69 (0.50)	
KL(a,b,θ,α)	1.49 (4.07)	5.29 (9.20)	2.35 (0.57)	9.39 (25.99)	
EKL$(a,b,\theta,\alpha,\beta)$	1.09 (2.36)	3.17 (4.08)	3.04 (2.53)	11.92 (33.01)	0.79 (0.61)
BL(a,b,α,β)	2.74 (0.58)	4.93 (7.84)	3.79 (6.15)	13.31 (17.55)	

TABLE 5
Cramer-von Mises (W^*) and Anderson-Darling (A^*) statistics for dataset 1.

	EGuL	L	EL	EGL	KL	EKL	BL
W^*	0.0564	1.1472	0.1031	0.1022	0.0881	0.0899	0.0975
A^*	0.3673	5.8920	0.7121	0.7035	0.6090	0.6213	0.6810

6.2. Dataset 2

The dataset 2 is remission times (in months) of bladder cancer patients (Table 6). The data was obtained from Almheidat *et al.* (2015).

TABLE 6
Data on the remission times (in months) of bladder cancer patients.

0.080	0.200	0.400	0.500	0.510	0.810	0.900	1.050
1.190	1.260	1.350	1.400	1.460	1.760	2.020	2.020
2.070	2.090	2.230	2.260	2.460	2.540	2.620	2.640
2.690	2.690	2.750	2.830	2.870	3.020	3.250	3.310
3.360	3.360	3.480	3.520	3.570	3.640	3.700	3.820
3.880	4.180	4.230	4.260	4.330	4.340	4.400	4.500
4.510	4.870	4.980	5.060	5.090	5.170	5.320	5.320
5.340	5.410	5.410	5.490	5.620	5.710	5.850	6.250
6.540	6.760	6.930	6.940	6.970	7.090	7.260	7.280
7.320	7.390	7.590	7.620	7.630	7.660	7.870	7.930
8.260	8.370	8.530	8.650	8.660	9.020	9.220	9.470
9.740	10.06	10.34	10.66	10.75	11.25	11.64	11.79
11.98	12.02	12.03	12.07	12.63	13.11	13.29	13.80
14.24	14.76	14.77	14.83	15.96	16.62	17.12	17.14
17.36	18.10	19.13	20.28	21.73	22.69	23.63	25.74
25.82	26.31	32.15	34.26	36.66	43.01	46.12	79.05

The shape of the hazard rate function of dataset 2 is unimodal, as shown by its TTT plot (a) in Figure 3, which is concave then convex. This suggests that EGuL distribution is appropriate for modeling dataset 2. Table 7 gives the Maximum Likelihood estimates and standard errors in parenthesis of EGuL, L, EL, EGL, KL, EKL, and BL distributions. The results from Table 8 indicates that EGuL has the smallest values of the Goodness of fit criteria considered. Hence EGuL can be said to be the best model for dataset 2. The plots of the histogram of dataset 2 and all densities, as shown in plot (b) of Figure 3, reveal that the estimated density of EGuL distribution fits the histogram better than the other models. Furthermore, the plot (c) of the estimated cdfs of all the considered distributions and the empirical cdf of dataset 2 validate the conclusion drawn from the results of the goodness of fit criteria.

TABLE 7
MLEs of EGuL parameters and other competing models for dataset 2 (standard errors in parenthesis).

Distributions	Estimates				
EGuL $(a, b, B, \alpha, \sigma)$	6.90 (10.03)	3.04 (6.88)	5.13 (3.89)	2.64 (7.42)	6.18 (5.49)
L(a, b)	13.96 (15.46)	121.24 (143.41)			
EL(a, b, θ)	4.58 (2.22)	24.67 (16.59)	1.59 (0.28)		
EGL(a, b, θ, α)	15.32 (73.95)	24.74 (16.69)	0.29 (1.44)	1.59 (0.28)	
KL(a, b, θ, α)	0.54 (2.46)	13.19 (16.50)	1.52 (0.27)	8.29 (43.09)	
EKL$(a, b, \theta, \alpha, \beta)$	0.67 (2.76)	11.39 (16.99)	1.82 (2.33)	6.52 (30.83)	0.83 (1.05)
BL(a, b, α, β)	1.58 (0.28)	3.23 (26.56)	1.39 (11.21)	21.23 (16.67)	

TABLE 8
Cramer-von Mises (W^*) and Anderson-Darling (A^*) statistics for dataset 2 .

	EGuL	L	EL	EGL	KL	EKL	BL
W^*	0.0130	0.2124	0.0262	0.0263	0.0233	0.0234	0.0264
A^*	0.0807	1.3759	0.1797	0.1800	0.1605	0.1608	0.1811

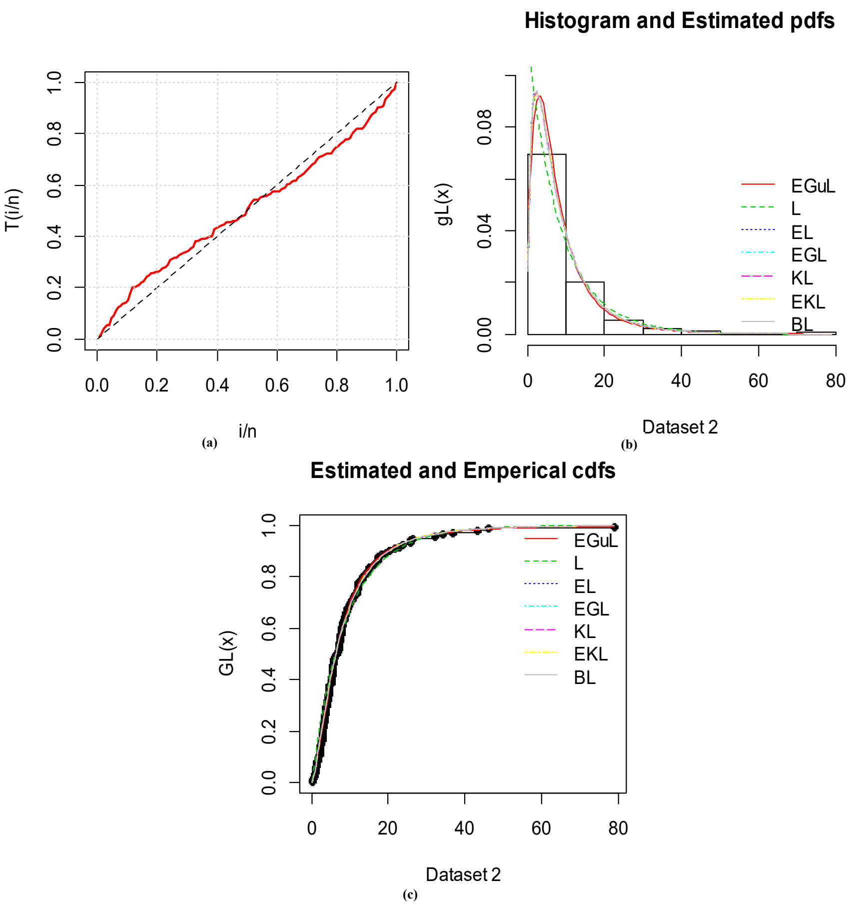

Figure 3 – Plots of TTT for the second dataset, estimated pdfs, empirical cdf with estimated cdfs: a) TTT plot b) Estimated pdfs of competing distributions c) Estimated cdfs of competing distributions with empirical cdf

7. Conclusions

A five-parameter distribution called EGuL distribution is proposed in this paper. Two extra shape parameters are added to the baseline distribution to enhance its flexibility. The pdf of the proposed distribution has various shapes while the hazard rate function has reverse-J shaped, inverted-bathtub shaped and J-shaped making EGuL useful in modelling survival and lifetime data. Two applications of EGuL distribution in modelling survival and lifetime data is used to show that it consistently provides a better fit than other distributions with the same baseline. We hope that the new distribution attracts a wider applicability in modelling survival and lifetime data.

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