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CHARACTERIZATION OF GENERALIZED DISTRIBUTION BY DOUBLY TRUNCATED MOMENT

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1. INTRODUCTION

The characterization of probability distribution plays an important role in statistical studies. A characterization is a certain distributional or statistical property of a statistic or statistics that uniquely determines the associated stochastic model. Distributions are characterized using the properties of sample moments, truncated moments, order statistics, record statistics, and reliability functions etc.

For the last few decades, there has been a great interest among researchers in the characterizations of probability distributions by truncated moments. The development of the general theory of the characterizations of probability distributions by truncated moment began with the work of Galambos and Kotz (1978). For further development one may refer to Kotz and Shanbhag (1980); Glanzel *et al.* (1984); Gupta (1985); Glanzel (1987, 1990); Khan and Abu-Salih (1989); Ali and Khan (1998); Su and Huang (2000); Khan and Athar (2004); Gupta and Ahsanullah (2006); Yildiz and Bairamov (2008); Ahsanullah (2009); Ahsanullah *et al.* (2016, 2017); Kilany (2017); Kilany and Hassanein (2018); Athar and Abdel-Aty (2020) and references therein.

In this study, the characterization properties based on conditional expectation of a continuous function of random variable are studied when truncation is from both the sides, left and right. The organization of paper is as follows. In Section 2, first we proved

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two propositions, then applied these results to characterize a general class of distributions $F(x) = [ah(x)+b]^c$ by doubly truncated k^{th} moment. Some of its deductions and particular cases are also discussed here. Section 3 deals with characterization of some continuous distributions like power function, Pareto, exponential and exponentiated Pareto. In Section 4, simulation study is performed to check the accuracy of characterization results. Further, in this section two real life data sets are used to demonstrate the application of proposed characterization results.

2. CHARACTERIZATION THEOREMS

PROPOSITION 1. *Let for an absolutely continuous (w.r.t) Lebesgue measure random variable X with cdf $F(x)$ and pdf $f(x)$, such that $F(\alpha) = 0$ and $F(\beta) = 1$. Suppose that $f'(x)$ and $E[\xi(X)|x \leq X \leq y]$ exist, where $\xi(x)$ is a continuous and differentiable function of x. If*

$$E[\xi(X)|x \leq X \leq y] = g(x,y)\eta(x,y), \tag{1}$$

where, $g(x,y)$ is differentiable function of $x, y \in (\alpha, \beta)$ and $\eta(x,y) = \frac{f(y)}{F(y)-F(x)}$ then

$$f(y) = K_1 \exp\left\{\int \frac{\xi(y) - \frac{\partial}{\partial y} g(x,y)}{g(x,y)} dy\right\}, \tag{2}$$

where K_1 can be determined using the relation

$$\int_{\alpha}^{\beta} f(y)dy = 1.$$

PROOF. We know that

$$E[\xi(X)|x \leq X \leq y] = \frac{1}{F(y)-F(x)} \int_x^y \xi(u)f(u)du.$$

Therefore,

$$\frac{1}{F(y)-F(x)} \int_x^y \xi(u)f(u)du = \frac{g(x,y)f(y)}{F(y)-F(x)},$$

which implies

$$\int_x^y \xi(u)f(u)du = g(x,y)f(y). \tag{3}$$

Differentiating both the sides $w.r.t\ y$, we have

$$\xi(y)f(y) = g(x,y)f'(y) + f(y)\frac{\partial}{\partial y} g(x,y),$$

which on simplification gives

$$\frac{f'(y)}{f(y)}=\frac{\xi(y)-\frac{\partial}{\partial y}g(x,y)}{g(x,y)}. \tag{4}$$

Now integrate the above expression to get (2). Hence the proof. □

PROPOSITION 2. *Under the conditions as stated in Proposition 1. If*

$$E[\xi(X)|x\le X\le y]=g(x,y)\nu(x,y), \tag{5}$$

where, $g(x,y)$ *is differentiable function of* $x,y\in(\alpha,\beta)$ *and* $\nu(x,y)=\frac{f(x)}{F(y)-F(x)}$ *then*

$$f(x)=K_2\ \exp\left\{-\int\frac{\xi(x)-\frac{\partial}{\partial x}g(x,y)}{g(x,y)}dx\right\}, \tag{6}$$

where K_2 *can be determined using the relation*

$$\int_{\alpha}^{\beta}f(x)dx=1.$$

PROOF. In view of (5), we have

$$\int_x^y\xi(u)f(u)du=g(x,y)f(x). \tag{7}$$

Differentiating Eq. (7) $w.r.t.\ x$, we get

$$\frac{f'(x)}{f(x)}=-\frac{\xi(x)+\frac{\partial}{\partial x}g(x,y)}{g(x,y)}. \tag{8}$$

Now integrate (8) $w.r.t.\ x$ to get (6) which establishes the proposition. □

LEMMA 3. *For any positive integers a, b, c and j*

$$\frac{\sum_{j=0}^{c-1}\binom{c-1}{j}(c-j-1)a^{c-j}\,b^j\,h^{c-j-2}(y)}{\sum_{j=0}^{c-1}\binom{c-1}{j}a^{c-j}\,b^j\,h^{c-j-1}(y)}=\frac{a(c-1)}{ah(y)+b}. \tag{9}$$

PROOF. The *LHS* of (9) can also be expressed as

$$LHS=\frac{a(c-1)\sum_{j=0}^{c-2}\binom{c-2}{j}a^{c-j-2}b^j h^{c-j-2}(y)}{\sum_{j=0}^{c-1}\binom{c-1}{j}a^{c-j-1}b^j h^{c-j-1}(y)}$$

$$= \frac{a(c-1)[ah(y)+b]^{c-2}}{[ah(y)+b]^{c-1}}$$
$$= \frac{a(c-1)}{ah(y)+b}.$$

Hence the lemma. □

THEOREM 4. *Suppose an absolutely continuous (w.r.t. Lebesgue measure) random variable X has cdf $F(x)$ and pdf $f(x)$ with $F(\alpha)=0$ and $F(\beta)=1$. Further, if $f'(x)$ and $E(X^k|x \le X \le y)$ exist for every x and y, $\alpha \le x < y \le \beta$. Then for a continuous and twice differentiable function $h(.)$ and $k=1,2,...$*

$$E[X^k|x \le X \le y] = g(x,y)\eta(x,y), \tag{10}$$

where, $\eta(x,y) = \frac{f(y)}{F(y)-F(x)}$ *and*

$$g(x,y) = \frac{\sum_{j=0}^{c-1}\binom{c-1}{j}\frac{a^{c-j}b^j}{(c-j)}\left\{y^k h^{c-j}(y) - x^k h^{c-j}(x) - k\int_x^y u^{k-1}h^{c-j}(u)du\right\}}{\sum_{j=0}^{c-1}\binom{c-1}{j}a^{c-j}b^j h^{c-j-1}(y)h'(y)} \tag{11}$$

if and only if

$$F(y) = \left[ah(y)+b\right]^c, a \neq 0, y \in (\alpha,\beta). \tag{12}$$

PROOF. Necessary part: We have

$$g(x,y) = \frac{\int_x^y \xi(u)f(u)du}{f(y)}.$$

For the cdf given in (12), we have the pdf given as

$$f(y) = F'(y) = c\sum_{j=0}^{c-1}\binom{c-1}{j}a^{c-j}b^j h^{c-j-1}(y)h'(y).$$

Now consider $\xi(u) = u^k$, then we have

$$g(x,y) = \frac{\sum_{j=0}^{c-1}\binom{c-1}{j}a^{c-j}b^j \int_x^y u^k h^{c-j-1}(u)h'(u)du}{\sum_{j=0}^{c-1}\binom{c-1}{j}a^{c-j}b^j h^{c-j-1}(y)h'(y)}.$$

Now integrating the above expression by parts, we get the required result.

To prove sufficiency part, in view of Eq. (4), we have

$$\frac{f'(y)}{f(y)} = \frac{y^k - \frac{\partial}{\partial y}g(x,y)}{g(x,y)}. \tag{13}$$

Suppose $g(x,y) = A(y) - B(x,y) - C(x,y)$, where

$$A(y) = \frac{y^k \sum_{j=0}^{c-1} \binom{c-1}{j} \frac{a^{c-j} b^j}{(c-j)} h^{c-j}(y)}{\sum_{j=0}^{c-1} \binom{c-1}{j} a^{c-j} b^j h^{c-j-1}(y) h'(y)},$$

$$B(x,y) = \frac{x^k \sum_{j=0}^{c-1} \binom{c-1}{j} \frac{a^{c-j} b^j}{(c-j)} h^{c-j}(x)}{\sum_{j=0}^{c-1} \binom{c-1}{j} a^{c-j} b^j h^{c-j-1}(y) h'(y)},$$

$$C(x,y) = \frac{k \sum_{j=0}^{c-1} \binom{c-1}{j} \frac{a^{c-j} b^j}{(c-j)} \int_x^y u^{k-1} h^{c-j}(u) du}{\sum_{j=0}^{c-1} \binom{c-1}{j} a^{c-j} b^j h^{c-j-1}(y) h'(y)}.$$

Then

$$\frac{\partial}{\partial y} g(x,y) = y^k - g(x,y) \left\{ \frac{h''(y)}{h'(y)} + \frac{h'(y) \sum_{j=0}^{c-1} \binom{c-1}{j} (c-j-1) a^{c-j} b^j h^{c-j-2}(y)}{\sum_{j=0}^{c-1} \binom{c-1}{j} a^{c-j} b^j h^{c-j-1}(y)} \right\}. \quad (14)$$

Now using value of $\frac{\partial}{\partial y} g(x,y)$ in Eq. (13), we get

$$\frac{f'(y)}{f(y)} = \frac{h''(y)}{h'(y)} + \frac{h'(y) \sum_{j=0}^{c-1} \binom{c-1}{j} (c-j-1) a^{c-j} b^j h^{c-j-2}(y)}{\sum_{j=0}^{c-1} \binom{c-1}{j} a^{c-j} b^j h^{c-j-1}(y)}. \quad (15)$$

Now on application of (9) in (15), we get

$$\frac{f'(y)}{f(y)} = \frac{h''(y)}{h'(y)} + \frac{a(c-1)h'(y)}{ah(y)+b}. \quad (16)$$

This implies

$$F(y) = [ah(y)+b]^c, \;\; y \in (\alpha, \beta).$$

Hence the required result. □

COROLLARY 5. *Let X be continuous random variable with cdf $F(x)$ and pdf $f(x)$ for $\alpha < x < \beta$. Further, if $f'(x)$ and $E(X^k | x \le X \le y)$ exist for all $x, y \in (\alpha, \beta)$, then for $k = 1, 2, \ldots$*

$$E[X^k | x \le X \le y] = g(x,y) \eta(x,y), \quad (17)$$

where, $\eta(x,y) = \frac{f(y)}{F(y)-F(x)}$ *and*

$$g(x,y) = \frac{y^k h(y) - x^k h(x) - k \int_x^y u^{k-1} h(u) du}{h'(y)} \quad (18)$$

if and only if

$$F(y) = ah(y) + b, a \neq 0, y \in (\alpha, \beta). \tag{19}$$

PROOF. Corollary can be established at $c = 1$ from (12). □

COROLLARY 6. *Let Y be continuous random variable with cdf $F(y)$ and pdf $f(y)$ for $\alpha < y < \beta$. Then for $k = 1, 2, \ldots$*

$$E[Y^k | Y \leq y] = g(y)\phi(y), \tag{20}$$

where, $\phi(y) = \frac{f(y)}{F(y)}$ and

$$g(y) = \frac{y^k h(y) + \frac{\alpha^k b}{a} - k \int_\alpha^y u^{k-1} h(u) du}{h'(y)} \tag{21}$$

if and only if

$$F(y) = ah(y) + b, a \neq 0, y \in (\alpha, \beta). \tag{22}$$

PROOF. The corollary can be proved easily if $x \to \alpha$ and $c = 1$ in Theorem 4. □

This result is also proved by Athar and Abdel-Aty (2020).

COROLLARY 7. *Under the condition as stated in Theorem 4*

$$E[h(X) | x \leq X \leq y] = g(x, y)\eta(x, y) = \frac{h(x) + h(y)}{2} \tag{23}$$

where,

$$g(x, y) = \frac{h^2(y) - h^2(x)}{2h'(x)}$$

and

$$\eta(x, y) = \frac{h'(x)}{h(y) - h(x)}$$

if and only if

$$F(x) = ah(x) + b, a \neq 0, x \in (\alpha, \beta). \tag{24}$$

PROOF. To prove necessary part, we have

$$g(x, y) = \frac{\int_x^y h(u) f(u) du}{f(x)}.$$

For the distribution given in (24), we have $f(x) = ah'(x)$.

Therefore,

$$g(x,y)=\frac{\int_x^y h(u)ah'(u)du}{ah'(x)}=\frac{h^2(y)-h^2(x)}{2h'(x)}.$$

Further, since for the distribution given in (24)

$$\eta(x,y)=\frac{f(x)}{F(y)-F(x)}=\frac{h'(x)}{h(y)-h(x)}.$$

This gives

$$E[h(X)|x\le X\le y]=\frac{h(x)+h(y)}{2}.$$

Hence the necessary part.

To prove sufficiency part, we have

$$g(x,y)=\frac{h^2(y)}{2h'(x)}-\frac{h^2(x)}{2h'(x)}. \tag{25}$$

Differentiate $g(x,y)$ partially $w.r.t$ x to get

$$\frac{\partial}{\partial x}g(x,y)=\frac{h^2(x)h''(x)}{2(h'(x))^2}-\frac{h^2(y)h''(x)}{2(h'(x))^2}-h(x).$$

Now using relation (8) with $\xi(x)=h(x)$, we get

$$\frac{f'(x)}{f(x)}=-\frac{h(x)+\frac{\partial}{\partial x}g(x,y)}{g(x,y)}=\frac{h''(x)}{h'(x)}.$$

This implies

$$F(x)=ah(x)+b,\ \ x\in(\alpha,\beta).$$

Hence the sufficiency part. □

REMARK 8. *Similar result was also established by Balasubramanian and Beg (1992); Khan and Athar (2004) in terms of order statistics, which is given as*

$$E\Big[h(X_{r+1:n})|X_{r:n}=x,X_{r+2}=y\Big]=E\Big[h(x)|x\le X\le y\Big]=\frac{h(x)+h(y)}{2}.$$

3. EXAMPLES

In this section characterization of some well known distributions like power function, Pareto, exponential and exponentiated Pareto based on Theorem 4 are presented.

3.1. Power function distribution

COROLLARY 9. *Let a random variable X has cdf $F(x)$ and pdf $f(x)$. Further, if $f'(x)$ and $E(X^k|x \le X \le y)$ exists for every $x, y \in (0,1)$. Then for $k = 1, 2, \ldots$*

$$E[X^k|x \le X \le y] = g(x,y)\eta(x,y), \tag{26}$$

where

$$g(x,y) = \frac{y^{k+p} - x^{k+p}}{(k+p)y^{p-1}}$$

and

$$\eta(x,y) = \frac{py^{p-1}}{y^p - x^p}$$

if and only if

$$F(y) = y^p, p > 0, y \in (0,1). \tag{27}$$

PROOF. On comparison of (27) with (19), we get $a = 1$, $h(y) = y^p$ and $b = 0$.

Thus, result follows from Corollary 5. □

REMARK 10. *Under condition as stated in Corollary 9 with $x \to 0$, we get the result for right truncated moment*

$$E[X^k|X \le y] = g(y)\eta(y) = \frac{py^k}{k+p}$$

with

$$g(y) = \frac{y^{k+1}}{k+p} \text{ and } \eta(y) = \frac{f(y)}{F(y)} = \frac{p}{y}$$

if and only if

$$F(y) = y^p,\ y \in (0,1), p > 0.$$

This result is also obtained by Athar and Abdel-Aty (2020).

3.2. Pareto distribution

COROLLARY 11. *Suppose a random variable X has an absolutely continuous cdf $F(x)$ and pdf $f(x)$. Further, assume that $f'(x)$ and $E(X^k|x \le X \le y)$ exist for every $x, y \in (1, \infty)$. Then for $k = 1, 2, \ldots$*

$$E[X^k|x \le X \le y] = g(x,y)\eta(x,y), \tag{28}$$

where

$$g(x,y) = \frac{y^{k-p} - x^{k-p}}{(k-p)y^{-(p+1)}}$$

and

$$\eta(x,y) = \frac{px^p}{y^{p+1} - x^p y}$$

if and only if

$$F(y) = 1 - y^{-p}, p > 1, y \in (1,\infty). \tag{29}$$

PROOF. Result can be seen in view of Corollary 5 with $a = -1$, $h(y) = y^{-p}$ and $b = 1$ in (19). □

REMARK 12. *Under condition as stated in Corollary 11 with $x \to 1$, we get the result for right truncated moment*

$$E[X^k | X \le y] = g(y)\eta(y) = \frac{p}{k-p}\frac{y^k - y^p}{y^p - 1}$$

with

$$g(y) = \frac{y^{k+1} - y^{p+1}}{k-p} \; and \; \eta(y) = \frac{f(y)}{F(y)} = \frac{py^{-(p+1)}}{1 - y^{-p}}$$

if and only if

$$F(y) = 1 - y^{-p}, \; y \in (1,\infty), p > 1.$$

Similar result is also obtained by Ahsanullah *et al.* (2016) and Athar and Abdel-Aty (2020).

3.3. *Exponential distribution*

COROLLARY 13. *Suppose a continuous random variable X has cdf $F(x)$ and pdf $f(x)$. Further, assume that $f'(x)$ and $E(X^k | x \le X \le y)$ exist for every $x, y \in (0,\infty)$. Then for $k = 1, 2, \ldots$*

$$E[X^k | x \le X \le y] = g(x,y)\eta(x,y), \tag{30}$$

where

$$g(x,y) = -\frac{y^k}{\theta} + \frac{x^k e^{-\theta(x-y)}}{\theta} + \frac{k}{\theta e^{-\theta y}} \int_x^y u^{k-1} e^{-\theta u} du$$

and

$$\eta(x,y) = \frac{\theta e^{-\theta y}}{e^{-\theta x} - e^{-\theta y}}$$

if and only if

$$F(y) = 1 - e^{-\theta y}, \theta > 0, y \in (0,\infty). \tag{31}$$

PROOF. On comparison of (31) with (19), we get $a=-1,\ h(y)=e^{-\theta y}$ and $b=1$.

Therefore, Corollary 13 can be established in view of Corollary 5. □

3.4. *Exponentiated Pareto distribution*

COROLLARY 14. *Under the conditions as stated in Corollary 11.*

$$E[X^k|x \leq X \leq y] = g(x,y)\eta(x,y), \tag{32}$$

where

$$g(x,y) = \frac{\sum_{j=0}^{[\theta-1]}(-1)^{\theta-j+1}\binom{[\theta-1]}{j}\frac{1}{k-p(\theta-j)}\left(y^{k-p(\theta-j)} - x^{k-p(\theta-j)}\right)}{\sum_{j=0}^{[\theta-1]}(-1)^{\theta-j+1}\binom{[\theta-1]}{j}y^{p(j-\theta)-1}}$$

and

$$\eta(x,y) = \frac{p\theta y^{-p-1}(1-y^{-p})^{\theta-1}}{(1-y^{-p})^{\theta}-(1-x^{-p})^{\theta}}$$

if and only if

$$F(y) = (1-y^{-p})^{\theta}, p>1,\ \theta>0, y\in(1,\infty). \tag{33}$$

PROOF. On comparison of (33) with (12), we get $a=-1,\ h(y)=y^{-p},\ b=1$ and $c=\theta$. Thus, corollary can be establish in view of Theorem 4. □

Similarly, with proper choice of a, b, c and $h(x)$ several distribution can be characterized using Theorem 4. For more distributions belonging to this class, one may refer to Khan and Abu-Salih (1989); Khan and Abouammoh (2000) and Athar and Akhter (2015).

4. APPLICATIONS

In this section, we provided numerical illustration to validate the accuracy of theoretical characterization results first via simulation and then applying these result to the real data sets. For these purposes we have considered power function, Pareto, exponential and exponentiated Pareto (Expo Pareto) distributions as examples of our main results. The different random left and right truncation points are chosen using random number generator and values between them are estimated. Finally, MSE are observed to trace out which distribution gives the best fit to the data. `R software` (R Core Team, 2020) has been used for computation.

The data set I given below represents the failure times of 50 components (per 1000h). For previous study on this data set see Merovei *et al.* (2020).

0.036	0.058	0.061	0.074	0.078	0.086	0.102	0.103	0.114	0.116
0.148	0.183	0.192	0.254	0.262	0.379	0.381	0.538	0.570	0.574
0.590	0.618	0.645	0.961	1.228	1.600	2.006	2.054	2.804	3.058
3.076	3.147	3.625	3.704	3.931	4.073	4.393	4.534	4.893	6.274
6.816	7.896	7.904	8.022	9.337	10.940	11.020	13.880	14.730	15.080

The values in above data set I do not lie in the interval $[0,1]$ for power function distribution. Therefore, the original values are divided by the maximum value (15.080) of the data set and transform them in the interval of $[0,1]$. Similarly, for Pareto and exponentiated Pareto distributions, original values are divided by minimum value (0.036) of the data set and transform them in the interval of $(1,\infty)$.

TABLE 1
Verification of the characterization results.

Distribution	Parameters	k	(x,y)	L.H.S.	R.H.S.	$\left\lvert\frac{L.H.S.-R.H.S.}{R.H.S.}\right\rvert$
Power Function	$p=0.5$	1	(0.00, 0.92)	0.33	0.33	0
	$p=1.0$	2	(0.02, 0.27)	0.03	0.03	0
	$p=1.5$	3	(0.04, 0.19)	0.00	0.00	0
	$p=2.0$	4	(0.02, 0.29)	0.00	0.00	0
Pareto	$p=2.0$	1	(2.83, 222.83)	5.60	5.60	0
	$p=2.5$	2	(10.58, 100.69)	379.81	379.81	0
	$p=5.0$	3	(17.17, 77.89)	12039.90	12039.90	0
	$p=8.0$	4	(7.06, 122.03)	4957.46	4957.46	0
Exponential	$\theta=0.5$	1	(0.08, 10.94)	2.03	2.03	0
	$\theta=1.0$	2	(0.26, 4.07)	2.05	2.05	0
	$\theta=1.5$	3	(0.62, 2.80)	2.76	2.76	0
	$\theta=2.0$	4	(0.25, 4.39)	2.34	2.34	0
Expo Pareto	$p=1.5,\theta=2$	1	(3.22, 189.33)	8.76	8.76	0
	$p=3.0,\theta=5$	2	(5.08, 135.92)	75.17	75.17	0
	$p=4.5,\theta=7$	3	(15.94, 85.44)	11185.20	11185.20	0
	$p=5.0,\theta=10$	4	(5.33, 125.94)	3875.80	3875.80	0

A simulation study is performed on data set I to validate the accuracy of theoretical characterization results for power function, Pareto, exponential and exponentiated Pareto distributions given in Corollary 9, Corollary 11, Corollary 13 and Corollary 14 respectively, which are considered as the examples of main characterization result of general form of distributions. In Table 1, it is observed that absolute relative difference between two sides of the characterizing Eqs. (28), (30), (32) and (33) is zero, which validates or confirms the accuracy of characterization results.

In Table 2, the original estimated values in column 5 for power function distribution is obtained by multiplying the transformed estimated value by 15.080 and for Pareto

TABLE 2
Application of characterization results.

Distribution	Parameters	(x, y)	X	$\hat{X}$	$(X-\hat{X})$	$(X-\hat{X})^2$	MSE
Power function	$p=0.5$	(0.004, 0.005)	0.061	0.068	-0.007	0.000	0.004
		(0.007, 0.008)	0.114	0.113	0.001	0.000	
		(0.524, 0.532)	7.904	7.962	-0.058	0.003	
	$p=1.0$	(0.005, 0.007)	0.086	0.091	-0.005	0.000	
		(0.036, 0.038)	0.570	0.558	0.012	0.000	
		(0.081, 0.133)	1.600	1.614	-0.014	0.000	
	$p=1.5$	(0.002, 0.004)	0.058	0.047	0.011	0.000	
		(0.039, 0.043)	0.618	0.618	0.000	0.000	
		(0.532, 0.725)	9.337	9.516	-0.179	0.032	
Pareto	$p=1.2$	(1.611, 2.056)	0.061	0.065	-0.004	0.000	0.001
		(2.861, 3.222)	0.114	0.109	0.005	0.000	
		(219.333, 222.833)	7.904	7.959	-0.055	0.003	
	$p=1.5$	(2.167, 2.833)	0.086	0.089	-0.003	0.000	
		(14.944, 15.944)	0.570	0.556	0.014	0.000	
		(34.111, 55.722)	1.600	1.544	0.056	0.003	
	$p=2.0$	(1.000, 1.694)	0.058	0.045	0.013	0.000	
		(16.389, 17.917)	0.618	0.616	0.002	0.000	
		(222.833, 303.889)	9.337	9.257	0.080	0.006	
Exponential	$\theta=0.5$	(0.058, 0.074)	0.061	0.066	-0.005	0.000	0.053
		(0.103, 0.116)	0.114	0.110	0.004	0.000	
		(7.896, 8.022)	7.904	7.958	-0.054	0.003	
	$\theta=1.0$	(0.078, 0.102)	0.086	0.090	-0.004	0.000	
		(0.538, 0.574)	0.570	0.556	0.014	0.000	
		(1.228, 2.006)	1.600	1.567	0.033	0.001	
	$\theta=1.5$	(0.036, 0.061)	0.058	0.048	0.010	0.000	
		(0.590, 0.645)	0.618	0.617	0.001	0.000	
		(8.022, 10.940)	9.337	8.652	0.685	0.470	
Expo Pareto	$p=1.5$, $\theta=2$	(1.611, 2.056)	0.061	0.066	-0.005	0.000	0.003
		(2.861, 3.222)	0.114	0.109	0.005	0.000	
		(219.333, 222.833)	7.904	7.959	-0.055	0.003	
	$p=2.0$, $\theta=3$	(2.167, 2.833)	0.086	0.089	-0.003	0.000	
		(14.944, 15.944)	0.570	0.555	0.015	0.000	
		(34.111, 55.722)	1.600	1.524	0.076	0.006	
	$p=2.5$, $\theta=4$	(1.000, 1.694)	0.058	0.052	0.006	0.000	
		(16.389, 17.917)	0.618	0.616	0.002	0.000	
		(222.833, 303.889)	9.337	9.220	0.117	0.014	

and exponentiated Pareto distributions by 0.036. Further, it is observed that the MSE of Pareto distribution is least possible among others. Thus, Pareto distribution gives the best bit for the above given data Set I.

The data set II given below represents the life of fatigue fracture of Kevlar 373/epoxy that are subjected to constant pressure at the 90% stress level until all had failed. For earlier studies on this data set one can refer to Barlow *et al.* (1984); Andrew and Herzberg (1985); Gillariose and Tomy (2020).

0.025	0.089	0.089	0.250	0.311	0.345	0.476	0.565	0.567	0.657
0.675	0.675	0.675	0.770	0.838	0.839	0.843	0.865	0.885	0.911
0.912	0.984	1.048	1.060	1.077	1.173	1.257	1.277	1.299	1.321
1.350	1.355	1.460	1.488	1.573	1.573	1.708	1.726	1.746	1.763
1.775	1.828	1.838	1.850	1.881	1.888	1.888	1.932	1.956	2.005
2.041	2.090	2.109	2.133	2.210	2.246	2.288	2.320	2.347	2.351
2.495	2.526	2.991	3.026	3.268	3.405	3.485	3.743	3.746	3.914
4.807	5.401	5.444	5.530	6.554	9.096				

In the above data set II again it can be noticed that the values do not lie in the interval $[0, 1]$ for power function distribution. Therefore, the original values are divided by the maximum value (9.096) in the data set and transform them in the interval of $[0, 1]$. Similarly, for Pareto and exponentiated Pareto distributions, original values are divided by minimum value (0.025) in the data set and transform them in the interval of $(1, \infty)$.

In Table 3, the original estimated values in column 5 for power function distribution is obtained by multiplying the transformed estimated value by 9.096 and for Pareto and exponentiated Pareto distributions by 0.025. Further, it is also observed that out of four considered distributions MSE is least in the case of power function distribution. Thus, power function distribution gives the best bit for the above given data Set II.

TABLE 3
Application of characterization results.

Distribution	Parameters	(x, y)	X	$\hat{X}$	$(X-\hat{X})$	$(X-\hat{X})^2$	MSE
Power Function	$p = 0.5$	(0.028, 0.038)	0.311	0.297	0.014	0.000	0.007
		(0.203, 0.208)	1.881	1.868	0.013	0.000	
		(0.594, 0.608)	5.444	5.465	-0.021	0.000	
	$p = 1.0$	(0.062, 0.072)	0.567	0.611	-0.044	0.002	
		(0.215, 0.224)	2.005	1.998	0.007	0.000	
		(0.278, 0.333)	2.991	2.776	0.215	0.046	
	$p = 1.5$	(0.192, 0.195)	1.763	1.761	0.002	0.000	
		(0.383, 0.412)	3.743	3.616	0.127	0.016	
		(0.594, 0.608)	5.444	5.465	-0.020	0.000	
Pareto	$p = 1.2$	(9.964, 13.749)	0.311	0.292	0.019	0.000	0.008
		(73.717, 75.211)	1.881	1.869	0.012	0.000	
		(215.159, 220.299)	5.444	5.464	-0.020	0.000	
	$p = 1.5$	(22.510, 26.159)	0.567	0.608	-0.041	0.002	
		(77.920, 81.307)	2.005	1.998	0.007	0.000	
		(100.638, 120.542)	2.991	2.757	0.234	0.055	
	$p = 2.0$	(69.562, 70.701)	1.763	1.760	0.003	0.000	
		(138.829, 149.223)	3.743	3.610	0.133	0.018	
		(215.159, 220.299)	5.444	5.464	-0.020	0.000	
Exponential	$\theta = 0.5$	(0.250, 0.345)	0.311	0.297	0.014	0.000	0.009
		(1.850, 1.888)	1.881	1.884	-0.003	0.000	
		(5.401, 5.530)	5.444	5.464	-0.020	0.000	
	$\theta = 1.0$	(0.565, 0.657)	0.567	0.610	-0.043	0.002	
		(1.956, 2.041)	2.005	1.998	0.007	0.000	
		(2.526, 3.026)	2.991	2.755	0.236	0.056	
	$\theta = 1.5$	(1.746, 1.775)	1.763	1.760	0.003	0.000	
		(3.485, 3.746)	3.743	3.607	0.136	0.018	
		(5.401, 5.530)	5.444	5.463	-0.019	0.000	
Expo Pareto	$p = 1.3$, $\theta = 3$	(9.964, 13.749)	0.311	0.292	0.019	0.000	0.009
		(73.717, 75.211)	1.881	1.869	0.012	0.000	
		(215.159, 220.299)	5.444	5.464	-0.020	0.000	
	$p = 2.5$, $\theta = 4$	(22.510, 26.159)	0.567	0.607	-0.040	0.002	
		(77.920, 81.307)	2.005	1.997	0.008	0.000	
		(100.638, 120.542)	2.991	2.750	0.241	0.058	
	$p = 3$, $\theta = 5$	(69.562, 70.701)	1.763	1.760	0.003	0.000	
		(138.829, 149.223)	3.743	3.609	0.134	0.018	
		(215.159, 220.299)	5.444	5.464	-0.020	0.000	

5. Conclusion

The characterization of probability distribution has significant contribution in statistical studies. It is used to check whether the proposed model fits the requirement of given probability distribution or not. The characterization using truncated moments limits the observations and hence researchers may save their time and cost. The same is observed through numerical illustration to the natural data. The proposed characterization result may be useful for the researchers, who are in the field of natural and allied sciences.

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Appendix

Supplementary information

```
#Power Function Distribution
p = 1.5; k = 1; x = 0.039; y = 0.043
#Calculation of LHS
A = p/(y^p - x^p)
f<-function(t){t^(k+p-1)}
B<-integrate(f,lower=x,upper=y,stop.on.error = FALSE)
C=B$value
lhs=A*C
#-----------------------------------------------------
#Calculation of RHS
A1 = p/(k + p)
B1 = (y^(k + p) - x^(k + p))/(y^p - x^p)
rhs = A1*B1
#-----------------------------------------------------
lhs
rhs
======================================================
#Pareto Distribution
p = 2.0; k = 1; x = 222.833; y = 303.889
#Calculation of LHS
A = p/(x^(-p) - y^(-p))
```

```
f<-function(t){t^(k-p-1)}
B<-integrate(f,lower=x,upper=y,stop.on.error = FALSE)
C=B$value
lhs=A*C
#-----------------------------------------------------
#Calculation of RHS
A1 = p/(k - p)
B1 = (x^p*y^k - x^k*y^p)/(y^p - x^p)
rhs = A1*B1
#-----------------------------------------------------
lhs
rhs
======================================================
#Exponential Distribution
th = 0.5; k = 1; x = 0.103; y = 0.116
#Calculation of LHS
A = th/(exp(-th*x)-exp(-th*y))
f<-function(t){t^(k)*exp(-th*t)}
B<-integrate(f,lower=x,upper=y,stop.on.error = FALSE)
C=B$value
lhs=A*C
#----------------------------------------------------
#Calculation of RHS
A1=-(y^k/th)
B1=((x^k)*exp(-th*(x-y)))/th
C1=k/(th*exp(-th*y))
f1<-function(u){u^(k-1)*exp(-th*u)}
D1<-integrate(f1,lower=x,upper=y,stop.on.error = FALSE)
E1=D1$value
gxy=A1+B1+C1*E1
nxy=(th*exp(-th*y))/(exp(-th*x)-exp(-th*y))
rhs=gxy*nxy
#----------------------------------------------------
lhs
rhs
=====================================================
#Exponentiated Pareto Distribution
p=3.0;th=5;k=1;x=215.1594;y=220.2988
#Calculation of LHS
f<-function(t){t^(k-p-1)*(1-t^(-p))^(th-1)}
B<-integrate(f,lower=x,upper=y,stop.on.error = FALSE)
C=B$value
A = (p*th)/((1 - y^(-p))^th - (1 - x^(-p))^th)
```

```
A
C
lhs=A*C
lhs
#------------------------------------------------------
#Calculation of RHS
sumn=0;sumd=0
for (j in 0:th-1){
a=(-1)^(th-j+1)
b=choose(th-1,j)
c=1/(k-p*(th-j))
sumn=sumn+a*b*c*(y^(k-p*(th-j))-x^(k-p*(th-j)))
sumd=sumd+a*b*y^(p*(j-th)-1)
}
sumn;sumd
gxy=sumn/sumd
a1=p*th*y^(-p-1)*(1-y^(-p))^(th-1)
b1=(1-y^(-p))^th-(1-x^(-p))^th
nxy=a1/b1
gxy;nxy
rhs=gxy*nxy
rhs
#------------------------------------------------------
lhs
rhs
======================================================
# r sample without replacement from vector
> sample (c(1:50), size=36, replace=F)
```

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SUMMARY

In this paper characterization properties based on conditional expectation of a continuous function of random variable are studied when truncation is from both the sides, left and right. Then, these results are applied to obtain the k-th doubly truncated moment for a general class of distribution. Further, some examples and particular cases based on this general class of distributions are also demonstrated. The results are obtained in simple and explicit manner which also unifies the earlier results obtained by several authors. In the end, simulation study is performed to validate the correctness of theoretical characterization results and then two real life data sets are used to demonstrate the applications of these results.

Keywords: Truncated moment; Characterization; Pareto distribution; Power function distribution; Exponential distribution; Exponentiated Pareto distribution.