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THE MARSHALL-OLKIN GOMPERTZ DISTRIBUTION: PROPERTIES AND APPLICATIONS

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1. INTRODUCTION

In spite of hundreds of lifetime Poisson models in literature, several new flexible models are still being developed to address real life scenarios. The Gompertz distribution is a continuous model proposed in Gompertz (1824). The Gompertz distribution is used to model mortality rate, survival time, failure rate in computer codes, lifespan in gerontology and customer lifetime value. However, despite the numerous researches in modeling lifetime Poisson process in literature with constant failure rate, real life scenarios are unfortunately non-monotonic increasing functions. Hence, a linear monotone increasing failure rate like the Gompertz distribution is introduced to account for the deficiency.

This study presents a bathtub, increasing, decreasing and skewed shaped class of statistical distribution called Marshall-Olkin Gompertz (MO-G) distribution with a better fit for real life data than existing well-known distributions. However, the results obtained from existing literature such as Gompertz, Alpha power Gompertz distribution (APGz) (Eghwerido *et al.*, 2021b), transmuted Gompertz distribution (TGz) (Khan *et al.*, 2016b), transmuted generalized Gompertz distribution, (TGGz) (Khan *et al.*, 2016a) and Marshall-Olkin extended generalized Gompertz distribution, (MOEGG) (Benkhelifa, 2016) stimulate this article.

Eghwerido *et al.* (2020b) proposed the inverse odd Weibull generated family of distribution. Eghwerido *et al.* (2021a) proposed the Shifted Exponential-G family of Distributions. Unal *et al.* (2018) proposed the alpha power inverted exponential distribution. Alizadeh *et al.* (2018a) proposed transmuted Weibull-G distribution. Alizadeh *et al.*

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(2018b) proposed the Poisson-G distribution. Alizadeh *et al.* (2020) proposed the transmuted odd log-logistics-G distribution. Eghwerido and Agu (2021) proposed the shifted Gompertz-G model. Alzaatreh *et al.* (2013) proposed a family of generating a new distribution called alpha power. Mahdavi and Kundu (2017) proposed a new method of generating distribution with application to exponential distribution. Weibull Fréchét distribution, (WFr) was proposed in Afify *et al.* (2016), alpha power inverse Weibull distribution (APIW) in Basheer (2019), Kumuraswamy alpha power inverted exponential distribution (KAPIE) in Zelibe *et al.* (2020), the alpha power Teissier distribution in Eghwerido (2021), Weibull alpha power inverted exponential distribution (WAPIE) in Efe-Eyefia *et al.* (2020). Aryal and Tsokos (2011) proposed the transmuted Weibull, Bourguignon *et al.* (2014) proposed the Weibull-G. Eghwerido *et al.* (2021c) proposed the alpha power Marshall-Olkin-G Distribution, Cordeiro *et al.* (2017) proposed the exponentiated Weibull-H, Granzotto *et al.* (2017) proposed the cubic rank transmuted. Merovci *et al.* (2017) proposed the exponentiated transmuted-G. Mahmoud and Mandouh (2013) proposed transmuted Frechet. Eghwerido *et al.* (2020a) proposed the Gompertz extended generalized exponential distribution. Nofal *et al.* (2017) proposed the generalized transmuted-G. Rahman *et al.* (2018) proposed general transmuted family. Yousof *et al.* (2015) and Yousof *et al.* (2017) proposed the transmuted exponentiated generalized-G and transmuted Topp-Leone-G and Gompertz alpha power inverted exponential (GAPIE) distributions were considered in Eghwerido *et al.* (2020c) .

The MO-G distribution was proposed based on the Marshall-Olkin characterizations.

This article draws a bead on a three parameter model called Marshall-Olkin Gompertz (MO-G) distribution for lifetime Poisson processes. The statistical structural properties of the proposed distribution are established in this paper. The maximum likelihood estimates (MLEs) of the MO-G parameters were derived in a closed form.

The Gompertz probability density function is given as

$$w(x) = \alpha \exp\left[\mu x + \frac{\alpha}{\mu}(1 - \exp(\mu x))\right] \quad x,\ \alpha,\ \ \mu > 0. \tag{1}$$

The cumulative distribution function that corresponds to Equation (1) is defined as

$$W(x) = 1 - \exp\left[-\frac{\alpha}{\mu}(\exp(\mu x) - 1)\right] \quad x, \alpha, \quad \mu > 0, \tag{2}$$

where α is the shape parameter and μ is the scale parameter.

Suppose, $w(x)$ and $W(x)$ are the density and cumulative functions of the baseline distribution or model. Then, Marshall and I. (1997) proposed a transformation for adding a parameter called Marshall-Olkin with the cdf given as

$$U(x) = \frac{W(x)}{\beta + (1-\beta)W(x)}, \quad \beta > 0, \tag{3}$$

where $(1-\beta)$ is the tilt parameter. However, the corresponding pdf is expressed as

$$u(x)=\frac{\beta w(x)}{[\beta+(1-\beta)W(x)]^2}, \qquad \beta>0. \tag{4}$$

In this article, the MO-G distribution together with its mathematical properties will be thoroughly and diligently treated. However, the empirical flexibility and proficiency of the proposed model is examined by application to glass fibre data obtained by workers at the UK National Physical Laboratory and National Highway Traffic Safety Administration data on fatal accidents that occur on roads in the United States. The data represent the number of vehicle fatalities for 39 counties in South Carolina for 2012 (www-fars.nhtsa.dot.gov/States) as used in Mann (2016).

This study is organized as follows: Section 2 defines the MO-G model together with its statistical properties. Section 3 discusses the mathematical linear representation of the proposed density. Section 4 examines statistical structural properties of the proposed MO-G distribution is discussed. The estimates of the parameters are obtained in Section 5. Section 6 presents real life applications to validate the flexibility and efficiency of the proposed MO-G model. The results obtained are compared with existing models in statistical literature in this Section. Section 7 is the conclusions.

2. THE MO-G DISTRIBUTION

In this Section, the new proposed MO-G three parameters of the Gompertz distribution is presented. Let X be a continuous random variable , then the pdf of the MO-G distribution is expressed as

$$u_{\text{MO-G}}(x)=\frac{\beta\ \alpha\ \exp\left[\mu\ x-\frac{\alpha}{\mu}(\exp(\mu x)-1)\right]}{\left[1-(1-\beta)\left[\exp\left[-\frac{\alpha}{\mu}(\exp(\mu x)-1)\right]\right]\right]^2} \qquad x,\ \beta,\ \mu,\ \alpha>0. \tag{5}$$

The corresponding cumulative distribution function (cdf) is defined as

$$U_{\text{MO-G}}(x)=\frac{1-\exp\left[-\frac{\alpha}{\mu}(\exp(\mu x)-1)\right]}{\left[1-(1-\beta)\left(\exp\left[\frac{\alpha}{\mu}(1-\exp(\mu x))\right]\right)\right]} \qquad x,\ \alpha,\ \beta,\ \mu>0, \tag{6}$$

where β is additional extra shape parameter.

Figure 1 is the plot for some parameters values cases for α, μ and β. The plots show that the MO-G density can be decreasing, skewed to the right, and unimodal.

The following were observed from Equation (5):

- $\beta=1$, we obtain the Gompertz distribution.
- $\beta=1$, μ tends to zero, we obtain the exponential distribution.

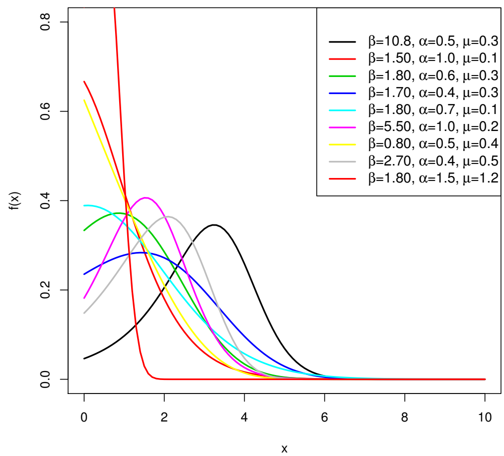


Figure 1 – The density plot for MO-G distribution.

- μ tends to zero, we obtain the Marshall-Olkin exponential distribution.

The survival function of the MO-G distribution is defined as

$$S_{\text{MO-G}}(x) = 1 - \frac{1 - \exp\left[\frac{\alpha}{\mu}(1 - \exp(\mu x))\right]}{\left[1 - (1-\beta)\left(\exp\left[-\frac{\alpha}{\mu}(\exp(\mu x) - 1)\right]\right)\right]}. \tag{7}$$

The hazard rate function (hrf) of the MO-G distribution is expressed as

$$\text{hrf}_{\text{MO-G}}(x) = \frac{\alpha \ \exp(\mu x)}{1 - (1-\beta)\exp\left[-\frac{\alpha}{\mu}(\exp(\mu x) - 1)\right]}. \tag{8}$$

Figure 2 is the plot for the MO-G hazard rate function for some parameter values cases. The plot reveals that the MO-G distribution is increasing and bathtub shaped.

3. Mathematical mixture representation

This Section derives the algebraic expression for the MO-G distribution. The mixture representation obtained would help to simplify the properties of the proposed MO-G model explicitly. More so, it would assist in expressing the proposed MO-G distribution in terms of Gompertz distribution. However, for $|g| < 1$ and $\eta > 0$, then,

$$(1 - g)^{-\eta} = \sum_{\rho=0}^{\infty} \binom{\eta + \rho - 1}{\rho} g^{\rho}, \tag{9}$$

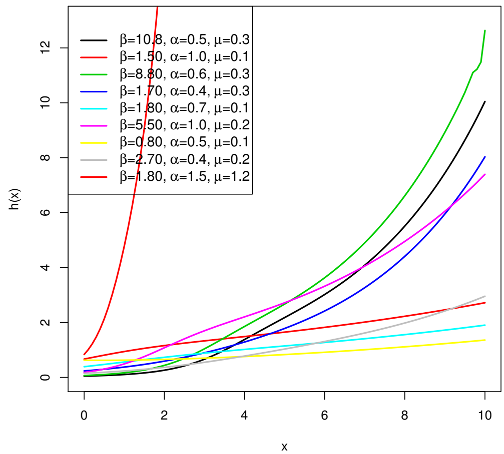


Figure 2 – The MO-G hazard rate function.

for a real non-integer. However, for η integer, ρ stops at $\eta - 1$.

Let the quantity in the denomination of equation (5) be B, Hence B can be enumerated as

$$B = \sum_{\rho=0}^{\infty} \binom{\rho+1}{\rho} (1-\beta)^{\rho} \left[\exp\left(\frac{\alpha}{\mu} (1 - \exp(\mu x)) \right) \right]^{\rho}. \tag{10}$$

Thus, as a power series, the MO-G distribution can be defined as

$$u_{\text{MO-G}}(x) = \sum_{\rho=0}^{\infty} \binom{\rho+1}{\rho} (1-\beta)^{\rho} \alpha \beta \exp\left[\mu x - \frac{\alpha}{\mu} (\exp(\mu x) - 1)(\rho+1) \right]. \tag{11}$$

The Odds function that corresponds to this distribution is given as

$$O(x) = \frac{1 - \exp\left[\frac{\alpha}{\mu} (1 - \exp(\mu x)) \right]}{\beta \exp\left[-\frac{\alpha}{\mu} (\exp(\mu x) - 1) \right]}. \tag{12}$$

4. Structural statistical properties of the MO-G distribution

In this Section, some of the statistical structural properties of the MO-G distribution are derived and investigated. These include the moments, generating function, quantile function, entropies, probability weighted moment, moments of the residual and order statistics.

4.1. Quantile function of the MO-G distribution

Suppose X is a random variable as $X \sim$ MO-G(α, μ, β). Then, the quantile function of X for $m \in [0,1]$ is given as

$$Q_m = \mu^{-1} \log\left[1 - \frac{\mu}{\alpha} \log\left[\frac{m-1}{m(1-\beta)-1}\right]\right] \quad 0 < m < 1. \tag{13}$$

However, by setting m as 0.5 in Equation (13), we have the median (M) of X as

$$M = \mu^{-1} \log\left[1 - \frac{\mu}{\alpha} \log\left[\frac{-0.5}{0.5(1-\beta)-1}\right]\right]. \tag{14}$$

Figures 3 and 4 show the Bowley's skewness and Moor's kurtosis of the MO-G modal.

However, the 25^{th} percentile and 75^{th} percentile for the random variable X is obtained as

$$Q_1 = \mu^{-1} \log\left[1 - \frac{\mu}{\alpha} \log\left[\frac{-0.75}{0.25(1-\beta)-1}\right]\right]; \tag{15}$$

$$Q_3 = \mu^{-1} \log\left[1 - \frac{\mu}{\alpha} \log\left[\frac{-0.25}{0.75(1-\beta)-1}\right]\right]. \tag{16}$$

The Bowley's skewness is obtained in the quantile function as

$$S_k = \frac{Q_{0.75} - 2Q_{0.50} + Q_{0.25}}{Q_{0.75} - Q_{0.25}}. \tag{17}$$

The Moor's kurtosis is expressed as

$$M_k = \frac{Q_{0.875} - Q_{0.625} - Q_{0.375} + Q_{0.125}}{Q_{0.75} - Q_{0.25}}. \tag{18}$$

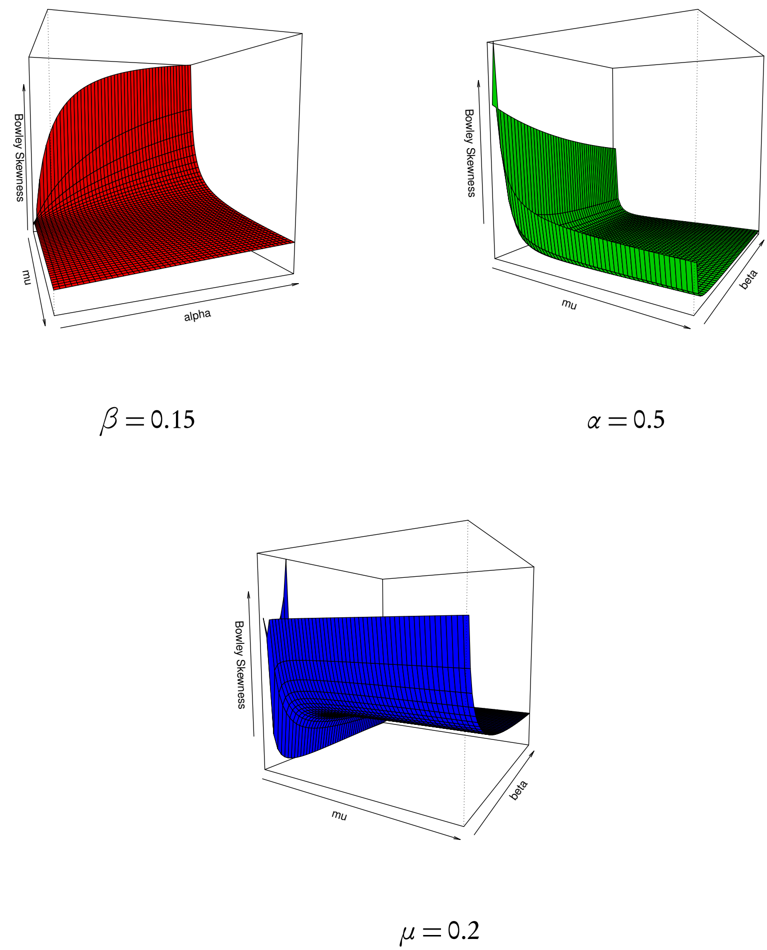
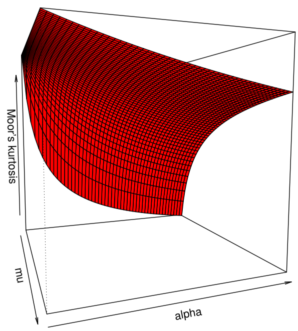
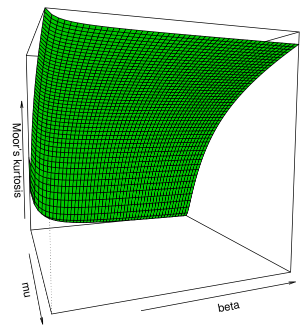


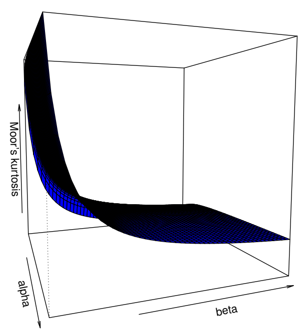
Figure 3 – The 3D plots of the Bowley's skewness.



$\beta = 0.15$



$\alpha = 0.5$



$\mu = 0.2$

Figure 4 – The 3D plots of the Moor's kurtosis.

4.2. *The MO-G distribution moments*

The MO-G distribution moments are given in a Laplace transform as

$$L(s)=\sum_{\rho=0}^{\infty}\binom{\rho+1}{\rho}(1-\beta)^{\rho}\beta\int_{0}^{\infty}\alpha\exp\left[\mu x-\frac{\alpha}{\mu}(\exp(\mu x)-1)(\rho+1)\right]\exp(-sx)\mathrm{d}x. \tag{19}$$

Let $\upsilon=\exp(\mu x)$, then Equation (19) can be expressed as

$$L(s)=\sum_{\rho=0}^{\infty}\binom{\rho+1}{\rho}(1-\beta)^{\rho}\beta\frac{\alpha}{\mu}\exp\left(\frac{\alpha}{\beta}(\rho+1)\right)\int_{1}^{\infty}\alpha\exp\left[-\frac{\alpha}{\mu}(\rho+1)v\right]v^{-\frac{s}{\mu}}dv. \tag{20}$$

However, by Abramowitz and Stegun (1965),

$$E_{n}(x)=\int_{1}^{\infty}\frac{e^{-zh}}{h^{\eta}}dh,\quad \eta>0,\quad Re(z)>0, \tag{21}$$

where $z=\frac{\alpha}{\mu}(\rho-1)$ and $\eta=\frac{S}{\mu}$. Thus

$$L(s)=\sum_{\rho=0}^{\infty}\binom{\rho+1}{\rho}(1-\beta)^{\rho}\beta\frac{\alpha}{\mu}\exp\left(\frac{\alpha}{\beta}(\rho+1)\right)E_{\frac{S}{\mu}}\left[\frac{\alpha}{\mu}(\rho+1)\right],\quad \alpha,\ \beta,\ \mu>0. \tag{22}$$

More so, the MO-G r^{th} moment of the random variable X is defined as

$$\begin{aligned}E[X^{r}]=&\sum_{\rho=0}^{\infty}\binom{\rho+1}{\rho}(1-\beta)^{\rho}\frac{\beta\alpha}{\mu}\exp\left(\frac{\alpha}{\mu}(\rho+1)\right)\times\\&\times\int_{1}^{\infty}\frac{r}{\mu^{r}}x^{-1}\exp\left(-\frac{\alpha}{\mu}(\rho+1)x\right)[\ln(x)]^{r-1}\mathrm{d}x.\end{aligned} \tag{23}$$

Thus, by Milgram (1985), the generalized integro-exponential function and the integral representation in Equation (23), can be defined as

$$E_{s}^{f}(z)=\int_{1}^{\infty}\ln[x]^{f}x^{-s}\frac{1}{\Gamma(f+1)}\exp(-zx)\mathrm{d}x, \tag{24}$$

where the quantity E_{s}^{f} is given as

$$E_{s}^{f}(z)=\frac{(-1)^{f}}{f!}\frac{\partial^{f}}{\partial s^{f}}E_{s}(z). \tag{25}$$

Thus, the MO-G r^{th} moment is expressed as

$$E[X^{r}]=\sum_{\rho=0}^{\infty}\binom{\rho+1}{\rho}(1-\beta)^{\rho}\beta\frac{r!}{\mu^{\eta}}\exp\left(\frac{\alpha}{\mu}(\rho+1)\right)E_{1}^{r-1}\left(\frac{\alpha}{\mu}(j+1)\right), \tag{26}$$

with the quantity $E_1^{r-1}\left(\frac{\alpha}{\mu}(j+1)\right)$ given as

$$E_1^{r-1}=\sum_{i=1}^{\infty}\frac{1}{(-i)^r}\frac{\left(-\frac{\alpha}{\mu}(\rho+1)\right)^i}{i!}+ \\ +\frac{(-1)^r}{r!}\sum_{i=0}^{\eta}\frac{r!}{(r-i)!i!}\ln\left(\frac{\alpha}{\mu}(\rho+1)\right)^{r-i}\lim_{j\to 0}\frac{d^i}{d\upsilon^i}\Gamma(1-\upsilon). \tag{27}$$

4.3. Generating Function

The MO-G probability generating function for a random variable X is defined as

$$M(t)=\int_{-\infty}^{\infty}t^x u_{\text{MO-G}}(x)\mathrm{d}x \\ =\sum_{i=0}^{\infty}x^i\frac{(\log t)^i}{i!}u_{\text{MO-G}}(x)\mathrm{d}x \qquad for \quad |t|>1, \qquad x>0. \tag{28}$$

This implies that

$$M(t)=\sum_{\rho=0}^{\infty}\sum_{i=0}^{\infty}w_{\rho i}\int_1^{\infty}x^i\alpha\exp\left(\mu x-\frac{\alpha}{\mu}(\exp(\mu x)-1)(\rho+1)\right)\mathrm{d}x, \tag{29}$$

with

$$w_{\rho i}=\frac{(\log t)^i}{i!}\binom{\rho+1}{\rho}(1-\beta)^\rho\beta.$$

Simplifying after integrating gives

$$M(t)=\sum_{\rho=0}^{\infty}\sum_{i=0}^{\infty}w_{\rho i}\frac{i!}{\mu^i}\exp\left(\frac{\alpha}{\mu}(\rho+1)\right)E_1^{i-1}\left(\frac{\alpha}{\mu}(\rho+1)\right). \tag{30}$$

However, the MO-G moment generating function (mgf) is expressed as

$$M_X(t)=\int_{-\infty}^{\infty}e^{tx}u_{\text{MO-G}}(x)\mathrm{d}x \\ =\sum_{\rho=0}^{\infty}w_\rho\int_0^{\infty}\exp\left(\mu x-\frac{\alpha}{\mu}(\exp(\mu x)-1)(\rho+1)+tx\right)\mathrm{d}x, \tag{31}$$

where $w_\rho=\binom{\rho+1}{\rho}(1-\beta)^\rho\beta$.
Thus, by the same substitution of Equation (19), the mgf, say $M_x(t)$ is given as

$$M_x(t)=\sum_{\rho=0}^{\infty}w_\rho\frac{\alpha}{\mu}\exp\left(\frac{\alpha}{\mu}(\rho+1)\right)\int_0^{\infty}\exp\left(-\frac{\alpha}{\mu}(\rho+1)\upsilon\right)\upsilon^{\frac{s}{\mu}}d\upsilon. \tag{32}$$

On simplifying Equation (32), we have

$$M_x(t)=\sum_{\rho=0}^{\infty} w_\rho \frac{\alpha}{\mu}\exp\left(\frac{\alpha}{\mu}(\rho+1)\right)\psi_{s/\mu}\left(\frac{\alpha}{\mu}(\rho+1)\right) \qquad \alpha,\ \beta,\quad \mu>0, \tag{33}$$

where

$$\psi_{s/\mu}\left(\frac{\alpha}{\mu}(\rho+1)\right)=\int_0^{\infty}\exp\left(-\frac{\alpha}{\mu}(\rho+1)\upsilon\right)\upsilon^{\frac{s}{\mu}}d\upsilon. \tag{34}$$

4.4. *Probability weighted moments (PWM)*

The $(s,r)^{th}$ PWM of MO-G model is defined as

$$\begin{aligned}P(s,r)=&E[x^r F(x)^s]\\ =&\int_0^{\infty} x^r U^s_{\text{MO-G}}(x)u_{\text{MO-G}}(x)\mathrm{d}x.\end{aligned} \tag{35}$$

This implies that

$$P(s,r)=\int_0^{\infty} x^r \frac{\beta\alpha\left[1-\exp\left(-\frac{\alpha}{\mu}\left(\exp(\mu x)-1\right)\right)\right]^s\left[\mu x-\frac{\alpha}{\mu}\left(\exp(\mu x)-1\right)\right]}{\left[1-\left(1-\beta\right)\exp\left(-\frac{\alpha}{\mu}\left(\exp(\mu x)-1\right)\right)\right]^{s+2}}\mathrm{d}x. \tag{36}$$

However, simplifying (36) we have

$$P(s,r)=\sum_{\rho=1}^{s}\sum_{\upsilon=1}^{s+2}\Omega_{\rho,\upsilon}\exp\left(\frac{\alpha}{\mu}(\rho+s+3)\right)E_1^{r-1}\left(\frac{\alpha}{\mu}(\rho+s+3)\right), \tag{37}$$

$$\Omega_{\rho,\upsilon}=\frac{r!}{\mu^r}\beta\binom{s}{\rho}\binom{s+2}{\upsilon}^{-1}\left(-1\right)^{2s-\upsilon-\rho+2}\left(1-\beta\right)^{-(s+2)}. \tag{38}$$

The PWM function can be used to obtain the parameters and quantiles of a distribution that may not be explicitly obtained.

4.5. *Entropies*

The Renyi entropy of the distribution MO-G is expressed as

$$R_\delta(x)=\sum_{\rho=0}^{\infty}\log\int_{-\infty}^{\infty} c_\rho\exp\left(\delta\mu x-\frac{\alpha\delta}{\mu}(\rho+1)(\exp(\mu x)-1)\right)\mathrm{d}x \quad \delta>0,\ \ \delta\neq 0, \tag{39}$$

where

$$c_\rho = \frac{1}{1-\rho}\binom{\rho+1}{\rho}^{\delta}(1-\beta)^{\rho\delta}(1-\beta)^{\rho\delta}\alpha^{\delta}\beta^{\delta}. \tag{40}$$

On integrating and simplifying gives

$$R_\delta(x) = \sum_{\rho=0}^{\infty}\left(\log\left(y_\rho + \left(\delta\mu x - \frac{\alpha\delta}{\mu}(\rho+1)(\exp(\mu x)-1)\right)\right)\right), \tag{41}$$

where

$$y_\rho = \frac{c_\rho}{\delta\beta - \mu\delta(\rho+1)\exp(\mu x)}. \tag{42}$$

The MO-G shannon entropy is defined as

$$E[-\log u_{\text{MO-G}}(x)] = \sum_{\rho=0}^{\infty}\left(\log n\rho - \frac{\alpha}{\mu}(\rho+1)E(\exp(\mu x)-1) + \mu E[x]\right), \tag{43}$$

where $E[x] = -\frac{d}{ds}L(s)\mid_{s=0}$.

4.6. *Moment of the residual*

The η^{th} moment of the residual life, say $d_n(t) = E[(x-t)^n \mid x > t]$ for $n = 1, 2, \ldots$ uniquely determines $U_{\text{MO-G}}(x)$. (see Navarro *et al.*, 1998). However, the η^{th} moment of the residual life is given as

$$d_n(t) = \frac{1}{1-U_{\text{MO-G}}(x)}\int_t^{\infty}(x-t)^n d\,U_{\text{MO-G}}(x)\mathrm{d}x. \tag{44}$$

However, on applying binomial expansion, we have

$$\begin{aligned} d_n(t) = &\frac{1}{1-U_{\text{MO-G}}(x)} \times \\ &\times \sum_{\rho=0}^{\infty}\sum_{k=0}^{n} D_{\rho k} t^{n-k}\int_t^{\infty}\alpha x^n \exp\left(\mu x - \frac{\alpha}{\mu}(\exp(\mu x)-1)(\rho+1)\right)\mathrm{d}x \end{aligned} \tag{45}$$

where $D_{\rho k} = \binom{\rho+1}{\rho}(1-\beta)^{\rho}(-1)^{n-k}\binom{n}{k}\beta$. Hence,

$$d_n(t) = \frac{1}{1-U_{\text{MO-G}}(x)}\sum_{\rho=0}^{\infty}\sum_{k=0}^{\infty} D_{\rho k} t^{n-k}\frac{n!}{\mu^n}\exp\left(\frac{\alpha}{\mu}\right)E_1^{n-1}\left(\frac{\alpha}{\mu}(\rho+1)\right) \tag{46}$$

for $\mid \arg t \mid < \pi$.

4.7. *Order Statistics*

The order statistics of the MO-G model is defined as

$$f_{i:n}(x) = \frac{u(x)}{B(i, n-i+1)} \sum_{j=0}^{n-i} (-1)^j \binom{n-1}{j} U(x)^{i+j-1}. \tag{47}$$

Clearly, we have

$$f_{i:n}(x) = \frac{1}{B(i, n-i+1)} \sum_{j=0}^{n-i} \sum_{\tau=0}^{\infty} \sum_{k=0}^{\infty} \vartheta_{j\tau k} \exp\Big[\mu x - \frac{\alpha}{x}(1+k+\tau)(\exp(\mu x)-1)\Big], \tag{48}$$

where $\vartheta_{j\tau k} = \beta\alpha\binom{n-1}{j}\binom{i+j-1}{k}\binom{i+j+\tau}{\tau}(-1)^{k+\tau}(1-\beta)^{\tau}$.

5. Estimation of MO-G parameters

Several approaches have been employed for parameter estimation in literature. In this article, the maximum likelihood method was adopted to obtain the parameters of the MO-G distribution.

Let $\mathbf{x} = (x_1, \ldots, x_n)$ be a random sample from the MO-G model with unknown parameter vector $\theta = (\alpha, \mu, \beta)^T$. Then log-likelihood function ℓ of the MO-G can be expressed as

$$\ell = \eta \log \beta + \sum_{\rho=1}^{\eta} \Big[\mu x_\rho - \frac{\alpha}{\mu}\big(e^{(\mu x_\rho)} - 1\big)\Big] - 2 \log \sum_{\rho=1}^{\eta} \omega_\rho + \eta \log \alpha, \tag{49}$$

where

$$\omega_\rho = \Big[1 - (1-\beta)\exp\big(\frac{\alpha}{\mu}(1 - \exp(\mu x_\rho))\big)\Big]. \tag{50}$$

However, taking the partial derivative of equation (49) with respect to each parameter and equating to zero is expressed as

$$\frac{\partial \ell}{\partial \beta} = \frac{\eta}{\beta} - 2 \sum_{\rho=1}^{\eta} \frac{\omega'_{\rho,\beta}}{\omega_\rho} = 0, \tag{51}$$

where $\omega'_\rho(\cdot)$ is the first partial derivative.

$$\frac{\partial \ell}{\partial \alpha} = \frac{\eta}{\alpha} - 2 \sum_{\rho=1}^{\eta} \frac{\omega'_{\rho,\alpha}}{\omega_\rho} - \sum_{\rho=1}^{\eta} \frac{1}{\mu}\Big(\exp(\mu x_\rho) - 1\Big) = 0, \tag{52}$$

$$\frac{\partial \ell}{\partial \mu} = \sum_{\rho=1}^{\eta} x_\rho - 2 \sum_{\rho=1}^{\eta} \frac{\omega'_{\rho,\mu}}{\omega_\rho} + \frac{\alpha}{\mu^2} \sum_{\rho=1}^{\eta} \Big[\exp(\mu x_\rho) - 1\Big] - \sum_{\rho=1}^{\eta} \frac{x_\rho \alpha}{\mu}\Big[\exp(\mu x_\rho) - 1\Big] = 0. \tag{53}$$

Newton-Raphson algorithm is used to obtain the solution to the estimates of the vector analytically using Software like, `R`, `MATLAB` and `MAPLE`.

5.1. MO-G regression analysis

Parametric models are widely used to estimate survival functions for univariate models and for regression problems for censored datasets. However, these parametric models tend to provide a good fit for survival times data.

Let X be an MO-G random variable. Then, for a regression model with scale (τ) and location $(\phi(v))$ parameters, then $D = \log(X)$ has log MO-G distribution defined as

$$D = \phi(v) + \tau Z; \quad \tau > 0, \tag{54}$$

such that Z does not depend on v. Thus, for a support $d \subseteq \Re$, the density of D is defined as

$$u(d) = \frac{\beta w\left(\frac{d-\phi(v)}{\tau}\right)}{\left[\beta + (1-\beta) W\left(\frac{d-\phi(v)}{\tau}\right)\right]^2}, \ \beta > 0. \tag{55}$$

The corresponding cdf is expressed as

$$U(d) = \frac{W\left(\frac{d-\phi(v)}{\tau}\right)}{\beta + (1-\beta) W\left(\frac{d-\phi(v)}{\tau}\right)}, \ \beta > 0. \tag{56}$$

Hence, the log MO-G regression model can be defined as

$$u(d) = \frac{\beta \tau^{-1} \exp\left(\exp\left(\frac{d-\phi(v)}{\tau}\right) - \tau^{-1} \exp\left(\frac{\phi(v)}{\tau}\right)\left(\exp\left(\exp\left(\frac{d-\phi(v)}{\tau}\right)\right) - 1\right)\right)}{\left[\beta + \left(1-\beta\right)\left(1 - \exp\left(-\tau^{-1} \exp\left(\frac{\phi(v)}{\tau}\right)\left(\exp\left(\exp\left(\frac{d-\phi(v)}{\tau}\right)\right) - 1\right)\right)\right)\right]^2}, \ \beta > 0. \tag{57}$$

The cdf that corresponds is defined as

$$U(d) = \frac{1 - \exp\left[-\tau^{-1} \exp\left(\frac{\phi(v)}{\tau}\right)\left(\exp\left(\exp\left(\frac{d-\phi(v)}{\tau}\right)\right)\right) - 1\right)\right]}{\left[\beta + \left(1-\beta\right)\left(1 - \exp\left(-\tau^{-1} \exp\left(\frac{\phi(v)}{\tau}\right)\left(\exp\left(\exp\left(\frac{d-\phi(v)}{\tau}\right)\right) - 1\right)\right)\right)\right]}, \ \beta > 0, \tag{58}$$

where $\phi \in \Re$ is the location parameter.

The survival function of D is defined as

$$S(d) = \frac{\beta \exp\left[-\tau^{-1} \exp\left(\frac{\phi(v)}{\tau}\right)\left(\exp\left(\exp\left(\frac{d-\phi(v)}{\tau}\right)\right)\right) - 1\right)\right]}{\left[\beta + \left(1-\beta\right)\left(1 - \exp\left(-\tau^{-1} \exp\left(\frac{\phi(v)}{\tau}\right)\left(\exp\left(\exp\left(\frac{d-\phi(v)}{\tau}\right)\right) - 1\right)\right)\right)\right]}, \ \beta > 0. \tag{59}$$

The standardized pdf of Z is expressed as

$$\pi(z)=\frac{\beta\tau^{-1}\exp\Big(\exp(z)-\tau^{-1}\exp(\frac{\phi(v)}{\tau})(\exp(\exp(z))-1)\Big)}{\Big[\beta+\Big(1-\beta\Big)\Big(1-\exp(-\tau^{-1}\exp(\frac{\phi(v)}{\tau})(\exp(\exp(z))-1))\Big)\Big]^2},\quad \beta>0. \tag{60}$$

However, a linear location-scale regression is proposed as

$$D_i=v_i^T\psi+\tau Z_i \quad ,i=1,2,3,\ldots,n, \tag{61}$$

where D_i is the response variable. $v_i^T=(v_{i1},v_{i2},v_{i3},\ldots,v_{ip})$ is the explanatory variable vector. Also, ψ is $p\times 1$ vector of parameter and Z_i is th i^{th} random error with density (60).

The purpose of the log MO-G is to enable .the possibilities of fitting variety of models to different datasets that are more flexible and practicable.

Suppose $D_{i1},D_2,D_3,\ldots,D_n$ are random sample with size n, such that each response variable is defined as

$$D_i=\min\Big\{\log(X_i),\log(m_i)\Big\}, \tag{62}$$

with m_i as the censure for the i^{th} measurement such that the censoring times and observed lifetimes are independent for non-informative censoring. Let C and F be the log-censoring and log-lifetime for sets of individuals for d_i. Then, the log-likelihood function for vector of parameters $\theta=(\psi^T,\tau,\beta,\phi)^T$ from model (61) is defined as

$$\ell(\theta)=\sum_{i\in F}\log\Big[u(d_i)\Big]+\sum_{i\in F}\log\Big[S(d_i)\Big], \tag{63}$$

where $u(d_i)$ is the density function in Equation (57) and $s(d_i)$ is the survival function in Equation (59) of D_i.

Thus, the likelihood can be expressed as

$$\ell(\theta)=\sum_{i\in F}\log\Bigg[\frac{\beta\tau^{-1}\exp\Big(\exp(\frac{d-\phi(v)}{\tau})-\tau^{-1}\exp(\frac{\phi(v)}{\tau})(\exp(\exp(\frac{d-\phi(v)}{\tau}))-1)\Big)}{\Big[\beta+\Big(1-\beta\Big)\Big(1-\exp(-\tau^{-1}\exp(\frac{\phi(v)}{\tau})(\exp(\exp(\frac{d-\phi(v)}{\tau}))-1))\Big)\Big]^2}\Bigg]$$
$$+\sum_{i\in F}\log\Bigg[\frac{\beta\exp\Big[-\tau^{-1}\exp(\frac{\phi(v)}{\tau})(\exp(\exp(\frac{d-\phi(v)}{\tau})))-1)\Big]}{\Big[\beta+\Big(1-\beta\Big)\Big(1-\exp(-\tau^{-1}\exp(\frac{\phi(v)}{\tau})(\exp(\exp(\frac{d-\phi(v)}{\tau}))-1))\Big)\Big]}\Bigg], \tag{64}$$

where the number of failure is denoted by ρ and $z_i = \frac{\left(d_i - v_i^T \psi\right)}{\tau}$ for $i = 1,2,3,\ldots,n$. The MLE of Equation (64) can be obtained for the unknown parameter by maximizing Equation (64). The estimates can be achieved using R package. The survival function for d_i can be estimated as

$$S(d_i) = \frac{\hat{\beta}\exp\Big[-\hat{\tau}^{-1}\exp(\frac{\hat{\phi}}{\hat{\tau}})(\exp(\exp(\frac{d_i - v_i^T\hat{\psi}}{\hat{\tau}}))) - 1)\Big]}{\Big[\hat{\beta} + \Big(1-\hat{\beta}\Big)\Big(1 - \exp(-\hat{\tau}^{-1}\exp(\frac{\hat{\phi}}{\hat{\tau}})(\exp(\exp(\frac{d_i - v_i^T\hat{\psi}}{\hat{\tau}})) - 1))\Big)\Big]}. \tag{65}$$

The asymptotic distribution of $(\hat{m} - m)$ under general regularity conditions is a multivariate normal $N_{\rho+1}(0, K(m)^{-1})$, where $K(m)$ is the expected information matrix with asyptotic covariance matrix $K(m)^{-1}$ of $\hat{m}$. This can be approximated using the inverse of $(\rho+3)\times(\rho+3)$ of observed information matrix $J(\theta)$ with inference based on the normal approximation $N_{\rho+3}(0.J(\hat{m})^{-1})$ distribution for $\hat{m}$.

5.2. *Bayesian inference with Stan package*

This Section discuss the Bayesian reliability analysis of the MO-G model for survival time datasets.

The Bayesian reliability analysis can be obtained using the pdf given in Equations (5) and (7). However, the prior is specified despite the parameter of interest before applying the pdf to analyze the experimental data. Although, several priors like the Uniform and Gaussian have been used in statistical literature. However, the uniform prior has been very useful in Bayesian analysis because it assumes that the value of the parameters for the prior is equally likely. Thus, the Half-Cauchy (HC) distribution with upper tail with a large mass that approaches zero for large values is preferred because, it exhibit the characteristics of the uniform distribution for a scale parameter of 25. Hence, the likelihood function can be expressed as

$$L = \prod_{d=1}^{D}\Big[u(y_d)\Big]^{\delta_d}\Big[S(y_d)\Big]^{1-\delta_d}, \tag{66}$$

such that for $\delta_d = 0$ for censored and $\delta_d = 1$ for uncensored. Hence,

$$\begin{aligned} L = \prod_{d=1}^{D}\Bigg[\frac{\beta\ \alpha\ \exp\Big[\mu\ y - \frac{\alpha}{\mu}(\exp(\mu y) - 1)\Big]}{\Big[1-(1-\beta)\Big[\exp\Big[-\frac{\alpha}{\mu}(\exp(\mu y) - 1)\Big]\Big]\Big]^2}\Bigg]^{\delta_d} \\ \times\Bigg[\frac{\beta\exp\Big[-\frac{\alpha}{\mu}(\exp(\mu y) - 1)\Big]}{\Big[1-(1-\beta)\Big(\exp\Big[-\frac{\alpha}{\mu}(\exp(\mu y) - 1)\Big]\Big)\Big]}\Bigg]^{1-\delta_d}. \end{aligned} \tag{67}$$

However, the joint posterior density can be expressed as

$$
\begin{aligned}
p\left(\alpha,\beta,\sigma|y,X\right) &\propto L\left(y,X|\alpha,\beta,\sigma\right)\times p(\alpha)\times p(\beta)\\
&\propto \prod_{d=1}^{D}\Bigg[\frac{\beta\ \alpha\ \exp\left[\mu\ y-\frac{\alpha}{\mu}(\exp(\mu y)-1)\right]}{\left[1-(1-\beta)\left[\exp\left[-\frac{\alpha}{\mu}(\exp(\mu y)-1)\right]\right]\right]^{2}}\Bigg]^{\delta_d}\\
&\times\Bigg[\frac{\beta\exp\left[-\frac{\alpha}{\mu}(\exp(\mu y)-1)\right]}{\left[1-(1-\beta)\left(\exp\left[-\frac{\alpha}{\mu}(\exp(\mu y)-1)\right]\right)\right]}\Bigg]^{1-\delta_d}\\
&\prod_{i}^{I}\frac{1}{\sqrt{2\times10^{3}\pi}}\exp\Big(-\frac{\sigma_i^2}{2\times10^{3}}\Big)\times\frac{50}{\pi(\alpha^2+625)}\times\frac{50}{\pi(\beta^2+625)},
\end{aligned}
\tag{68}
$$

with $\alpha\sim HC(0,25)$, $\beta\sim HC(0,25)$ and $\sigma\sim N(0,1000)$, for $i=1,2,3,4,\ldots,I$. The closed form of the Bayesian Equation (68) does not exist. However, the marginal (the basis of the Bayesian inference) posterior densities of the parameter cannot be obtained in a closed form. Hence, MCMC methods are used to evaluate the posterior parameters. The posterior parameters can be evaluated using the `rstan` package in `R` (R Core Team, 2019).

5.3. *Simulation studies*

A simulation study was performed to investigate the flexibility and proficiency of the class of MO-G distribution. `R` (R Core Team, 2019) was used for the statistical computing. Table 1 shows the results for various parameters values. The simulation was examined as follows:

- Data were generated using the MO-G quantile function as

$$
x=\mu^{-1}\log\left[1-\frac{\mu}{\alpha}\log\left[\frac{m-1}{m(1-\beta)-1}\right]\right]\quad 0<m<1. \tag{69}
$$

- The sample sizes of $n=50,100,150,200,250,300$ and 350 were taken for $\mu=0.5$, $\alpha=1.5$ and $\beta=0.5$.
- The sample size was replicated 5000 times.

The simulation study investigated the mean estimates (AEs), biases, variance and means squared errors (MSEs) of the maximum likelihood estimate MLEs.

The bias is calculated by (for M)

$$
\hat{\text{BIAS}}_M=\frac{1}{5000}\sum_{i=1}^{5000}\left(\hat{M}_i-M\right). \tag{70}
$$

Also, the MSE is obtained as

$$\hat{\text{MSE}}_M = \frac{1}{5000}\sum_{i=1}^{5000}\left(\hat{M}_i - M\right)^2. \tag{71}$$

TABLE 1
Monte Carlos simulation study for the MO-G Distribution.

Sample size	Parameter	AE	Bias	Variance	MSE
5	$\beta = 0.5$	0.0003	-0.4997	0.0000	0.2497
	$\alpha = 1.5$	1.3156	0.3156	0.0392	0.1388
	$\mu = 0.5$	0.5366	0.0366	0.0034	0.0048
10	$\beta = 0.5$	0.0003	-0.4997	0.0000	0.2497
	$\alpha = 1.5$	1.3284	0.3284	0.0347	0.1425
	$\mu = 0.5$	0.5219	0.0219	0.0024	0.0029
50	$\beta = 0.5$	0.0003	-0.4997	0.0000	0.2497
	$\alpha = 0.5$	1.3603	0.3603	0.0256	0.1554
	$\mu = 0.5$	0.5000	0.0000	0.0004	0.0004
100	$\beta = 0.5$	0.0003	-0.4997	0.0000	0.2497
	$\alpha = 1.5$	1.3727	0.3727	0.0244	0.1633
	$\mu = 0.5$	0.4997	-0.0003	0.0003	0.0003
150	$\beta = 0.5$	0.0003	-0.4997	0.0000	0.2497
	$\alpha = 1.5$	1.3806	0.3806	0.0238	0.1687
	$\mu = 0.5$	0.5004	0.0004	0.0002	0.0002
200	$\beta = 1.0$	0.0003	-0.4997	0.0000	0.2497
	$\alpha = 0.5$	1.3875	0.3875	0.0232	0.1733
	$\mu = 0.5$	0.5011	0.0011	0.0002	0.0002
250	$\beta = 0.5$	0.0003	-0.4997	0.0000	0.2497
	$\alpha = 1.5$	1.3919	0.3919	0.0227	0.1763
	$\mu = 0.5$	0.5016	0.0016	0.0002	0.0002

The mean estimate of μ, α, and β tend to the true parameter values in Table 1 as sample sizes increases. This indicate that the parameter estimate corresponds to the first-order asymptotic theory. Also, the biases, variance and MSEs of the MLEs of the parameters estimate decreases as the sample size increases and approach zero. This implies that the normal approximation can be improved by the bias adjustments of the estimates.

TABLE 1
Continued.

Sample size	Parameter	AE	Bias	Variance	MSE
300	$\beta = 0.5$	0.0003	-0.4997	0.0000	0.2497
	$\alpha = 1.5$	1.3982	0.3982	0.0220	0.1805
	$\mu = 0.5$	0.5022	0.0022	0.0002	0.0002
350	$\beta = 1.0$	0.0003	-0.4997	0.0000	0.2497
	$\alpha = 0.5$	1.4014	0.4014	0.0216	0.1827
	$\mu = 0.5$	0.5025	0.0025	0.0002	0.0002
400	$\beta = 0.5$	0.0003	-0.4997	0.0000	0.2497
	$\alpha = 1.5$	1.4065	0.4065	0.0209	0.1861
	$\mu = 0.5$	0.5030	0.0030	0.0002	0.0002
450	$\beta = 0.5$	0.0003	-0.4997	0.0000	0.2497
	$\alpha = 1.5$	1.4084	0.4084	0.0206	0.1874
	$\mu = 0.5$	0.5032	0.0032	0.0002	0.0002
500	$\beta = 0.5$	0.0003	-0.4997	0.0000	0.2497
	$\alpha = 1.5$	1.4141	0.4141	0.0198	0.1913
	$\mu = 0.5$	0.5038	0.0038	0.0002	0.0002
800	$\beta = 1.7$	1.6710	0.4299	0.0171	0.2498
	$\alpha = 0.5$	0.4650	-0.4998	0.0000	0.2019
	$\mu = 1.5$	1.4688	0.0054	0.0002	0.0002

6. Application

The empirically the efficiency, proficiency and flexibility of the MO-G model were illustrated using real life data sets. The fits of the MO-G, Kumaraswamy Gompertz (KUGz), Weibull Gompertz (WGz), Gompertz (G), transmuted Gompertz (TGz) (Khan *et al.*, 2016b), Topp Leone Gompertz (TLGz), transmuted generalized Gompertz (TGGz) (Khan *et al.*, 2016a), transmuted Weibull (TW), beta Weibull (BW), transmuted alpha power Gompertz (TAPO-GGz), alpha power Weibull (APW) (Nassar *et al.*, 2017), alpha power inverse Weibull (APIW) (Basheer, 2019), Marshall-Olkin extended generalized Gompertz (MOEGG), lognormal Burrxii (LoGBur) and Gompertz Weibull (GW) distributions were compared.

The first data set is made up of 63 observations of the strength of 1.5cm glass fibres obtained by workers at the UK National Physical Laboratory (Smith and Naylor, 1987) as used in Mead *et al.* (2019), Abouelmagd *et al.* (2018), Efe-Eyefia *et al.* (2020), Eghwerido *et al.* (2020c), Eghwerido *et al.* (2020d), Zelibe *et al.* (2020), Bourguignon *et al.* (2014) and Afify *et al.* (2016).

The second data set were obtained from the National Highway Traffic Safety Administration on fatal accidents that occured on roads in the United States. The data represent the number of vehicle fatalities for 39 counties in South Carolina for 2012 (www-fars.nhtsa.dot.gov/States) as used in Mann (2016) .The dataset are as follow:

22, 26, 17, 4, 48, 9, 9, 31, 27, 20, 12, 6, 5, 14, 9, 16, 3, 33, 9, 20, 68, 13, 51, 13, 2, 4, 17, 16, 6, 52, 50, 48, 23, 12, 13, 10, 15, 8, 1.

The descriptive statistics of the vehicle fatalities dataset is as shown in Table 2.

TABLE 2
Descriptive statistics for the vehicle fatalities dataset.

Mean	Mode	Median	St.D	Variance	IQR	Kurtosis	Skewness	$25^{th}percent$	$75^{th}percent$	$99^{th}percent$
19.54	9.00	14.00	16.51	272.47	15.50	0.61	1.24	9.00	24.50	61.92

The following criteria Akaike Information Criteria (AIC), Bayesian Information Criteria (BIC), Consistent Akaike Information Criteria (CAIC), and Hannan and Quinn Information Criteria (HQIC), the Anderson Darling (A) statistic, Cramér-von Mises statistic (W), Kolmogorov Smirnov (KS) statistic, and the p value were also provided. The test statistics are given as follows: $HQIC = -2\hat{\ell} + 2k\log(\log(n))$, $BIC = -2\hat{\ell} + k\log(n)$, $AIC = -2\hat{\ell} + 2k$, $CAIC = -2\hat{\ell} + \frac{2kn}{n-k-1}$, where k is the number of model parameters , $\hat{\ell}$ is minus twice the maximized log-likelihood and n is the sample size.

The test statistics are provided in Tables 3 and 4. The standard errors (in parenthesis) and model parameters MLEs were also included.

TABLE 3
The statistics rating for MO-G distribution with glass fibres dataset with standard errors in parentheses.

Distribution	Parameter MLEs	AIC	CAIC	BIC	HQIC	W	A	p-value	K-S
MO-G	$\hat{\alpha}=42.396(54.218)$ $\hat{\beta}=0.650(0.585)$ $\hat{\lambda}=1.430(0.551)$	**30.511**	**30.917**	**36.940**	**33.039**	**0.077**	**0.463**	**0.559**	**0.0997**
APGz	$\hat{\alpha}=25.819(34.740)$ $\hat{\beta}=0.088(0.0677)$ $\hat{\lambda}=2.459(0.474)$	32.579	32.986	39.009	35.108	0.107	0.629	0.410	0.112
TAPO-GGz	$\hat{\alpha}=36.137(55.020)$ $\hat{\lambda}=0.799(0.227)$ $\hat{\phi}=0.06152(0.06139)$ $\hat{\beta}=2.519(0.614)$	32.858	33.548	41.431	36.230	0.087	0.514	0.405	0.494
TLGz	$\hat{\alpha}=1.671(0.590)$ $\hat{\beta}=0.020(0.018)$ $\hat{\lambda}=2.817(0.534)$	34.300	34.707	40.729	36.829	0.166	0.932	0.433	0.212
TGGz	$\hat{a}=0.850(0.200)$ $\hat{b}=0.020(0.200)$ $\hat{\alpha}=2.950(0.660)$ $\hat{\beta}=1.580(0.540)$	34.850	35.540	43.420	38.220	0.1520	0.844	0.438	0.254
TGz	$\hat{\alpha}=-0.750(0.330)$ $\hat{\beta}=0.030(0.030)$ $\hat{\lambda}=2.920(0.490)$	34.890	35.090	39.180	36.580	0.131	0.754	0.402	0.239
MOEGG	$\hat{\alpha}=73.195(53.355)$ $\hat{\beta}=1.291(0.427)$ $\hat{\lambda}=0.988(0.228)$ $\hat{\mu}=1.568(0.420)$	36.207	36.897	44.780	39.579	0.106	0.598	0.444	0.109
WGz	$\hat{a}=0.032(0.085)$ $\hat{b}=3.227(1.305)$ $\hat{\alpha}=0.833(0.566)$ $\hat{\beta}=-0.002(0.361)$	36.833	37.523	45.406	40.205	0.181	1.012	0.376	0.139
TW	$\hat{\alpha}=-0.501(0.275)$ $\hat{\beta}=0.646(0.024)$ $\hat{\lambda}=5.150(0.668)$	36.672	37.362	45.245	40.044	0.204	1.118	0.185	0.137

TABLE 3
Continued.

Distribution	Parameter MLEs	AIC	CAIC	BIC	HQIC	W	A	p-value	K-S
KUGz	$\hat{a}=1.555(0.423)$ $\hat{b}=0.214(0.048)$ $\hat{\alpha}=0.097(0.018)$ $\hat{\beta}=3.115(0.015)$	37.476	38.165	46.048	40.847	0.180	1.010	0.109	0.152
BW	$\hat{\alpha}=0.620(0.247)$ $\hat{\beta}=32.426(901.401)$ $\hat{a}=7.762(2.022)$ $\hat{b}=2.768(9.968)$	37.176	37.865	45.748	40.547	0.196	1.089	0.140	0.145
APW	$\hat{\alpha}=6.559(8.031)$ $\hat{\beta}=0.154(0.096)$ $\hat{\lambda}=4.731(0.818)$	38.181	38.587	44.610	40.709	0.175	0.963	0.330	0.119
WFr	$\hat{\alpha}=3.612(0.80)$ $\hat{\beta}=25.186(0.29)$ $\hat{a}=0.162(4.78)$ $\hat{b}=0.213(20.49)$	38.796	39.486	47.369	42.168	0.247	1.357	0.096	0.155
GW	$\hat{\alpha}=0.224(0.812)$ $\hat{\beta}=0.009(0.046)$ $\hat{a}=0.797(0.514)$ $\hat{b}=5.618(0.510)$	38.377	39.067	46.949	41.749	0.233	1.283	0.309	0.152
APIW	$\hat{\alpha}=61.099(48.144)$ $\hat{\beta}=0.775(0.164)$ $\hat{\lambda}=3.805(0.298)$	82.585	82.992	89.014	85.114	0.985	5.296	0.358	0.024
BBur	$\hat{\alpha}=15.076(69.041)$ $\hat{\beta}=36.940(98.231)$ $\hat{a}=2.052(0.638)$ $\hat{b}=0.647(0.689)$	67.331	68.021	75.904	70.703	0.706	3.857	0.309	0.152
TBur	$\hat{\alpha}=-0.917(0.128)$ $\hat{\beta}=0.576(0.134)$ $\hat{\lambda}=5.796(1.203)$	85.364	85.771	91.793	87.893	0.973	5.329	0.075	0.284
G	$\hat{\alpha}=0.510(0.005)$ $\hat{\beta}=3.626(0.347)$	141.380	141.580	145.600	143.060	0.185	0.835	0.0382	0.146

TABLE 4
The statistics rating for MO-G distribution with vehicle fatalities dataset with standard errors in parentheses.

Distribution	Parameter MLEs	AIC	CAIC	BIC	HQIC	W	A	p-value	K-S
MO-G	$\hat{\alpha}=2.937(4.188)$ $\hat{\beta}=0.095(0.093)$ $\hat{\lambda}=-0.007(0.027)$	**314.068**	**314.754**	**319.059**	**315.859**	**0.003**	**0.045**	**0.508**	**0.002**
TAPO-GGz	$\hat{\alpha}=15.170(24.371)$ $\hat{\lambda}=0.269(0.652)$ $\hat{\phi}=0.0905(0.051)$ $\hat{\beta}=-0.008(0.017)$	315.234	316.411	321.889	317.622	0.046	0.338	0.058	0.094
BW	$\hat{\alpha}=2.410(2.129)$ $\hat{\beta}=1.328(0.000)$ $\hat{a}=0.790(0.152)$ $\hat{b}=12.892(0.000)$	315.556	316.732	322.210	317.943	0.136	0.464	0.022	0.090
WGz	$\hat{a}=5.262(8.514)$ $\hat{b}=1.329(0.310)$ $\hat{\alpha}=0.014(0.015)$ $\hat{\beta}=-0.016(0.017)$	315.895	317.072	322.550	318.283	0.046	0.363	0.089	0.091
APGz	$\hat{\alpha}=0.004(0.017)$ $\hat{\beta}=0.004(0.001)$ $\hat{\lambda}=1.324(0.217)$	316.147	316.832	321.137	317.937	0.140	0.495	0.090	0.093
APW	$\hat{\alpha}=0.004(0.017)$ $\hat{\beta}=0.004(0.001)$ $\hat{\lambda}=1.324(0.217)$	316.147	316.832	321.137	317.937	0.140	0.495	0.090	0.093
TW	$\hat{\alpha}=0.422(0.502)$ $\hat{\beta}=0.039(0.010)$ $\hat{\lambda}=1.335(0.168)$	316.401	317.087	321.392	318.192	0.084	0.869	0.067	0.092
KUGz	$\hat{a}=2.208(1.542)$ $\hat{b}=0.194(0.381)$ $\hat{\alpha}=0.281(0.477)$ $\hat{\beta}=0.004(0.014)$	317.771	318.947	324.425	320.158	0.049	0.396	0.097	0.089

TABLE 4
Continued.

Distribution	Parameter MLEs	AIC	CAIC	BIC	HQIC	W	A	p-value	K-S
GW	$\hat{\alpha} = 0.006(0.004)$ $\hat{\beta} = 4.729(2.460)$ $\hat{a} = 0.277(0.580)$ $\hat{b} = 0.186(0.037)$	318.142	319.319	324.796	320.530	0.127	0.794	0.066	0.013
APIW	$\hat{\alpha} = 69.681(107.776)$ $\hat{\beta} = 4.242(1.876)$ $\hat{\lambda} = 1.246(0.139)$	319.739	320.425	324.730	321.530	0.113	0.729	0.016	0.012
WFr	$\hat{\alpha} = 0.01310.010)$ $\hat{\beta} = -0.391(0.226)$ $\hat{a} = -0.299(0.055)$ $\hat{b} = 0.008(0.003)$	321.189	322.366	327.844	323.577	0.181	1.103	0.339	0.151
LoGBur	$\hat{\alpha} = 57.620(133.078)$ $\hat{\beta} = 33.783(52.945)$ $\hat{a} = 1.644(2.575)$ $\hat{b} = 0.227(0.306)$	326.126	327.303	332.780	328.514	0.765	0.918	0.352	0.148
BBur	$\hat{\alpha} = 90.294(214.169)$ $\hat{\beta} = 78.582(223.283)$ $\hat{a} = 0.802(1.897)$ $\hat{b} = 0.180(0.233)$	326.235	327.412	332.889	328.623	0.048	0.841	0.317	0.101
MOEGG	$\hat{\alpha} = 15.658(12.228)$ $\hat{\beta} = 0.301(0.084)$ $\hat{\lambda} = -0.044(0.013)$ $\hat{\mu} = 1.534(0.285)$	328.072	329.249	334.726	330.460	0.000	0.6370	0.000	14.143
G	$\hat{\alpha} = 0.050(0.091)$ $\hat{\beta} = 0.052(0.011)$	406.104	406.437	409.431	407.298	0.291	0.367	0.099	0.010

The estimated densities of the models under consideration are shown in Figure 5. Figure 6 shows the estimated cdfs plots. These plots indicate that the MO-G distribution provides a better fit than others models considered for both data.

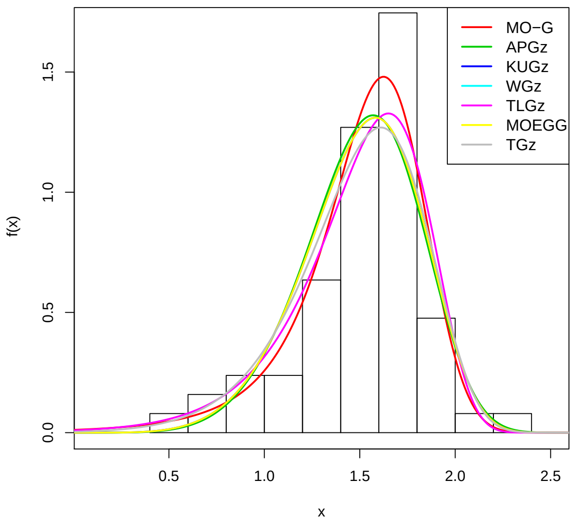


Figure 5 – The plots of estimated MO-G density.

The third data consist of the number of women breast cancer cases in the Western World Hospital as used in Khan and Khan (2018) and AbuJarad *et al.* (2020). Censored survival times are indicated as an asterisk .The data are represented as follows:

Negatively stained: 23, 47, 69, 70*, 71*, 100*, 101*, 148, 181, 198*, 208*, 212*, 224* Positively stained: 5, 8, 10, 13, 18, 24, 26, 26, 31, 35, 40, 41, 48, 50, 59, 61, 68, 71, 76*, 105*, 107*, 109*, 113, 116*, 118, 143*, 154*, 162*, 188*, 212*, 217*, 225*

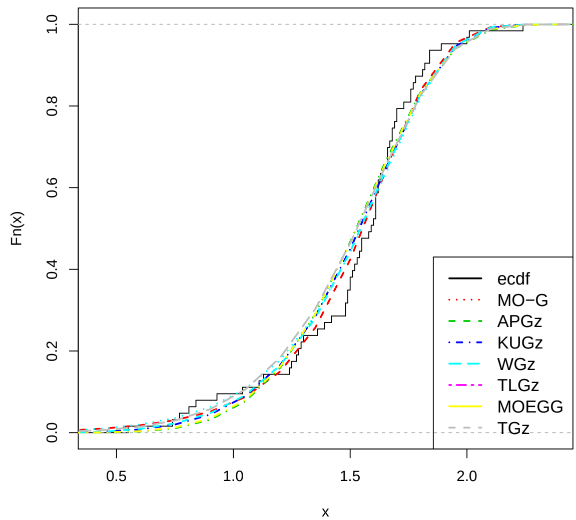


Figure 6 – The plots of estimated cdf of the MO-G model.

The Censored is denoted with `0` and uncensored is recorded as 1. The data are recorded as data in matrix form. The **`rstan`** code in `R` is shown in Appendix A. The summary results for the performance rating are shown in Table 5. In Table 5, the following abbreviations where used: posterior mean is denoted as mean, se-mean is the Monte Carlo standard errors, posterior standard deviation is denoted as std, numbers of effective sample size denoted as NE and spits is denoted as (Rhat). Figures 7 and 8 show the autocorrelation and traceplots plots for the model convergence and output for the MO-G model.

TABLE 5

Bayesian performance results of breast cancer with rstan function for MO-G Model.

	Std	Mean	se-mean	97.5 percent	75 percent	50 percent	25 percent	2.5 percent	Rhat	NE
dev	3.03	313.13	0.04	320.27	319.50	311.51	311.69	309.49	1.00	2107
Beta[1]	0.53	-1.31	0.02	-0.08	-0.48	-1.10	-1.15	-2.21	1.00	1907
Beta[0]	1.41	6.11	0.09	10.49	8.13	7.51	4.16	1.64	1.00	1004
$lp-$	1.48	-125.24	0.06	-129.51	-130.71	-131.28	-132.42	-149.55	1.00	1211
shape	416.51	17.10	5.62	55.25	5.59	0.78	0.35	0.25	1.00	2207
scale	6.10	0.65	0.11	7.26	0.77	0.31	0.10	0.02	1.00	2207

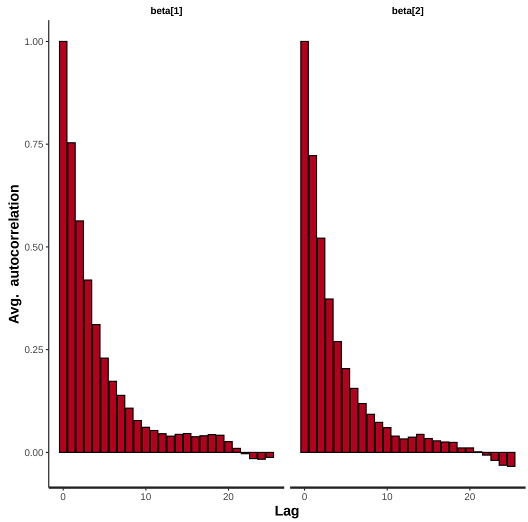


Figure 7 – The plots of the autocorrelation of the MO-G model.

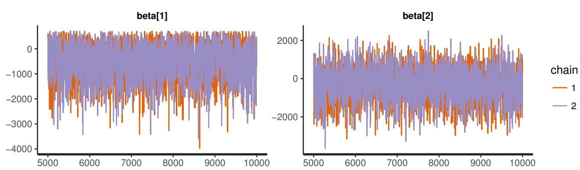


Figure 8 – The plots of trace of the MO-G model.

6.1. Discussion

The performance of a model is determined by the value that corresponds to the lowest Akaike Information Criteria (AIC). Alternatively, the model with the highest Log-likelihood value is regarded as the best model. In the two real life cases considered in this study, the MO-G distributions have the lowest AIC value in glass fibres data and vehicle fatalities data respectively. Hence, the proposed model competes favourably with other existing models for the data used.

More so, Table 5 shows the Stan results for individual and merged chains. The posterior Bayesian estimate of Beta_0 is 6.11 ± 1.41 with percentage confidence of $1.64, 10.49$ with Rhat 1.00. This implies that it is significant. Also, The posterior Bayesian estimate of Beta[1] is -1.31 ± 0.53 with percentage confidence of $-2.21, -0.08$ with . This implies that it is significant.

7. Conclusions

The concept of the MO-G distribution has been defined and studied. The mathematical properties for the pdf and cdf were carefully examined. We also derived some of the mathematical properties of the MO-G distribution among quantile and median function were established. The model parameter estimation was obtained using the maximum likelihood estimation (MLE) approach. The PWMs and entropies of the MO-G model were also derived. A simulation study of the MO-G model was also illustrated. The new distribution was applied to a real life data to examine its flexibility. It shows that the MO-G distribution performed better than, APGz, KUGz, WGz, TGz, TGGz, TLGz, WFr, APIW, APW, KUGz, BBur, LoGBur, TAPO-GGz, BW, TW, MOEGG, GW, APIW, APIW, TBur and G models.

Appendix

A. Codes

```
#pdf
f<-function(x,b,a,m){
((b*a*exp(m*x-(a/m)*(exp(m*x) - 1)))/(1 - (1 -b)
*(exp(-(a/m)*(exp(m*x)-1))))^2)
}

curve(f(x,10.8,0.5,0.3),main="",ylab="f(x)"
,xlab="x",ylim=c(0,0.8),0,10,lwd=2)
curve(f(x,1.5,1.0,0.1),lty=1,col=2,add=T,lwd=2)
curve(f(x,1.8,0.6,0.3),lty=1,col=3,add=T,lwd=2)
curve(f(x,1.7,0.4,0.3),lty=1,col=4,add=T,lwd=2)
curve(f(x,1.8,0.7,0.1),lty=1,col=5,add=T,lwd=2)
curve(f(x,5.5,1.0,0.2),lty=1,col=6,add=T,lwd=2)
curve(f(x,0.8,0.5,0.4),lty=1,col=7,add=T,lwd=2)
curve(f(x,2.7,0.4,0.5),lty=1,col=8,add=T,lwd=2)
curve(f(x,1.8,1.5,1.2),lty=1,col=10,add=T,lwd=2)
legend("topright",title=expression(""),
c(expression(beta*"=10.8,"*~alpha*"=0.5,"*~mu*"=0.3"),
expression(beta*"=1.50,"*~alpha*"=1.0,"*~mu*"=0.1"),
expression(beta*"=1.80,"*~alpha*"=0.6,"*~mu*"=0.3"),
expression(beta*"=1.70,"*~alpha*"=0.4,"*~mu*"=0.3"),
expression(beta*"=1.80,"*~alpha*"=0.7,"*~mu*"=0.1"),
expression(beta*"=5.50,"*~alpha*"=1.0,"*~mu*"=0.2"),
expression(beta*"=0.80,"*~alpha*"=0.5,"*~mu*"=0.4"),
expression(beta*"=2.70,"*~alpha*"=0.4,"*~mu*"=0.5"),
expression(beta*"=1.80,"*~alpha*"=1.5,"*~mu*"=1.2")),
```

```
cex=1.1,lty=c(1),lwd=2,col=c(1,2,3,4,5,6,7,8,10))

win.graph(width=10,height=10)
plot.new()
#hazard
f<-function(x,b,a,m){
((((b*a*exp(m*x-(a/m)*(exp(m*x) - 1)))/
(1 - (1 -b)*(exp(-(a/m)*(exp(m*x)-1))))^2) )/
(1 - (((1 - (exp(-(a/m)*(exp(m*x) - 1))))/(1 - (1 -b)
*(exp(-(a/m)*(exp(m*x)-1)))))) ))
}

curve(f(x,10.8,0.5,0.3),main="",ylab="h(x)",
xlab="x",ylim=c(0,13),0,10,lwd=2)
curve(f(x,1.5,1.0,0.1),lty=1,col=2,add=T,lwd=2)
curve(f(x,8.8,0.6,0.3),lty=1,col=3,add=T,lwd=2)
curve(f(x,1.7,0.4,0.3),lty=1,col=4,add=T,lwd=2)
curve(f(x,1.8,0.7,0.1),lty=1,col=5,add=T,lwd=2)
curve(f(x,5.5,1.0,0.2),lty=1,col=6,add=T,lwd=2)
curve(f(x,0.8,0.5,0.1),lty=1,col=7,add=T,lwd=2)
curve(f(x,2.7,0.4,0.2),lty=1,col=8,add=T,lwd=2)
curve(f(x,1.8,1.5,1.2),lty=1,col=10,add=T,lwd=2)
legend("topleft",title=expression(""),
c(expression(beta*"=10.8,"*~alpha*"=0.5,"*~mu*"=0.3"),
expression(beta*"=1.50,"*~alpha*"=1.0,"*~mu*"=0.1"),
expression(beta*"=8.80,"*~alpha*"=0.6,"*~mu*"=0.3"),
expression(beta*"=1.70,"*~alpha*"=0.4,"*~mu*"=0.3"),
expression(beta*"=1.80,"*~alpha*"=0.7,"*~mu*"=0.1"),
expression(beta*"=5.50,"*~alpha*"=1.0,"*~mu*"=0.2"),
expression(beta*"=0.80,"*~alpha*"=0.5,"*~mu*"=0.1"),
expression(beta*"=2.70,"*~alpha*"=0.4,"*~mu*"=0.2"),
expression(beta*"=1.80,"*~alpha*"=1.5,"*~mu*"=1.2")),
cex=1.1,lty=c(1),lwd=2,col=c(1,2,3,4,5,6,7,8,10))
```

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SUMMARY

This article introduces three parameters class for lifetime Poisson processes in the Marshall-Olkin transformation family that are increasing, bathtub and skewed. Some structural mathematical properties of the Marshall-Olkin Gompertz (MO-G) model were derived. The MO-G model parameters were established by maximum likelihood approach. The flexibility, efficiency, and behavior of the MO-G model estimators were examined through simulation. The empirical applicability, flexibility and proficiency of the MO-G model was scrutinized by a real-life dataset. The proposed MO-G model provides a better fit when compared to existing models in statistical literature and can serve as an alternative model to those appearing in modeling Poisson processes.

Keywords: Bayesian analysis; Gompertz failure rate; Gompertz distribution; Gompertz mortality rate; Marshall-Olkin distribution; Regression analysis.