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ON INDUCED GENERALIZED RECORD RANKED SET SAMPLING AND ITS ROLE IN BIVARIATE MODEL BUILDING

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SUMMARY

A new variety of Ranked Set Sampling (RSS), namely Induced Generalized Record Ranked Set Sampling (IGRRSS), is introduced. In the proposed methodology, ranking is implemented by considering generalized (k) record values on the auxiliary variable X from each sequence of units. The selected units are further screened for measuring the variable of primary interest Y. Further, we propose estimators based on IGRRSS for the unknown parameters associated with the variable Y when the parent bivariate distribution belongs to the Morgenstern family of distributions. The proposed sampling scheme is utilized to collect primary data on the usable timber volume Y based on the ranking of units by generalized (2) record values on tree height X of acacia trees. Accordingly, Morgenstern type bivariate logistic distribution has been modelled for the distribution of the population random vector (X, Y) and estimated the average usable timber volume of the population.

Keywords: Generalized (k) record values; Record ranked set sampling; Induced generalized upper record ranked set sampling (IGURRSS); Induced generalized lower record ranked set sampling; Concomitants of generalized record values; Morgenstern family of distributions; Morgenstern type bivariate logistic distribution; Best linear unbiased estimation; Modelling bivariate distribution by IGURRSS data.

1. INTRODUCTION

The concept of ranked set sampling (RSS) was introduced by McIntyre (1952) as a process of increasing the precision of the sample mean as an estimator of the population mean. McIntyre's method of RSS consists of choosing randomly n^2 units, arranging them randomly in n sets of n units each, ranking the units in each set by a judgment method, choosing the i-th ranked unit from the i-th set and making a measurement

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of the variable of interest on the selected unit for $i = 1, 2, \ldots, n$. For detailed discussion about the development and applications of RSS see Stokes (1977, 1995), Patil *et al.* (1994), Lam *et al.* (1994, 1996),Patil (2002), Chen *et al.* (2004), Modarres and Zheng (2004), Wolfe (2004), Zheng and Modarres (2006), Priya and Thomas (2013, 2016), and Paul and Thomas (2017).

It is to be noted that any imperfect ranking of the units by judgment ranking in RSS can have the effect that the statistics constructed from them incurs large mean square error. Stokes (1977) has introduced another RSS in which an easily and inexpensively measurable auxiliary variable X is used for ranking the units and for identifying appropriate units for measuring the variable Y of primary interest, provided X and Y are jointly distributed with a bivariate probability density function (pdf) $f(x, y)$. The advantage of this method of RSS is that no ranking error occurs in this case, and the resulting observations are exactly distributed as concomitants of order statistics in which the information on the ranks of the observations on the auxiliary variable are captured and impounded probabilistically in those distributions. Chen *et al.* (2004) have portrayed a detailed account of applications of RSS as devised by Stoke's (1977). For some recent applications of this RSS see also Muttlak and McDonald (1990), Kaur *et al.* (1996), Muttlak (1998), Sinha (2005), Chacko and Thomas (2007, 2008), Lesitha *et al.* (2010), Lesitha and Thomas (2013), Thomas *et al.* (2014), and Philip and Thomas (2015).

It may be noted that the construction of record values of a sequence of observations does severe filtering in the sequence and thereby produces a significantly reduced set of units consisting of the most sensitive and extraordinary members of the original sequence. One may refer to Paul and Thomas (2017) in which classical record values on an easily measurable auxiliary variable are used to rank the units and to define a suitable RSS. But a difficulty one encounters in using the data resulting from the RSS defined by Paul and Thomas (2017) to statistical inference problems regards their limited occurrence, as the expected values of interarrival times of records is infinite Glick (1978). However, the k-th record values, as introduced by Dziubdziela and Kopocinski (1976), occur more frequently than those of the classical records. Those records were conveniently called generalized (k) record values in Minimol and Thomas (2013, 2014), Paul and Thomas (2013, 2015, 2016), Paul (2014), Thomas *et al.* (2014) and Thomas and Paul (2014). As the above authors, we call k-th record values of Dziubdziela and Kopocinski (1976) as generalized (k) record values all through this paper.

Initially, it seems that some disinterest was developed among statisticians to deal with generalized upper (k) record values (GURVs) by the message conveyed by the fact that the distributions of GURVs arising from a distribution with cumulative distribution function (cdf) $F(x)$ is the same as the distribution of classical upper record values arising from the distribution with cdf $1-(1-F(x))^k$ (for details see Arnold *et al.*, 1998, pp. 43-44). However, Paul and Thomas (2015) pointed out that a characterization result based on classical record values arising from $1-(1-F(x))^k$ cannot become a characterization result for $F(x)$ or vice versa. Another advantage in utilizing GURVs is that for a given value of k, the process of constructing GURVs evolves with an inbuilt system to eliminate the $k-1$ number of probable large values in the data, which may turn out to be outliers. Paul and Thomas (2015) have also pointed out cases when estimates based on GURVs are more efficient than those based on classical record values. Thus GURVs are useful to address certain problems more efficiently than addressing them by classical record values.

It is interesting to note that Salehi and Ahmadi (2014) have defined record ranked set sampling (RRSS), in which n different sequences of units from a population are considered, an inexpensive mechanism such as a judgment method is used to determine the record value sequence on the judgment scores of each of the n sequences formed, and from the i-th sequence, that unit corresponding to the i-th record value (with respect to judgment scores) is chosen for actual measurement of the variable of primary interest for $i = 1, 2, \ldots, n$. The RRSS, defined by Salehi and Ahmadi (2014), also suffers due to imperfect ranking, as in the case of McIntyre's RSS.

In this paper we consider a situation when an experimenter is facing the following constraints:

C_1 : the sampling is too costly or painful, so that strong economic considerations in sampling become necessary,

C_2 : requirement for ranking the units by depending on an easily measurable auxiliary variable X that is jointly distributed with the variable Y of primary interest,

C_3 : requirement for designing a mechanism to keep away from the sample some fixed number of possible outliers (of high value) on X observations so as to check the outliers on Y as well in an indirect way,

C_4 : requirement of capturing in the sample the most valuable units that are different from the suspected outliers on X directly (and hence on Y indirectly).

Under the above constraints, we aim to define an RSS similar to Stokes (1977), but it uses the units with generalized upper (k) record values on the auxiliary variable X for further screening the units.

Suppose (X_i, Y_i), $i = 1, 2, \ldots$ is a sequence of independent and identically distributed (iid) bivariate random variables each distributed identically as the random variable (X, Y), which has an absolutely continuous cdf $F_{X,Y}(x, y)$ and pdf $f_{X,Y}(x, y)$. Let $F_X(x)$ and $f_X(x)$, respectively, be the marginal cdf and pdf of X, and $F_Y(y)$ and $f_Y(y)$, respectively, be the marginal cdf and pdf of Y. Now we consider the marginal sequence of random variables X_i, $i = 1, 2, \ldots$. Then for a positive integer $k \geq 1$, the sequence of generalized upper (k) record times $\{T_{U(n,k)}, n \geq 1\}$ of $\{X_i\}$ is defined as (see Nevzorov, 2001, p. 82)

$$T_{U(1,k)} = k$$

and

$$T_{U(n+1,k)} = \min\{j : j > T_{U(n,k)}, X_j > X_{T_{U(n,k)}-k+1:T_{U(n,k)}}\} \quad \text{for } n \geq 1,$$

where $X_{i:m}$ denotes the i-th order statistic in a sample of size m. Now, if we write

$$X_{U(n,k)} = X_{T_{U(n,k)}-k+1:T_{U(n,k)}} \quad \text{for } n = 1,2,\ldots,$$

then $\{X_{U(n,k)}\}$ is known as the sequence of the generalized upper (k) record values. Now we can identify the ordered pair (X,Y) from the sequence (X_i,Y_i), $i=1,2,\ldots$ in which $X = X_{U(n,k)}$, so that the Y component in that ordered pair is denoted by $Y_{U[n,k]}$ and is known as the concomitant of the n-th generalized upper (k) record value. If we write $f_{Y|X}(y|x)$ to denote the conditional pdf of Y given $X = x$, then the pdf of $Y_{U[n,k]}$ is given by (for more details see, Chacko and Mary, 2013)

$$f_{Y_{U[n,k]}}(y) = \frac{k^n}{\Gamma(n)} \int_{-\infty}^{\infty} [-\ln\{1-F_X(x)\}]^{n-1}[1-F_X(x)]^{k-1} f_{Y|X}(y|x) f_X(x)\mathrm{d}x. \tag{1}$$

Similarly, the joint pdf of $Y_{U[m,k]}$ and $Y_{U[n,k]}$ for $m<n$ is then given by

$$f_{Y_{U[m,k]},Y_{U[n,k]}}(y_1,y_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{x_2} f_{Y|X}(y_1|x_1) f_{Y|X}(y_2|x_2) f_{X_{U(m,k)},X_{U(n,k)}}(x_1,x_2)\mathrm{d}x_1\mathrm{d}x_2,$$

where

$$\begin{aligned} f_{X_{U(m,k)},X_{U(n,k)}}(x_1,x_2) &= \frac{k^n\left\{-\ln[\bar{F}_X(x_1)]\right\}^{m-1}}{\Gamma(m)\Gamma(n-m)}\left\{\ln[\bar{F}_X(x_1)-\ln[\bar{F}_X(x_2)]]\right\}^{n-m-1} \\ &\quad \times[\bar{F}_X(x_2)]^{k-1}\frac{f_X(x_1)f_X(x_2)}{\bar{F}_X(x_1)}, \end{aligned} \tag{2}$$

for $x_1 < x_2$, $1 \le m < n$ and $n \ge 2$, where $\bar{F}_X(t) = 1 - F_X(t)$.

This paper is organized as follows. Section 2 introduces the newly proposed IGURRSS. The general estimation theory of location and scale parameters of the marginal distribution of the variable of primary interest of any bivariate Morgenstern family of distributions using the observations of IGURRSS is discussed in Section 3. In Section 4 we consider a specific member of the Morgenstern family of distributions viz. Morgenstern type bivariate logistic distribution (MTBLD) to illustrate the applications of the results derived in Section 3. We have devoted Section 5 to demonstrate the newly proposed IGRRSS method to a real life problem, and we have further utilized the collected observations by this new sampling scheme to identify the parental bivariate model. Further, we obtain the estimate of the average timber volume of the population of trees surveyed.

2. Induced Generalized Upper Record Ranked Set Sampling

Suppose we have n sequences of independent units drawn from an infinite population. The interest is in carrying out a study on a variable Y of primary interest whose measurement on the units is comparatively expensive or time consuming whereas measurement of an auxiliary variable X is quite easy, under constraints C_1 to C_4 as stated in the introduction. To deal with this problem, we define a new sampling method namely Induced Generalized Record Ranked Set Sampling below.

Definition 1. *Draw n sequences of independent units from a population with Y as the variable of primary interest and X as an auxiliary variable on the units such that X and Y are jointly distributed. We consider a situation in which measurement of X on the units is inexpensive and easy while measuring Y on the units is difficult and costly. Make a measurement of X on the units of each sequence and thereby use those observations within each sequence to construct the sequence of generalized upper (k) record values. Now from the i-th sequence of generalized upper (k) record values, select the unit corresponding to the i-th generalized upper (k) record value on X and make measurement on the variable Y of primary interest on this unit. Let the resulting measurement be denoted by $Y_{U[i,k]i}$ for $i = 1, 2, \ldots, n$. Then the observations $Y_{U[1,k]1}, Y_{U[2,k]2}, \ldots, Y_{U[n,k]n}$ taken together is called the induced generalized upper record ranked set sample (**igurrss**). The sampling strategy that yields this sample is called Induced Generalized Upper Record Ranked Set Sampling (IGURRSS).*

Remark 2. *If we construct the generalized lower (k) record values of marginal X observations, and define the analogous ranked set sampling of Definition 1, then the obtained sample is known as the induced generalized Lower Record Ranked Set Sample (**iglrrss**). The sampling strategy used in this case is called Induced Generalized Lower Record Ranked Set Sampling $(IGLRRSS)$.*

If $f_{X,Y}(x, y)$ is the joint pdf of the population bivariate random vector (X, Y) with marginal pdf $f_X(x)$ for X and marginal pdf $f_Y(y)$ for Y, then it is of interest to note that the i-th observation $Y_{U[i,k]i}$ generated in IGURRSS is distributed as the concomitant of i-th generalized upper (k) record values (on the variable X), and its pdf is denoted by $f_{Y_{U[i,k]i}}(y)$, which can be obtained from Eq. (1) putting $n = i$.

As in the case of the mean of observations of the RSS defined by McIntyre (1952), the mean of the observations of IGURRSS as defined in Definition 1 may not estimate as such the parameter corresponding to $E(Y)$ of the variable of primary interest. However it is interesting to note that if the distribution of the parent random vector (X, Y) belongs to the Morgenstern family of distributions with pdf

$$f_{X,Y}(x,y) = f_X(x) f_Y(y)\{1+\gamma[1-2F_X(x)][1-2F_Y(y)]\}, \quad -1 \le \gamma \le 1, \tag{3}$$

then from Chacko and Mary (2013) we have

$$f_{Y_{U[i,k]i}}(y) = f_Y(y)\left[1+\gamma\left(1-2\left(\frac{k}{(k+1)}\right)^i\right)\{2F_Y(y)-1\}\right], \quad (4)$$

and consequently we obtain the asymptotic form of the mean $\frac{\sum_{i=1}^{n} f_{U[i,k]i}(y)}{n}$ of pdf's $f_{U[i,k]i}(y)$, $i=1,2,\ldots,n$ as follows. From Eq. (4) we have

$$\frac{\sum_{i=1}^{n} f_{U[i,k]i}(y)}{n} = f_Y(y)+\gamma f_Y(y)(2F_Y(y)-1)-\frac{2\gamma k}{n}\left[1-\left(\frac{k}{k+1}\right)^n\right](2F_Y(y)-1).$$

Then the limiting form of the above mean function is given by

$$\begin{aligned}\lim_{n\to\infty}\frac{\sum_{i=1}^{n} f_{U[i,k]i}(y)}{n} &= f_Y(y)[1+\gamma(2F_Y(y)-1)]\\ &= \{1-\gamma\}f_Y(y)+\gamma f_{2:2}(y).\end{aligned}$$

Clearly, the right side of the above equation is again a pdf, which seems to be a modified mixture of the pdf's $f_Y(y)$ and $f_{2:2}(y)$ (as the range of γ is $-1<\gamma<1$), where $f_{2:2}(y)=2F_Y(y)f_Y(y)$ is the pdf of the largest order statistic of a random sample of size two arising from the marginal distribution of Y.

3. Estimation of Some Parameters of Morgenstern Type Bivariate Distribution Using IGURRSS

Suppose a population of units is such that on each unit the measurement on an auxiliary variable X can be made easily, whereas measurement on the variable Y of primary interest is somewhat difficult or expensive. Suppose (X,Y) is such that X and Y are distributed marginally with cdf's $F_X\left(\frac{x-\theta_1}{\lambda_1}\right)$ and $F_Y\left(\frac{y-\theta_2}{\lambda_2}\right)$, respectively, and jointly distributed as a Morgenstern type distribution given by the cdf

$$\begin{aligned}F_{X,Y}(x,y;\theta_1,\theta_2,\lambda_1,\lambda_2,\gamma) &= F_X\left(\frac{x-\theta_1}{\lambda_1}\right)F_Y\left(\frac{y-\theta_2}{\lambda_2}\right)\{1+\gamma\\ &\quad\times\left[1-F_X\left(\frac{x-\theta_1}{\lambda_1}\right)\right]\left[1-F_Y\left(\frac{y-\theta_2}{\lambda_2}\right)\right]\Bigg\}, \quad (5)\end{aligned}$$

where $-\infty<x<\infty$, $-\infty<y<\infty, -\infty<\theta_i<\infty$, $\lambda_i>0, i=1,2$, $-1\le\gamma\le 1$. Clearly, γ is the association parameter, θ_1,λ_1 are the location and scale parameters, respectively, of $F_X\left(\frac{x-\theta_1}{\lambda_1}\right)$, and θ_2,λ_2 are the location and scale parameters, respectively, of $F_Y\left(\frac{y-\theta_2}{\lambda_2}\right)$. The joint pdf corresponding to the cdf defined in Eq. (5) is given by,

$$f_{X,Y}(x,y)=\frac{f_X\left(\frac{x-\theta_1}{\lambda_1}\right)f_Y\left(\frac{y-\theta_2}{\lambda_2}\right)}{\lambda_1\lambda_2}\left\{1+\gamma\left[1-2F_X\left(\frac{x-\theta_1}{\lambda_1}\right)\right]\left[1-2F_Y\left(\frac{y-\theta_2}{\lambda_2}\right)\right]\right\}. \tag{6}$$

The conditional distribution $f_{Y|X}(y|x)$ of the random variable Y given $X=x$ is then given by

$$f_{Y|X}(y|x)=\frac{f_Y\left(\frac{y-\theta_2}{\lambda_2}\right)}{\lambda_2}\left\{1+\gamma\left[1-2F_X\left(\frac{x-\theta_1}{\lambda_1}\right)\right]\left[1-2F_Y\left(\frac{y-\theta_2}{\lambda_2}\right)\right]\right\}. \tag{7}$$

The Morgenstern bivariate family of distributions is a well known and extensively large class of bivariate distributions, which are characterized by their marginal distributions. This family is further well known for its fine properties on the distributions of concomitants of order statistics or concomitants of records or concomitants of generalized records arising from it. For more details see Johnson *et al.* (1994), Veena and Thomas (2008), Thomas and Veena (2014), and Thomas *et al.* (2014). As measurement on X can be carried out inexpensively on any number of units while there is much difficulty in the measurement of the characteristic Y of interest, we are constrained to impose an observational economy consideration on Y, and hence we apply Definition 1 to generate an ***igurrss*** with observations $Y_{U[1,k]1}, Y_{U[2,k]2}, \ldots, Y_{U[n,k]n}$. Then, from Eq. (4), the pdf of the i-th ***igurrss*** observation $Y_{U[i,k]i}$ is given by

$$\begin{aligned} f_{Y_{U[i,k]i}}(y)=\int f_{Y|X}(y|x)\, f_{X_{U(i,k)}}(x)\,dx \quad &= \quad \frac{f_Y\left(\frac{y-\theta_2}{\lambda_2}\right)}{\lambda_2}\left[1+\gamma\left(1-2\left(\frac{k}{k+1}\right)^i\right)\right. \\ &\quad \left.\times\left\{2F_Y\left(\frac{y-\theta_2}{\lambda_2}\right)-1\right\}\right]. \end{aligned} \tag{8}$$

If we put

$$W_{U[i,k]i}=\frac{Y_{U[i,k]i}-\theta_2}{\lambda_2}, \tag{9}$$

then the pdf of $W_{U[i,k]i}$ is given by

$$\begin{aligned} f_{W_{U[i,k]i}}(w) &= f_W(w)\left[1+\gamma\left(1-2\left(\frac{k}{k+1}\right)^i\right)\{2F_W(w)-1\}\right] \\ &= f_W(w)+\gamma\left(1-2\left(\frac{k}{k+1}\right)^i\right)[2f_W(w)F_W(w)-f_W(w)] \\ &= f_W(w)+\gamma\left(1-2\left(\frac{k}{k+1}\right)^i\right)[f_{2:2}(w)-f_W(w)], \end{aligned} \tag{10}$$

where $f_{2:2}(w)$ denotes the density function of the second order statistic $W_{2:2}$ of a random sample of size 2 arising from $f_W(w)$ of $W = \frac{Y-\theta_2}{\lambda_2}$. Then for $i = 1,2,\ldots,n$, from Eq. (10) we have,

$$E\left[W_{U[i,k]i}\right] = \zeta_i, \tag{11}$$

where

$$\zeta_i = \alpha + \gamma\left(1-2\left(\frac{k}{k+1}\right)^i\right)[\alpha_{2:2}-\alpha], \tag{12}$$

in which α and $\alpha_{2:2}$ denote the expectations of W and $W_{2:2}$, respectively. Also if we write $\alpha^{(2)} = E(W^2)$ and $\alpha^{(2)}_{2:2} = E(W^2_{2:2})$, then for $i = 1,2,\ldots,n$ we have

$$E\left[W^2_{U[i,k]i}\right] = \alpha^{(2)} + \gamma\left(1-2\left(\frac{k}{k+1}\right)^i\right)\left[\alpha^{(2)}_{2:2}-\alpha^{(2)}\right]. \tag{13}$$

From Eq. (11) to Eq. (13) we have for $i = 1,2,\ldots,n$,

$$\mathrm{Var}\left[W_{U[i,k]i}\right] = \beta_i, \tag{14}$$

where

$$\beta_i = \alpha^{(2)} + \frac{\zeta_i - \alpha}{\alpha_{2:2}-\alpha}\left(\alpha^{(2)}_{2:2}-\alpha^{(2)}\right) - \zeta_i^2. \tag{15}$$

Clearly, ζ_i and β_i for $i = 1,2,\ldots,n$ are free from the parameters provided γ is known. Since for $i \neq j$, $Y_{U(i,k)i}$ and $Y_{U(j,k)j}$ are independently distributed (as they arise from different independent sequences), it follows that

$$\mathrm{Cov}\left[W_{U[i,k]i}, W_{U[j,k]j}\right] = 0. \tag{16}$$

Thus, if we write $\boldsymbol{Y}_{U[n,k]} = \left(Y_{U[1,k]1}, Y_{U[2,k]2}, \ldots, Y_{U[n,k]n}\right)'$, then from Eq. (9) and Eq. (11) to (16), the mean vector $E\left[\boldsymbol{Y}_{U[n,k]}\right]$ and dispersion matrix $D\left[\boldsymbol{Y}_{U[n,k]}\right]$ of $\boldsymbol{Y}_{U[n,k]}$ are given by

$$E\left[\boldsymbol{Y}_{U[n,k]}\right] = \theta_2\mathbf{1} + \lambda_2\boldsymbol{\zeta}, \tag{17}$$

$$D\left[\boldsymbol{Y}_{U[n,k]}\right] = \lambda_2^2\boldsymbol{G}, \tag{18}$$

where $\boldsymbol{\zeta} = (\zeta_1, \zeta_2, \ldots, \zeta_n)'$, $\boldsymbol{G}$=diag$(\beta_1, \beta_2, \ldots, \beta_n)$, and $\mathbf{1}$ is a column vector of n ones. Now if γ is known, then the elements of $\boldsymbol{\zeta}$ and $\boldsymbol{G}$ are known real numbers. Consequently, Equations (17) and (18) together define a generalized Gauss-Markov setup, and hence the best linear unbiased estimators (BLUEs) $\theta^*_{2,k}$ and $\lambda^*_{2,k}$ of θ_2 and λ_2 are obtained as

$$\theta^*_{2,k} = D^{-1}\left[\boldsymbol{\zeta}'\boldsymbol{G}^{-1}\left(\boldsymbol{\zeta}\mathbf{1}' - \mathbf{1}\boldsymbol{\zeta}'\right)\boldsymbol{G}^{-1}\right]\boldsymbol{Y}_{U[n,k]} \tag{19}$$

and

$$\lambda^*_{2,k} = D^{-1}\left[\mathbf{1}'\mathbf{G}^{-1}\left(\mathbf{1}\zeta' - \zeta\mathbf{1}'\right)\mathbf{G}^{-1}\right]\mathbf{Y}_{U[n,k]}, \tag{20}$$

with variances given by

$$\mathrm{Var}(\theta^*_{2,k}) = \lambda_2^2\left(\zeta'\mathbf{G}^{-1}\zeta\right)/D \tag{21}$$

and

$$\mathrm{Var}(\lambda^*_{2,k}) = \lambda_2^2\left(\mathbf{1}'\mathbf{G}^{-1}\mathbf{1}\right)/D, \tag{22}$$

where $D = \left(\zeta'\mathbf{G}^{-1}\zeta\right)\left(\mathbf{1}'\mathbf{G}^{-1}\mathbf{1}\right) - \left(\zeta'\mathbf{G}^{-1}\mathbf{1}\right)^2$. The expressions in Equations (19) and (20) can be further simplified as

$$\theta^*_{2,k} = \sum_{r=1}^{n}\left\{\frac{\beta_r^{-1}\left(\sum_{i=1}^{n}\zeta_i^2\beta_i^{-1}\right) - \zeta_r\beta_r^{-1}\left(\sum_{i=1}^{n}\zeta_i\beta_i^{-1}\right)}{\left(\sum_{i=1}^{n}\beta_i^{-1}\right)\left(\sum_{i=1}^{n}\zeta_i^2\beta_i^{-1}\right) - \left(\sum_{i=1}^{n}\zeta_i\beta_i^{-1}\right)^2}\right\} Y_{U[r,k]r} \tag{23}$$

and

$$\lambda^*_{2,k} = \sum_{r=1}^{n}\left\{\frac{\zeta_r\beta_r^{-1}\left(\sum_{i=1}^{n}\beta_i^{-1}\right) - \beta_r^{-1}\left(\sum_{i=1}^{n}\zeta_i\beta_i^{-1}\right)}{\left(\sum_{i=1}^{n}\beta_i^{-1}\right)\left(\sum_{i=1}^{n}\zeta_i^2\beta_i^{-1}\right) - \left(\sum_{i=1}^{n}\zeta_i\beta_i^{-1}\right)^2}\right\} Y_{U[r,k]r}. \tag{24}$$

Clearly, the estimators given in Equations (23) and (24) are linear in $Y_{U[r,k]r}$ for $r = 1, 2, \ldots, n$, and can be written as

$$\theta^*_{2,k} = \sum_{r=1}^{n} a_{r,n,k} Y_{U[r,k]r}, \tag{25}$$

$$\lambda^*_{2,k} = \sum_{r=1}^{n} b_{r,n,k} Y_{U[r,k]r}, \tag{26}$$

where $a_{r,n,k}$ and $b_{r,n,k}$, $r = 1, 2, \ldots, n$ are appropriate constants defined from Equations (23) and (24), respectively. The variances given in Equations (21) and (22) can also be simplified as

$$\mathrm{Var}(\theta^*_{2,k}) = \frac{\sum_{i=1}^{n}\zeta_i^2\beta_i^{-1}}{\left(\sum_{i=1}^{n}\beta_i^{-1}\right)\left(\sum_{i=1}^{n}\zeta_i^2\beta_i^{-1}\right) - \left(\sum_{i=1}^{n}\zeta_i\beta_i^{-1}\right)^2}\lambda_2^2 \tag{27}$$

and

$$\mathrm{Var}(\lambda^*_{2,k}) = \frac{\sum_{i=1}^{n}\beta_i^{-1}}{\left(\sum_{i=1}^{n}\beta_i^{-1}\right)\left(\sum_{i=1}^{n}\zeta_i^2\beta_i^{-1}\right) - \left(\sum_{i=1}^{n}\zeta_i\beta_i^{-1}\right)^2}\lambda_2^2. \tag{28}$$

REMARK 3. *In real life applications, IGURRSS as explained in Definition 1 has its own limitations when used with large n, as one may not be successful to realize many generalized (k) records on the measurements of the auxiliary variable from a given sequence of units of the population. Hence the **igurrss** size n is usually chosen small. However, if one insists in more precision of the **igurrss** based estimates θ_2^* and λ_2^* as given in Equations (23) and (24), respectively, then repeated cycles of IGURRSS are recommended. Thus, if N repeated cycles of IGURRSS are carried out, and the estimates for $\theta_{2,k}$ and $\lambda_{2,k}$ based on the j-th **igurrss** are denoted by $\theta^*_{2,k,j}$ and $\lambda^*_{2,k,j}$, respectively, for $j = 1,2,\ldots,N$, then the improved estimates of θ_2 and λ_2 are defined by $\tilde{\theta}_{2,k} = \frac{\sum_{j=1}^{N}\theta^*_{2,k,j}}{N}$ and $\tilde{\lambda}_{2,k} = \frac{\sum_{j=1}^{N}\lambda^*_{2,k,j}}{N}$, respectively, with $Var(\tilde{\theta}_{2,k}) = \frac{Var(\theta^*_{2,k})}{N}$ and $Var(\tilde{\lambda}_{2,k}) = \frac{Var(\lambda^*_{2,k})}{N}$, where $Var(\theta^*_{2,k})$ and $Var(\lambda^*_{2,k})$ are defined as in Equations (27) and (28), respectively. The choice of N depends on the desired extent of the precision of the estimates $\tilde{\theta}_{2,k}$ and $\tilde{\lambda}_{2,k}$.*

In the following subsections we further deal with two special cases of the problem discussed above. Subsection 3.1 deals with the BLUE of θ_2 when λ_2 is known. Subsection 3.2 deals with the BLUE of λ_2 when θ_2 is known.

3.1. *Case I: Estimation of θ_2 when λ_2 is known*

Now, we consider the problem of obtaining the BLUE of θ_2 based on the observations of an ***igurrss*** arising from an arbitrary distribution belonging to the Morgenstern family of distributions for the case, when λ_2 and γ are known. For convenience we take $\lambda_2 = 1$. If $Y_{U[i,k]i}, i = 1,2,\ldots,n$ are the observations of the ***igurrss*** arising from Eq. (6) for $\lambda_2 = 1$, then we have $E\left[Y_{U[i,k]i}\right] = \theta_2 + \zeta_i$ and

$$E\left[Z_{U[i,k]i}\right] = \theta_2, \tag{29}$$

where $Z_{U[i,k]i} = Y_{U[i,k]i} - \zeta_i$. Also we have

$$\text{Var}\left[Z_{U[i,k]i}\right] = \beta_i, \tag{30}$$

for $i = 1,2,\ldots,n$, and

$$\text{Cov}\left[Z_{U[i,k]i}, Z_{U[j,k]j}\right] = 0, \quad i \neq j, \tag{31}$$

where ζ_i and β_i are as defined in Equations (12) and (15), respectively.

Let $\mathbf{Z}_{U[n,k]} = \left(Z_{U[1,k]1}, Z_{U[2,k]2}, \ldots, Z_{U[n,k]n}\right)'$. Then, from Eq. (29) and Eq. (31), we can write the mean vector and dispersion matrix of $\mathbf{Z}_{U[n,k]}$ as

$$E\left[\mathbf{Z}_{U[n,k]}\right] = \theta_2 \mathbf{1}$$

and

$$D\left[Z_{U[n,k]}\right] = G,$$

where $\mathbf{1}$ is a column vector of n ones and $\boldsymbol{G}$=diag$(\beta_1,\beta_2,\ldots,\beta_n)$. If γ is known, then the equations for $E\left[Z_{U[n,k]}\right]$ and $D\left[Z_{U[n,k]}\right]$ together define a generalized Gauss-Markov set up, and consequently the BLUE $\theta_{2,k}^{(0)}$ of θ_2 is given by

$$\theta_{2,k}^{(0)} = \left(\mathbf{1}'\boldsymbol{G}^{-1}\mathbf{1}\right)^{-1}\mathbf{1}'\boldsymbol{G}^{-1}\boldsymbol{Z}_{U[n,k]} \tag{32}$$

and

$$\mathrm{Var}(\theta_{2,k}^{(0)}) = \left(\mathbf{1}'\boldsymbol{G}^{-1}\mathbf{1}\right)^{-1}. \tag{33}$$

Clearly, from Eq. (32) and Eq. (33) we can further write

$$\theta_{2,k}^{(0)} = \left(\sum_{i=1}^{n}\beta_i^{-1}\right)^{-1}\sum_{r=1}^{n}\beta_r^{-1}Z_{U[r,k]r} = \sum_{r=1}^{n}c_{r,n,k}Z_{U[r,k]r}, \tag{34}$$

where $c_{r,n,k}$, $r=1,2,\ldots,n$ are appropriate constants and

$$\mathrm{Var}(\theta_{2,k}^{(0)}) = \left(\sum_{i=1}^{n}\beta_i^{-1}\right)^{-1}. \tag{35}$$

3.2. *Case II: Estimation of λ_2 when θ_2 is known*

Suppose θ_2 involved in the Morgenstern family of bivariate distributions with pdf in Eq. (6) is known. Then, for convenience, we write $\theta_2=0$. Now, we obtain the BLUE $\lambda_{2,k}^{(0)}$ of λ_2, when γ is known. Let $Y_{U[i,k]i}, i=1,2,\ldots,n$, be the observations of the ***igurrss*** drawn from Eq. (6) for $\theta_2=0$. Let $\boldsymbol{Y}_{U[n,k]}=\left(Y_{U[1,k]1},Y_{U[2,k]2},\ldots,\ Y_{U[n,k]n}\right)'$. Then, from Eq. (10) and Eq. (11), for $\theta_2=0$, the mean vector and dispersion matrix of $\boldsymbol{Y}_{U[n,k]}$ are given by

$$E\left[\boldsymbol{Y}_{U[n,k]}\right] = \lambda_2\zeta, \tag{36}$$

$$D\left[\boldsymbol{Y}_{U[n,k]}\right] = \lambda_2^2\boldsymbol{G}, \tag{37}$$

where $\zeta=(\zeta_1,\zeta_2,\ldots,\zeta_n)'$ and $\boldsymbol{G}$=diag$(\beta_1,\beta_2,\ldots,\beta_n)$ in which ζ_r and β_r are defined as in Eq. (12) and Eq. (15), respectively. If γ is known, then Equations (36) and (37) together define a generalized Gauss-Markov set up, and hence the BLUE $\lambda_{2,k}^{(0)}$ of λ_2 is obtained as

$$\lambda_{2,k}^{(0)} = \left(\zeta'\boldsymbol{G}^{-1}\zeta\right)^{-1}\zeta'\boldsymbol{G}^{-1}\boldsymbol{Y}_{U[n,k]} \tag{38}$$

with

$$\mathrm{Var}(\lambda_{2,k}^{(0)}) \;=\; \left(\zeta' G^{-1}\zeta\right)^{-1}\lambda_2^2. \tag{39}$$

The above expressions can be further simplified to

$$\lambda_{2,k}^{(0)} \;=\; \left(\sum_{i=1}^{n}\zeta_i^2\beta_i^{-1}\right)^{-1}\sum_{r=1}^{n}\zeta_r\beta_r^{-1}Y_{U[r,k]r} = \sum_{r=1}^{n} d_{r,n,k}Y_{U[r,k]r}, \tag{40}$$

where $d_{r,n,k}$, $r = 1, 2, \ldots, n$ are appropriate constants and

$$\mathrm{Var}(\lambda_{2,k}^{(0)}) \;=\; \left(\sum_{i=1}^{n}\zeta_i^2\beta_i^{-1}\right)^{-1}\lambda_2^2. \tag{41}$$

The expression for the BLUEs of the parameters associated with the distribution of the variable of primary interest for various cases presented in this section based on the observations of an ***igurrss*** have much importance, since these results are applicable as such to a very large class of bivariate distributions called Morgenstern family of bivariate distributions.

4. Estimation of Parameters of MTBLD

A bivariate random vector (X, Y) is said to have a MTBLD if its cdf is of the form (see, Kotz *et al.*, 2000)

$$F_{X,Y}(x,y) = \frac{1}{\left(1+e^{-\frac{(x-\theta_1)}{\lambda_1}}\right)}\frac{1}{\left(1+e^{-\frac{(y-\theta_2)}{\lambda_2}}\right)}\left[1+\gamma\frac{e^{-\frac{(x-\theta_1)}{\lambda_1}}e^{-\frac{(y-\theta_2)}{\lambda_2}}}{\left(1+e^{-\frac{(x-\theta_1)}{\lambda_1}}\right)\left(1+e^{-\frac{(y-\theta_2)}{\lambda_2}}\right)}\right], \tag{42}$$

where $-\infty < x < \infty$, $-\infty < y < \infty, -\infty < \theta_i < \infty$, $\lambda_i > 0, i = 1, 2$ and $-1 \leq \gamma \leq 1$. The pdf $f_{X,Y}(x,y)$ corresponding to the cdf of Eq. (42) is then given by

$$f_{X,Y}(x,y) = \frac{e^{-\frac{(x-\theta_1)}{\lambda_1}}}{\lambda_1\left\{1+e^{-\frac{(x-\theta_1)}{\lambda_1}}\right\}^2}\frac{e^{-\frac{(y-\theta_2)}{\lambda_2}}}{\lambda_2\left\{1+e^{-\frac{(y-\theta_2)}{\lambda_2}}\right\}^2}\left[1+\gamma\frac{\left\{1-e^{-\frac{(x-\theta_1)}{\lambda_1}}\right\}\left\{1-e^{-\frac{(y-\theta_2)}{\lambda_2}}\right\}}{\left\{1+e^{-\frac{(x-\theta_1)}{\lambda_1}}\right\}\left\{1+e^{-\frac{(y-\theta_2)}{\lambda_2}}\right\}}\right]. \tag{43}$$

Lam *et al.* (1996) considered the problem of estimating the parameters θ_2 and λ_2 of a logistic distribution using ranked set sampling. They also improved these estimators by identifying units having maximum information. Chacko and Thomas (2009) have derived the BLUEs of θ_2 and λ_2 based on the observations of RSS. Lesitha *et al.* (2010) have derived the Fisher information in concomitants of order statistics and thereby identified the units having maximum Fisher information. Using this knowledge, they suggested an extreme ranked set sampling (ERSS) procedure and thereby modified the estimates of the parameters obtained by Chacko and Thomas (2009).

Chacko and Thomas (2006) have dealt with the distribution of the concomitants of record values arising from an MTBLD. Chacko and Mary (2013) studied the distribution of concomitants of GURVs arising from an MTBLD and estimated some of its parameters. We utilize the basic distribution theory of concomitants of GURVs arising from an MTBLD as given in Chacko and Mary (2013) for developing a technique of estimating the parameters θ_2 and λ_2 based on IGURRSS.

Now, we consider a population of units on which the measurement of an auxiliary variable X and that on a variable Y of primary interest are such that they are jointly distributed as an MTBLD with cdf given in Eq. (42). We consider a situation where it is inexpensive and straightforward to measure the auxiliary variable X on any number of units. However, it is hard and/or expensive to make measurements on variable Y of primary interest. Hence in this situation, there is much relevance for application of IGURRSS from the population under discussion.

Applying Definition 1 on a population, if measurements made on units for (X,Y) follow a distribution with pdf in Eq. (43), then it results in an ***igurrss*** involving observations $Y_{U[i,k]i}$, $i=1,2,\ldots,n$. Clearly, $Y_{U[i,k]i}$ is distributed as the concomitant of the i-th GURV arising from (43). Chacko and Mary (2013) have derived the means, variances and covariances of concomitants of record values arising from Eq. (43). Hence, from Chacko and Mary (2013) we write the following:

$$\begin{aligned}
\alpha &= E\left(\frac{Y-\theta_2}{\lambda_2}\right)=0, \quad \alpha^{(2)}=E\left[\left(\frac{Y-\theta_2}{\lambda_2}\right)^2\right]=\frac{\pi^2}{3}, \\
\alpha_{2:2} &= E\left(\frac{Y_{2:2}-\theta_2}{\lambda_2}\right)=\psi(2)-\psi(1)
\end{aligned}$$

and

$$\alpha_{2:2}^{(2)} = E\left[\left(\frac{Y_{2:2}-\theta_2}{\lambda_2}\right)^2\right]=\psi'(2)+\psi'(1)+\{\psi(2)-\psi(1)\}^2,$$

where $\psi(.)$ and $\psi'(.)$ are the well known digamma and trigamma functions, respectively, with the property that $\psi(2)-\psi(1)=1$ and $\psi'(2)+\psi'(1)=\frac{\pi^2}{3}-1$. Then we have

$$E\left[Y_{U[r,k]r}\right] = \theta_2+\zeta_r\lambda_2, \tag{44}$$

$$\mathrm{Var}\left[Y_{U[r,k]r}\right] = \beta_r\lambda_2^2, \tag{45}$$

where

$$\zeta_r = \gamma\left(1-2\left(\frac{k}{k+1}\right)^r\right), \tag{46}$$

and

$$\beta_r = \frac{\pi^2}{3} - \gamma^2\left(1-2\left(\frac{k}{k+1}\right)^r\right)^2. \tag{47}$$

Now, by substituting the values of ζ_r and β_r as defined in Equations (46) and (47), respectively, in the estimators of location and scale parameters derived in Section 3 and each of its Subsections 3.1 and 3.2, we get the required estimators of the parameters. Those results are given in the following Subsections.

4.1. Case I: Estimators of θ_2 and λ_2

Let $Y_{U[i,k]i}, i = 1,2,\ldots,n$ be the ***igurrss*** obtained from an MTBLD with pdf as in Eq. (43). Then the BLUEs for the location parameter θ_2 and scale parameter λ_2 are obtained by substituting the values of ζ_r and β_r defined by Eq. (46) and Eq. (47), respectively in Equations (23) and (24). The corresponding variances are also obtained by substituting the values of ζ_r and β_r defined by Equations (46) and (47) in Equations (27) and (28), respectively.

REMARK 4. *If we write the estimators of θ_2 and λ_2 based on an* ***igurrss*** *from an MTBLD as given in Equations (23) and (24) as $\theta^*_{2,k}(\gamma)$ and $\lambda^*_{2,k}(\gamma)$ (provided γ is known), respectively, then the estimators of θ_2 and λ_2 when γ is replaced by $-\gamma$ will be $\theta^*_{2,k}(\gamma)$ and $-\lambda^*_{2,k}(\gamma)$, respectively. The reason for this is that when γ is replaced by $-\gamma$, the value of ζ_r become $-\zeta_r$ while β_r remains unchanged for $n = 1,2,\ldots,n$. Clearly, from Eq. (27) and Eq. (28) we have $Var[\theta^*_{2,k}(-\gamma)] = Var[\theta^*_{2,k}(\gamma)]$ and $Var[\lambda^*_{2,k}(-\gamma)] = Var[\lambda^*_{2,k}(\gamma)]$.*

We have evaluated the coefficients $a_{i,n,k}$ and $b_{i,n,k}$ of $Y_{U[i,k]i}$, $1 \le i \le n$ involved in the BLUEs $\theta^*_{2,k}$ and $\lambda^*_{2,k}$ and the values of $\lambda_2^{-2}\mathrm{Var}(\theta^*_{2,k})$ and $\lambda_2^{-2}\mathrm{Var}(\lambda^*_{2,k})$ for $n = 2(1)10$, $\gamma = 0.25(0.25)0.75$ and $k = 2$. Results are presented in Tables 1, 2, and 3. By using Remark 4 we can use the same tables to obtain the coefficients $a_{i,n,k}$ and $b_{i,n,k}$ of $Y_{U[i,k]i}$, $1 \le i \le n$ in $\theta^*_{2,k}$ and $\lambda^*_{2,k}$ and their variances for $n = 2(1)10$, $\gamma = -0.25, -0.50, -0.75$ and $k = 2$, as well.

Chacko and Mary (2013) considered the problem of concomitants of GURVs arising from Eq. (43). They have also tabulated the coefficients of the concomitants of GURVs involved in the BLUEs as well as the numerical values of $\lambda^{-2}\mathrm{Var}(\hat{\theta}_{2,k})$ and $\lambda^{-2}\mathrm{Var}(\hat{\lambda}_{2,k})$ for $k = 2$. In order to analyze the performance of the estimators $\theta^*_{2,2}$ and $\lambda^*_{2,2}$ derived in this section, we have obtained the values of $\lambda^{-2}\mathrm{Var}(\theta^*_{2,2})$, $\lambda^{-2}\mathrm{Var}(\lambda^*_{2,2})$, relative efficiency $e_1 = \frac{\mathrm{Var}(\hat{\theta}_{2,2})}{\mathrm{Var}(\theta^*_{2,2})}$ of $\theta^*_{2,2}$ relative to $\hat{\theta}_{2,2}$, relative efficiency $e_2 = \frac{\mathrm{Var}(\hat{\lambda}_{2,2})}{\mathrm{Var}(\lambda^*_{2,2})}$ of $\lambda^*_{2,2}$ relative to $\hat{\lambda}_{2,2}$. These results are also included in Table 3. From Table 1 we can observe that the estimate $\theta^*_{2,2}$ based on IGURRSS is uniformly better than the BLUE $\hat{\theta}_{2,2}$ based on

TABLE 1

The coefficients $a_{i,n,2}$ of $Y_{U[i,1]i}$ in the BLUE $\theta^*_{2,2} = \sum_{i=1}^{n} a_{i,n,2} Y_{U[i,2]i}$ of θ_2.

		The coefficients $a_{i,n,2}$ in the BLUE $\theta^*_{2,2} = \sum_{i=1}^{n} a_{i,n,2} Y_{U[i,2]i}$									
n	γ	$a_{1,n,2}$	$a_{2,n,2}$	$a_{3,n,2}$	$a_{4,n,2}$	$a_{5,n,2}$	$a_{6,n,2}$	$a_{7,n,2}$	$a_{8,n,2}$	$a_{9,n,2}$	$a_{10,n,2}$
2	0.25	0.250	0.750								
	0.5	0.250	0.750								
	0.75	0.250	0.750								
3	0.25	0.421	0.322	0.257							
	0.5	0.422	0.320	0.258							
	0.75	0.423	0.318	0.259							
4	0.25	0.460	0.284	0.167	0.089						
	0.5	0.461	0.282	0.167	0.090						
	0.75	0.462	0.280	0.167	0.091						
5	0.25	0.467	0.281	0.158	0.075	0.020					
	0.5	0.467	0.279	0.158	0.076	0.020					
	0.75	0.469	0.277	0.158	0.077	0.020					
6	0.25	0.464	0.281	0.160	0.080	0.026	-0.011				
	0.5	0.465	0.280	0.161	0.081	0.026	-0.012				
	0.75	0.465	0.277	0.162	0.082	0.026	-0.013				
7	0.25	0.459	0.282	0.165	0.086	0.034	-0.001	-0.025			
	0.5	0.459	0.280	0.165	0.088	0.035	-0.001	-0.026			
	0.75	0.460	0.278	0.166	0.090	0.036	-0.002	-0.028			
8	0.25	0.454	0.282	0.168	0.093	0.042	0.008	-0.015	-0.031		
	0.5	0.454	0.280	0.169	0.094	0.043	0.008	-0.016	-0.032		
	0.75	0.454	0.278	0.170	0.096	0.045	0.008	-0.017	-0.035		
9	0.25	0.449	0.282	0.171	0.097	0.048	0.015	-0.007	-0.022	-0.032	
	0.5	0.449	0.280	0.172	0.099	0.049	0.015	-0.008	-0.023	-0.034	
	0.75	0.449	0.278	0.173	0.102	0.052	0.016	-0.008	-0.025	-0.037	
10	0.25	0.445	0.281	0.173	0.101	0.053	0.020	-0.001	-0.016	-0.025	-0.032
	0.5	0.445	0.280	0.174	0.103	0.054	0.021	-0.001	-0.016	-0.027	-0.033
	0.75	0.445	0.277	0.175	0.106	0.057	0.023	-0.001	-0.017	-0.029	-0.036
6	**1**	**0.467**	**0.274**	**0.163**	**0.085**	**0.027**	**-0.016**				

TABLE 2
The coefficients $b_{i,n,2}$ of $Y_{U[i,2]i}$ in the BLUE $\lambda^*_{2,2} = \sum_{i=1}^{n} b_{i,n,2} Y_{U[i,2]i}$ of λ_2, for $n = 2(1)10$ and $\gamma = 0.25(0.25)0.75$.

		The coefficients $b_{i,n,2}$ in the BLUE $\lambda^*_{2,2} = \sum_{i=1}^{n} b_{i,n,2} Y_{U[i,2]i}$									
n	γ	$b_{1,n,2}$	$b_{2,n,2}$	$b_{3,n,2}$	$b_{4,n,2}$	$b_{5,n,2}$	$b_{6,n,2}$	$b_{7,n,2}$	$b_{8,n,2}$	$b_{9,n,2}$	$b_{10,n,2}$
2	0.25	-9.000	9.000								
	0.5	-4.500	4.500								
	0.75	-3.000	3.000								
3	0.25	-5.683	0.708	4.975							
	0.5	-2.840	0.349	2.490							
	0.75	-1.891	0.228	1.663							
4	0.25	-4.250	-0.695	1.676	3.269						
	0.5	-2.122	-0.351	0.829	1.643						
	0.75	-1.410	-0.238	0.542	1.106						
5	0.25	-3.488	-1.063	0.551	1.635	2.365					
	0.5	-1.739	-0.533	0.265	0.814	1.194					
	0.75	-1.154	-0.358	0.165	0.538	0.809					
6	0.25	-3.030	-1.175	0.059	0.886	1.443	1.817				
	0.5	-1.510	-0.588	0.019	0.435	0.723	0.920				
	0.75	-1.000	-0.393	0.001	0.281	0.483	0.627				
7	0.25	-2.732	-1.207	-0.194	0.486	0.942	1.249	1.455			
	0.5	-1.361	-0.603	-0.106	0.234	0.468	0.629	0.738			
	0.75	-0.901	-0.401	-0.081	0.145	0.308	0.424	0.506			
8	0.25	-2.527	-1.211	-0.338	0.248	0.641	0.905	1.082	1.200		
	0.5	-1.258	-0.604	-0.177	0.115	0.315	0.453	0.547	0.610		
	0.75	-0.832	-0.402	-0.127	0.066	0.204	0.302	0.371	0.419		
9	0.25	-2.380	-1.206	-0.427	0.094	0.445	0.680	0.838	0.943	1.013	
	0.5	-1.185	-0.601	-0.221	0.038	0.216	0.338	0.421	0.477	0.515	
	0.75	-0.784	-0.399	-0.155	0.015	0.136	0.223	0.284	0.326	0.354	
10	0.25	-2.271	-1.198	-0.487	-0.010	0.309	0.524	0.668	0.764	0.828	0.871
	0.5	-1.130	-0.597	-0.250	-0.014	0.148	0.259	0.334	0.386	0.420	0.443
	0.75	-0.748	-0.395	-0.174	-0.019	0.091	0.169	0.223	0.261	0.287	0.305
6	1	**-0.743**	**-0.295**	**-0.012**	**0.200**	**0.363**	**0.488**				

concomitants of GURVs obtained from a single sequence of units from the given population. From Table 3 we observe a similar tendency, though there is a slight fall in the relative efficiency on the estimate $\lambda^*_{2,2}$ based on IGURRSS with four or less observations the relative efficiency surpassed the performance of its competitor $\hat{\lambda}_{2,2}$ for $n > 6$. It may be noted that the evaluation of $\hat{\theta}_{2,2}$ and $\hat{\lambda}_{2,2}$ involves a variance-covariance matrix of order n with all elements non-zero and its inverse, whereas the evaluation of $\theta^*_{2,2}$ and $\lambda^*_{2,2}$ involves a variance-covariance matrix that is just a diagonal matrix of order n. Hence there is a real advantage in using $\theta^*_{2,2}$ and $\lambda^*_{2,2}$ instead of $\hat{\theta}_{2,2}$ and $\hat{\lambda}_{2,2}$ as estimators of θ_2 and λ_2.

TABLE 3

Values of $\lambda_2^{-2}\,Var(\hat{\theta}_{2,2})$, $\lambda_2^{-2}\,Var(\theta^*_{2,2})$, e_1 = relative efficiency of $\theta^*_{2,2}$ relative to $\hat{\theta}_{2,2}$, $\lambda_2^{-2}\,Var(\hat{\lambda}_{2,2})$, $\lambda_2^{-2}\,Var(\lambda^*_{2,2})$, e_2 = relative efficiency of $\lambda^*_{2,2}$ relative to $\hat{\lambda}_{2,2}$, $n = 2(1)10$ and $\gamma = 0.25(0.25)0.75$.

		Variances		Efficiency	Variances		Efficiency
n	γ	$\lambda_2^{-2}\mathrm{Var}(\hat{\theta}_{2,2})$	$\lambda_2^{-2}\mathrm{Var}(\theta^*_{2,2})$	e_1	$\lambda_2^{-2}\mathrm{Var}(\hat{\lambda}_{2,2})$	$\lambda_2^{-2}\mathrm{Var}(\lambda^*_{2,2})$	e_2
2	0.25	2.055	2.059	1.002	532.334	530.834	0.997
	0.5	2.053	2.067	1.007	132.615	131.115	0.989
	0.75	2.048	2.080	1.015	58.593	57.093	0.974
3	0.25	1.140	1.145	1.005	188.859	188.497	0.998
	0.5	1.134	1.155	1.019	46.854	46.492	0.992
	0.75	1.124	1.171	1.042	20.556	20.194	0.982
4	0.25	1.078	1.083	1.005	105.021	104.895	0.999
	0.5	1.072	1.092	1.019	25.955	25.829	0.995
	0.75	1.061	1.106	1.042	11.312	11.186	0.989
5	0.25	1.075	1.080	1.005	71.589	71.544	0.999
	0.5	1.069	1.089	1.018	17.642	17.597	0.997
	0.75	1.059	1.103	1.042	7.650	7.606	0.994
6	0.25	1.075	1.080	1.005	54.748	54.740	1.000
	0.5	1.069	1.088	1.019	13.466	13.459	0.999
	0.75	1.059	1.103	1.042	5.819	5.813	0.999
7	0.25	1.072	1.077	1.005	45.001	45.014	1.000
	0.5	1.066	1.086	1.019	11.055	11.069	1.001
	0.75	1.055	1.101	1.043	4.767	4.782	1.003
8	0.25	1.068	1.073	1.005	38.814	38.842	1.001
	0.5	1.061	1.082	1.020	9.530	9.558	1.003
	0.75	1.051	1.097	1.044	4.104	4.133	1.007
9	0.25	1.063	1.069	1.005	34.621	34.658	1.001
	0.5	1.057	1.078	1.020	8.498	8.536	1.004
	0.75	1.046	1.094	1.045	3.659	3.697	1.011
10	0.25	1.059	1.065	1.005	31.634	31.679	1.001
	0.5	1.053	1.074	1.020	7.765	7.810	1.006
	0.75	1.042	1.090	1.047	3.343	3.389	1.014
6	**1**	**1.036**	**1.118**	**1.08**	**2.202**	**2.233**	**1.014**

4.2. Case II: Estimation of θ_2 when λ_2 is known

Suppose the scale parameter λ_2 involved in the MTBLD defined by the pdf in Eq. (43) is known. For convenience, we may take $\lambda_2 = 1$. Let $Y_{U[i,k]i}, i = 1,2,\ldots,n$ be an ***igurrss*** drawn from Eq. (43) for $\lambda_2 = 1$. Define $Z_{U[r,k]r} = Y_{U[r,k]r} - \zeta_r, r = 1,2,\ldots,n,$

where the constant ζ_r is defined by Eq. (46). By making use of the theory developed in Subsection 3.1, we can derive the BLUE $\theta_{2,2}^{(0)}$ of θ_2 based on $Z_{U[r,k]r}, r = 1,2,\ldots,n$, and its variance $\mathrm{Var}(\theta_{2,2}^{(0)})$.

REMARK 5. *If we write the estimator of θ_2 based on an* ***igurrss*** *drawn from an MTBLD with $\lambda_2 = 1$ as $\theta_{2,2}^{(0)}(\gamma)$, then the estimator of θ_2 for γ replaced by $-\gamma$ will be also $\theta_{2,2}^{(0)}(\gamma)$. As discussed already, the reason for this is that when γ is replaced by $-\gamma$, the value of β_r involved in Eq. (34) remains unchanged for all $r = 1,2,\ldots,n$. The same argument further proves that $Var(\theta_{2,2}^{(0)}(\gamma)) = Var(\theta_{2,2}^{(0)}(-\gamma))$. This establishes that the coefficients c_i of $Z_{U[i,k]i}$ in the BLUE $\theta_{2,2}^{(0)}$ of θ_2 and $Var(\theta_{2,2}^{(0)})$ remain unchanged when we replace γ by $-\gamma$.*

The coefficients $c_{i,n,2}$ of $Z_{U[i,2]i}, i = 1,2,\ldots,n$ in $\theta_{2,2}^{(0)}$ and $\mathrm{Var}(\theta_{2,2}^{(0)})$ for $n = 2(1)10$, $\gamma = 0.25(0.25)0.75$ are evaluated and presented in Tables 4 and 5.

TABLE 4
The coefficients $c_{i,n,2}$ of $Z_{U[i,2]i}$ in the BLUE $\theta_{2,2}^{(0)} = \sum_{i=1}^{n} c_{i,n,2} Z_{U[i,2]i}$ of θ_2, for $n = 2(1)10$ and $\gamma = 0.25(0.25)0.75$.

		The coefficients $c_{i,n,2}$ in the BLUE $\theta_{2,2}^{(0)} = \sum_{i=1}^{n} c_{i,n,2} Z_{U[i,2]i}$									
n	γ	$c_{1,n,2}$	$c_{2,n,2}$	$c_{3,n,2}$	$c_{4,n,2}$	$c_{5,n,2}$	$c_{6,n,2}$	$c_{7,n,2}$	$c_{8,n,2}$	$c_{9,n,2}$	$c_{10,n,2}$
2	0.25	0.500	0.500								
	0.5	0.502	0.498								
	0.75	0.504	0.496								
3	0.25	0.333	0.333	0.334							
	0.5	0.334	0.331	0.335							
	0.75	0.334	0.328	0.337							
4	0.25	0.250	0.249	0.250	0.251						
	0.5	0.249	0.247	0.250	0.254						
	0.75	0.248	0.243	0.250	0.259						
5	0.25	0.200	0.199	0.200	0.200	0.201					
	0.5	0.198	0.197	0.199	0.202	0.205					
	0.75	0.195	0.192	0.197	0.204	0.211					
6	0.25	0.166	0.166	0.166	0.167	0.167	0.168				
	0.5	0.164	0.163	0.165	0.167	0.170	0.172				
	0.75	0.161	0.158	0.162	0.168	0.174	0.178				
7	0.25	0.142	0.142	0.142	0.143	0.143	0.144	0.144			
	0.5	0.140	0.139	0.140	0.143	0.145	0.146	0.147			
	0.75	0.136	0.134	0.137	0.142	0.147	0.151	0.154			
8	0.25	0.124	0.124	0.124	0.125	0.125	0.126	0.126	0.126		
	0.5	0.122	0.121	0.122	0.124	0.126	0.127	0.128	0.129		
	0.75	0.118	0.116	0.119	0.123	0.127	0.130	0.133	0.135		
9	0.25	0.110	0.110	0.110	0.111	0.111	0.112	0.112	0.112	0.112	
	0.5	0.108	0.107	0.108	0.110	0.112	0.113	0.114	0.114	0.115	
	0.75	0.103	0.102	0.104	0.108	0.112	0.115	0.117	0.119	0.120	
10	0.25	0.099	0.099	0.099	0.100	0.100	0.100	0.100	0.101	0.101	0.101
	0.5	0.097	0.096	0.097	0.099	0.100	0.101	0.102	0.103	0.103	0.103
	0.75	0.092	0.091	0.093	0.097	0.100	0.102	0.104	0.106	0.107	0.108

Remark 5 will help to use the same coefficients $c_{i,n,2}$ of $Z_{U[i,2]i}$ to obtain $\theta_{2,2}^{(0)}$ for negative values of γ as well. Chacko and Mary (2013) derived the BLUE $\hat{\theta}_{2,0}$ of θ_2 based on concomitants of GURVs arising from Eq. (43) with $\lambda_2 = 1$ using a single sequence of units for known γ. They have also tabulated the numerical coefficients of concomitants of GURVs involved in the BLUE $\hat{\theta}_{2,0}$ and $\mathrm{Var}(\hat{\theta}_{2,0})$ for $n = 2(1)10$, $\gamma = 0.25(0.25)0.75$ and $k = 2$. We compared our estimator $\theta^{(0)}_{2,2}$ derived in this section with $\hat{\theta}_{2,0}$ by computing the relative efficiency $e_3 = \frac{\mathrm{Var}(\hat{\theta}_{2,0})}{\mathrm{Var}(\theta^{(0)}_{2,2})}$. These results are also given in Tables 4 and 5.

TABLE 5

Values of $Var(\theta^{(0)}_{2,2})$, $Var(\hat{\theta}_{2,0})$, e_3 = relative efficiency of $\theta^{(0)}_{2,2}$ relative to $\hat{\theta}_{2,0}$, $\lambda_2^{-2} Var(\hat{\lambda}_{2,0})$ of λ_2, $\lambda_2^{-2} Var(\lambda^{(0)}_{2,2})$, e_4 = relative efficiency of $\lambda^{(0)}_{2,2}$ relative to $\hat{\lambda}_{2,0}$, $n = 2(1)10$ and $\gamma = 0.25(0.25)0.75$.

		Variances		Efficiency	Variances		Efficiency
n	γ	$\lambda_2^{-2}\mathrm{Var}(\hat{\theta}_{2,0})$	$\lambda_2^{-2}\mathrm{Var}(\theta^{(0)}_{2,2})$	e_3	$\lambda_2^{-2}\mathrm{Var}(\hat{\lambda}_{2,0})$	$\lambda_2^{-2}\mathrm{Var}(\lambda^{(0)}_{2,2})$	e_4
2	0.25	1.648	1.643	1.003	424.826	425.547	0.998
	0.5	1.656	1.637	1.011	105.047	105.771	0.993
	0.75	1.669	1.627	1.026	45.823	46.553	0.984
3	0.25	1.100	1.095	1.005	181.084	181.385	0.998
	0.5	1.110	1.089	1.020	44.686	44.987	0.993
	0.75	1.127	1.078	1.045	19.426	19.728	0.985
4	0.25	0.825	0.820	1.006	79.933	79.910	1.000
	0.5	0.832	0.812	1.025	19.693	19.672	1.001
	0.75	0.844	0.799	1.056	8.534	8.515	1.002
5	0.25	0.659	0.655	1.007	43.676	43.611	1.001
	0.5	0.663	0.646	1.027	10.720	10.656	1.006
	0.75	0.669	0.630	1.062	4.614	4.552	1.014
6	0.25	0.549	0.545	1.007	27.826	27.769	1.002
	0.5	0.550	0.535	1.028	6.799	6.743	1.008
	0.75	0.551	0.518	1.063	2.902	2.847	1.019
7	0.25	0.470	0.467	1.007	19.634	19.591	1.002
	0.5	0.468	0.456	1.027	4.776	4.733	1.009
	0.75	0.465	0.438	1.061	2.021	1.980	1.021
8	0.25	0.410	0.408	1.006	14.857	14.824	1.002
	0.5	0.407	0.397	1.026	3.599	3.567	1.009
	0.75	0.401	0.379	1.058	1.512	1.481	1.021
9	0.25	0.364	0.362	1.006	11.813	11.789	1.002
	0.5	0.360	0.352	1.024	2.852	2.828	1.008
	0.75	0.352	0.334	1.055	1.190	1.167	1.020
10	0.25	0.327	0.326	1.006	9.742	9.723	1.002
	0.5	0.323	0.315	1.022	2.345	2.327	1.008
	0.75	0.313	0.298	1.051	0.973	0.956	1.018

Table 5 shows that the estimate $\theta^{(0)}_{2,2}$ based on ***igurrss*** performs better than that of $\hat{\theta}_{2,0}$ based on concomitants of GURVs of a single sequence of units for all values of $n \geq 2$.

It is to be noted that determination of $\mathrm{Var}(\hat{\theta}_{2,0})$ requires the additional evaluation of $\binom{n}{2}$ integrals for product moments of concomitants of generalized (2) record values when compared with the determination of integrals to obtain $\mathrm{Var}(\theta^{(0)}_{2,2})$. This makes our estimator $\theta^{(0)}_{2,2}$ superior when compared with $\hat{\theta}_{2,0}$.

4.3. Case III : Estimation of λ_2 when θ_2 is known

In this subsection we deal with the problem of estimating the BLUE of λ_2 involved in an MTBLD by using the procedure derived in Subsection 3.2. Let $Y_{U[i,k]i}, i = 1,2,\ldots,n$ be the ***igurrss*** obtained from an MTBLD for known θ_2 and γ. For convenience, we take $\theta_2 = 0$. Then the estimator $\lambda_{2,k}^{(0)}$ of λ_2 can be computed using Eq. (40) and Var($\lambda_{2,k}^{(0)}$) by Eq. (41). It is to be noted that ζ_r and β_r involved in Equations (40) and (41) are given in Equations (46) and (47), respectively.

REMARK 6. *If we write the estimator of λ_2 based on an **igurrss** from an MTBLD for $\theta_2 = 0$ as $\lambda_{2,k}^{(0)}(\gamma)$, given in Eq. (40), then the estimator of λ_2 for γ replaced by $-\gamma$ will be $-\lambda_{2,k}^{(0)}(\gamma)$. This is so, since if we replace γ by $-\gamma$, then the value of ζ_r becomes $-\zeta_r$ while β_r remains unchanged for $r = 1,2,\ldots,n$. Clearly, from Eq. (41), we further have $Var\left[\lambda_{2,k}^{(0)}(-\gamma)\right] = Var\left[\lambda_{2,k}^{(0)}(\gamma)\right]$.*

The numerical coefficients $d_{i,n,2}$ of $Y_{U[i,2]i}, i = 1,2,\ldots,n$ involved in the BLUE $\lambda_{2,2}^{(0)}$ and the values of $\lambda_2^{-2}\text{Var}(\lambda_{2,2}^{(0)})$, for $n = 2(1)10, \gamma = 0.25(0.25)0.75$ are evaluated and presented in Table 6. Clearly, Remark 6 helps us to use Tables 5 and 6 to obtain the coefficients of $d_{i,n,2}$ of $Y_{U[i,2]i}, i = 1,2,\ldots,n$ in $\lambda_{2,2}^{(0)}$ and its variance for $n = 2(1)10$, $\gamma = -0.25, -0.50, -0.75$ as well.

Once $\lambda_{2,2}^{(0)}$ is derived as an estimator of λ_2, an immediate quest is to consider the BLUE $\hat{\lambda}_{2,0}$ as an estimator of λ_2 based on concomitants of the first n GURVs of a single sequence of independent observations drawn from Eq. (43) for $\theta_2 = 0$ so as to have a comparison of $\lambda_{2,2}^{(0)}$ with $\hat{\lambda}_{2,0}$. Though Chacko and Mary (2013) have obtained the simultaneous BLUEs $\hat{\theta}_{2,k}$ and $\hat{\lambda}_{2,k}$ involved in (43) based on the first n concomitants of GURVs, they have not derived the expression for the BLUE $\hat{\lambda}_{2,0}$ of λ_2 in the corresponding cases when $\theta_2 = 0$. However using the derived expressions provided by Chacko and Mary (2013) for the means, variances and covariances of concomitants of record values arising from Eq. (43) for $\theta_1 = \theta_2 = 0$, $\lambda_1 = \lambda_2 = 1$, we have computed and tabulated $\lambda_2^{-2}\text{Var}(\hat{\lambda}_{2,0})$ for $n = 3(1)10, \gamma = 0.25(0.25)0.75$ in Tables 5 and 6.

The relative efficiency $e_4 = \frac{\text{Var}(\hat{\lambda}_{2,0})}{\text{Var}(\lambda_{2,2}^{(0)})}$ of $\lambda_{2,2}^{(0)}$ relative to $\hat{\lambda}_{2,0}$ also has been evaluated for $n = 3(1)10, \gamma = 0.25(0.25)0.75$, and the results are presented in Table 6. From these tables we observe that except for the case $n = 3$, $k = 2$, the estimate $\lambda_{2,2}^{(0)}$ based on IGURRSS is better than that of the estimate based on concomitants of GURVs of a single sequence of independent units. It is to be noted that the determination of Var($\hat{\lambda}_{2,0}$) requires the additional evaluation of $\binom{n}{2}$ integrals for product moments of concomitants of record values when compared with the determination of Var($\lambda_{2,2}^{(0)}$). This makes our estimator $\lambda_{2,2}^{(0)}$ superior when compared with $\hat{\lambda}_{2,0}$.

TABLE 6
The coefficients $d_{i,n,2}$ of $Y_{U[i,2]i}$ in the BLUE $\lambda_{2,2}^{(0)} = \sum_{i=1}^{n} d_{i,n,2} Y_{U[i,2]i}$ for $n = 2(1)10$ and $\gamma = 0.25(0.25)0.75$.

		The coefficients $d_{i,n,2}$ in the BLUE $\lambda_{2,2}^{(0)} = \sum_{i=1}^{n} d_{i,n,2} Y_{U[i,2]i}$									
n	γ	$d_{1,n,2}$	$d_{2,n,2}$	$d_{3,n,2}$	$d_{4,n,2}$	$d_{5,n,2}$	$d_{6,n,2}$	$d_{7,n,2}$	$d_{8,n,2}$	$d_{9,n,2}$	$d_{10,n,2}$
2	0.25	-10.802	3.594								
	0.5	-5.404	1.788								
	0.75	-3.606	1.182								
3	0.25	-4.604	1.532	5.633							
	0.5	-2.298	0.760	2.821							
	0.75	-1.528	0.501	1.886							
4	0.25	-2.028	0.675	2.482	3.699						
	0.5	-1.005	0.333	1.234	1.860						
	0.75	-0.660	0.216	0.814	1.253						
5	0.25	-1.107	0.368	1.354	2.019	2.467					
	0.5	-0.544	0.180	0.668	1.008	1.244					
	0.75	-0.353	0.116	0.435	0.670	0.843					
6	0.25	-0.705	0.235	0.862	1.285	1.571	1.762				
	0.5	-0.345	0.114	0.423	0.638	0.787	0.891				
	0.75	-0.221	0.072	0.272	0.419	0.527	0.606				
7	0.25	-0.497	0.165	0.608	0.907	1.108	1.243	1.334			
	0.5	-0.242	0.080	0.297	0.448	0.553	0.625	0.675			
	0.75	-0.153	0.050	0.189	0.291	0.367	0.421	0.460			
8	0.25	-0.376	0.125	0.460	0.686	0.838	0.941	1.010	1.056		
	0.5	-0.182	0.060	0.224	0.337	0.417	0.471	0.509	0.534		
	0.75	-0.115	0.038	0.142	0.218	0.274	0.315	0.344	0.364		
9	0.25	-0.299	0.100	0.366	0.546	0.667	0.748	0.803	0.839	0.864	
	0.5	-0.144	0.048	0.177	0.267	0.330	0.374	0.403	0.424	0.437	
	0.75	-0.090	0.030	0.112	0.172	0.216	0.248	0.271	0.287	0.298	
10	0.25	-0.247	0.082	0.302	0.450	0.550	0.617	0.662	0.692	0.713	0.726
	0.5	-0.119	0.039	0.146	0.220	0.272	0.307	0.332	0.349	0.360	0.367
	0.75	-0.074	0.024	0.091	0.141	0.177	0.203	0.222	0.235	0.244	0.250

5. Real Data Generation Using IGURRSS and Bivariate Model Construction

Usually, two types of problems arise when investigating a bivariate population. Less severity is experienced when information about the form of the distribution of the population random variable is available. In addition to the problem of determining the parameters of the available form of the bivariate distribution completely, if we face a constraint describing the difficulty on measuring the variable Y of primary interest, then, in this case, the results of this paper developed in the previous sections will be very useful. However, more complexity is involved in modelling the parent bivariate distribution if its form as well is not available. If a methodology evolves in this situation to model the parent bivariate distribution, then it will be ingenious and priceless. We describe such a methodology below.

Suppose that $f_{X,Y}(x,y)$ is an absolutely continuous bivariate density function of a random vector (X,Y) with pdf $f_X(x)$ and cdf $F_X(x)$ on the marginal random variable X. Then the pdf of the n-th concomitant of the generalized lower (k) record value $Y_{U[n,k]}$ arising from $f_{X,Y}(x,y)$ can be written as

$$f_{Y_{U[n,k]}}(y) = \frac{k^n}{\Gamma(n)} \int_x [-\log(1-F_X(x))]^{n-1}[1-F_X(x)]^{k-1} f_{X,Y}(x,y)\mathrm{d}x. \tag{48}$$

Thomas *et al.* (2014) have used an auxiliary family of pdf's $f_{Y_{A[\omega,k]}}(y)$ with reference to Eq. (48) as

$$f_{Y_{A[\omega,k]}}(y) = \frac{k^\omega}{\Gamma(\omega)} \int_x [-\log(1-F_X(x))]^{\omega-1}[1-F_X(x)]^{k-1} f_{X,Y}(x,y)\mathrm{d}x, \quad \omega > 0, \tag{49}$$

and established that $f_X(x)$ and $f_{Y_{A[\omega,k]}}(y)$ together determine uniquely the parent bivariate distribution $f_{X,Y}(x,y)$. As a consequence of the above theory, they further used a process of impounding $f_X(x)$ with $f_{Y_{A[\omega,k]}}(y)$ and through an inverse Mellin transform determined $f_{X,Y}(x,y)$.

Thus, if the measuring mechanism on Y is very costly or difficult, whereas one can take any number of observations on the auxiliary variable X, then to get the required data for applying the theory explained above for modelling $f_{X,Y}(x,y)$, it is enough to apply IGURRSS in several, say N, cycles. If the maximum order of generalized upper (k) record value that we intend to observe is n, then for given k and any m $(1 \le m \le n)$ we get a group of N independent observations $Y_{U[m,k]i}$ for $i = 1,2,\ldots,N$, and in this case we can use these observations for modelling the univariate distributions with pdf $f_{Y_{U[m,k]}}(y)$. If by the above process we could model $f_{Y_{U[m,k]}}(y)$ for $m = 1,2,\ldots,n$ and further model $f_X(x)$ using the marginal X observations observed on each of the enumerated units, then the methodology explained in the previous paragraph helps us to identify $f_{X,Y}(x,y)$ of the parent bivariate distribution.

Paul and Thomas (2017) have applied the above methodology for $k = 1$ to model $f_{X,Y}(x,y)$, where X is the height of acacia trees which can be measured very quickly from the ground by a hypsometer, whereas Y represents the volume of usable timber of acacia trees, which is very difficult to measure keeping the trees alive. It is to be noted that for $k = 1$, the generated data by IGURRSS are only classical upper record values. Thus, to obtain the required data, one has to enumerate a huge number of trees. The occurrence of outliers, in that case, arrests the further occurrence of higher-ranked classical record values. Hence in this illustration, we fix $k = 2$ and thereby carry out IGURRSS for $k = 2$ only. Thus, to apply the results given here to this modelling problem, we have measured the height of the randomly selected acacia trees belonging to each of 6 different lines of each of the six randomly chosen blocks of acacia (*Acacia Auriculiformis, A. Cunn. ex Benth.*) trees planted in an extensive area of barren lands of Kerala University campus, Trivandrum, using a hypsometer. The number of trees selected randomly for enumeration of their height in the i-th line of any block is up to a number that is enough to observe i generalized (2) record values for $i = 1,2,3,4,5,6$. We have engaged a labourer to take the perimeter of the usable pieces of timber (those pieces whose top parts have at least 20 inches perimeter) at each multiple of 5 feet height from the bottom of the tree in the i-th line whose height is the i-th generalized upper (2) record value for $i = 1,2,3,4,5,6$ in each block. We then computed the timber volume of each piece of a tree by approximating it as a cylinder whose radius is the mean radius of its top and bottom points. By adding the volume of each piece of a tree, the usable timber volume is calculated for each of the selected trees by the IGURRSS strategy. We may call the IGURRSS data collected from the j-th block as the data of the j-th cycle of IGURRSS (as defined in Remark 3). The details of the data collected are given below in Table 7.

It is well known that random variables such as height of trees and timber volume of trees often have an apporoximately normal distribution. Also the existing literature contains discussions on the similarity between normal and logistic distributions in terms of shape and some basic properties (for details see Balakrishnan, 1992, p. 8 and Johnson *et al.*, 1994, p. 119). Now for constructing an appropriate bivariate model for the distribution of the population random vector (X,Y) based on the data available with us, first we ascertain whether the Normal distribution with pdf

$$g_X(x;\nu,\alpha) = \frac{1}{\sqrt{2\pi}\alpha} e^{-\frac{(x-\nu)^2}{2\alpha^2}}, \tag{50}$$

or the logistic distribution with pdf

$$f_X(x;\theta_1,\lambda_1) = \frac{e^{-\frac{(x-\theta_1)}{\lambda_1}}}{\lambda_1\left(1+e^{-\frac{(x-\theta_1)}{\lambda_1}}\right)^2} \tag{51}$$

fits the marginal data on the variable X representing the height of acacia trees well. In the process of IGURRSS, in order to realize the required record values on X, altogether 308 trees were enumerated, while enumeration of the timber volume was made on only 36 trees. In the supplementary material the enumerated X values from 308 trees and the timber volume from 36 trees are provided. If we use the marginal X observations of 308 units considered in IGURRSS to fit the models in Equations (50) and (51) using the maximum likelihood method, then the estimated values of the parameters in the models and the associated K-S statistics with the p-values are given in Table 8.

From Table 8, we observe that, though both normal and logistic distributions fit the data on tree height well, it is evident that the logistic distribution can be taken as the most suitable model for the data on the height of acacia trees. Hence we may take

$$f_X(x;74.4681,11.2002) = \frac{1}{11.2002}\frac{e^{\frac{x-74.4681}{11.2002}}}{\left(1+e^{\frac{x-74.4681}{11.2002}}\right)^2} \tag{52}$$

TABLE 7
igurrss data on height and timber volume of acacia trees.

Cycle	Line no.	Generalized (2) Record Value rank of the selected trees (i)	Generalized (2) Record tree height $X_{U(i,2)}$ in feet	Timber volume $Y_{U[i,2]}$ in cubic feet
I	1	1	76.3780	8.1291
	2	2	89.3701	11.4060
	3	3	103.7730	13.7987
	4	4	104.9869	20.7435
	5	5	82.6772	3.24591
	6	6	102.4278	9.6731
II	1	1	78.2152	7.1187
	2	2	60.8596	10.7052
	3	3	83.6942	20.4190
	4	4	100.0656	9.7888
	5	5	101.0499	5.2839
	6	6	137.1391	23.6338
III	1	1	71.0302	4.5978
	2	2	79.1667	5.9110
	3	3	81.8898	25.1125
	4	4	72.6378	6.2172
	5	5	95.9646	29.2668
	6	6	124.0157	12.5485
IV	1	1	61.8110	8.4915
	2	2	70.7677	22.2393
	3	3	74.1470	2.1296
	4	4	79.8885	11.7794
	5	5	87.1063	8.0686
	6	6	97.1129	17.8589
V	1	1	57.9396	5.9888
	2	2	66.2730	2.2890
	3	3	96.1286	44.9545
	4	4	82.6443	47.8350
	5	5	77.2638	8.2584
	6	6	101.7060	10.0898
VI	1	1	51.8373	1.4341
	2	2	96.6207	16.0036
	3	3	88.4843	16.7355
	4	4	92.5197	12.5897
	5	5	103.8714	11.7407
	6	6	94.3242	11.3510

TABLE 8
Summary of the fitted distributions.

Model	Estimated values of the parameters	K-S statistic	p-value
Normal	ν=75.4443, $\alpha = 19.6555$	0.0639	0.1551
Logistic	θ_1=74.4681, λ_1=11.2002	0.0476	0.4736

as the fitted model for the pdf of X. We may expect trees' height and timber volume as similarly behaving variables. As X is observed to be distributed as logistic, we may expect the same type of distribution for Y as well. For the data given in Table 7 we have evaluated the Pearson correlation coefficient r between height and timber volume of trees as $r = 0.3253$, which is not a high correlation. The well-known Morgenstern bivariate distribution is an ideal bivariate model with a known form of distributions for the marginals and contains the knowledge that the marginal random variables are not highly correlated. This information guides us to postulate for the random vector (X, Y) a Morgenstern type bivariate logistic distribution with pdf given by

$$
\begin{aligned}
f_{X,Y}(x,y) &= \frac{e^{-\frac{(x-74.4681)}{11.2002}}}{11.2002 \times \left\{1+e^{-\frac{(x-74.4681)}{11.2002}}\right\}^2} \frac{e^{-\frac{(y-\theta_2)}{\lambda_2}}}{\lambda_2\left\{1+e^{-\frac{(y-\theta_2)}{\lambda_2}}\right\}^2} \\
&\quad \times \left[1+\gamma \frac{\left\{1-e^{-\frac{(x-74.4681)}{11.2002}}\right\}\left\{1-e^{-\frac{(y-\theta_2)}{\lambda_2}}\right\}}{\left\{1+e^{-\frac{(x-74.4681)}{11.2002}}\right\}\left\{1+e^{-\frac{(y-\theta_2)}{\lambda_2}}\right\}}\right],
\end{aligned} \tag{53}
$$

to the population from which the ***igurrss*** data as given in Table 7 were drawn. There are limitations to validate the model postulated as above using the data available in Table 7 as each X observation in the Table 7 is a generalized (2) record value and the corresponding Y observation is a concomitant of a generalized (2) record value. The 36 bivariate observations in the data are those which are enumerated from 36 different independent sets of units by choosing one unit from each set. The usual classical methods fail to provide a suitable technique to validate the model of Eq. (53) using the data of above nature at hand.

It is to be noted that if $X_{U(i,k)}$ is the i-th GURV observed on the variable X of a unit arising from a sequence of independent units of the population with a bivariate distribution defined in Eq. (53), then the pdf of the concomitant $Y_{U[i,k]}$ of the i-th GURV $X_{U(i,k)}$ on X is given by (for details see, Chacko and Mary, 2013)

$$
f_{Y_{U[i,k]i}}(y) = \frac{e^{-\frac{y-\theta_2}{\lambda_2}}}{\lambda_2\left[1+e^{-\frac{y-\theta_2}{\lambda_2}}\right]^2}\left\{1+\gamma\left(1-2\left(\frac{k}{k+1}\right)^i\right)\left[\frac{1-e^{-\frac{y-\theta_2}{\lambda_2}}}{1+e^{-\frac{y-\theta_2}{\lambda_2}}}\right]\right\}, \tag{54}
$$

for $i = 1,2,\ldots,$ and $k = 2$. Using the pdf in Eq. (53), we can evaluate the correlation coefficient of the MTBLD as $\rho = \frac{3\gamma}{\pi^2}$. Thus the method of moments type estimate of γ can be taken as (see Chacko and Thomas, 2006; Chacko and Mary, 2013,, for further details)

$$\hat{\gamma} = \begin{cases} -1, & \text{if } r \leq -3/\pi^2, \\ 1, & \text{if } r \geq 3/\pi^2, \\ \frac{r\pi^2}{3}, & \text{otherwise.} \end{cases} \tag{55}$$

Since the correlation coefficient for the data is r=0.3253, we obtain the estimate of γ as $\hat{\gamma} = 1.0$. Therefore, by considering the known value of γ as $\hat{\gamma} = 1$, we can utilize the estimators given in Subsection 4.1. We have evaluated the coefficients $a_{i,6,2}$, $i = 1,2,\ldots,6$, involved in $\theta^*_{2,2} = \sum_{i=1}^{6} a_{i,6,2} Y_{U[i,2]i}$ and the variance $\frac{\text{Var}\,(\theta^*_{2,2,i})}{\lambda^2}$ based on a single cycle. These are included in the last row of Tables 1 and 3. If we write $\theta^*_{2,2,i}$ as the estimate obtained from the i-th cycle, $i = 1,2,\ldots,6$, then those values and their overall means $\tilde{\theta}_2$ with its variance are given in Table 9.

TABLE 9

Estimate $\theta^*_{2,2,i}$ of θ_2 from the cycle i, $i = 1,\ldots,6$, = with the mean $\tilde{\theta}_2 = \sum_{i=1}^{6} \frac{\theta^*_{2,2,i}}{6}$ of all those estimates and $\lambda^{-2}\,Var(\tilde{\theta}_2)$.

Cycle No.	1	2	3	4	5	6	Mean	Variance
Estimates	$\theta^*_{2,2,1}$	$\theta^*_{2,2,2}$	$\theta^*_{2,2,3}$	$\theta^*_{2,2,4}$	$\theta^*_{2,2,5}$	$\theta^*_{2,2,6}$	$\tilde{\theta}_2$	$\frac{\text{Var}(\tilde{\theta}_2)}{\lambda^2}$
Values	10.86	10.19	8.98	11.35	14.87	8.99	10.87	0.17

If the interest is to obtain a similar estimate of λ_2 based on 6 cycles, then one can use the coefficients $b_{i,6,2}$ of $Y_{U[i,2]}$, $i = 1,2,\ldots,6$ in $\lambda^*_{2,2} = \sum_{i=1}^{6} b_{i,6,2} Y_{U[i,2]i}$, its variance $\frac{\text{Var}(\lambda^*_{2,2})}{\lambda^2}$, based on a single cycle as presented in Table 2. As in the case of estimating θ, if we write $\lambda^*_{2,2,i}$ to denote the estimate of λ_2 from cycle 2, then the refined estimate of λ_2 is obtained by $\tilde{\lambda}_2 = \sum_{i=1}^{6} \frac{\lambda^*_{2,2,i}}{6}$. The obtained estimate of λ_2 is 12.1156. Thus, the obtained estimates $\tilde{\theta}_2 = 10.8733$ for θ and $\tilde{\lambda}_2 = 12.1156$ are very valuable information for the University of Kerala, to ascertain its timber wealth. If Eq. (53) is the parent bivariate distribution for our population random variable (X,Y), then the $Y_{U[1,2]}$ values collected from all six cycles (in Table 7) viz. 8.129, 7.119, 4.598, 8.492, 5.989, 1.434, can be considered as a random sample of size 6 drawn from a distribution with pdf $f_{Y_{U[1,2]}}$, i.e., one obtained from Eq. (54) with $i = 1$. Based on the above 6 iid observations assumed to be drawn from $f_{Y_{U[1,2]}}$ we have worked out the K-S statistic between the empirical cdf and the distribution corresponding to $f_{Y_{U[1,2]}}$, and have observed that it is not significant at 5% level, indicating the validation of the assumed model $f_{Y_{U[1,2]}}$. In general, $Y_{U[i,2]}$ values collected from all six cycles can be considered as a random sample of size 6 drawn from a distribution with pdf $f_{Y_{U[i,2]}}(y)$, i.e., obtained from the distribution defined in Eq. (54). We have repeated the above stated procedure. The details are given in Table 10, and we noticed that data on $Y_{U[i,2]}$ validate the model $f_{Y_{U[i,2]}}$, for all $i = 1,2,\ldots,6$.

TABLE 10

Model suitability of the distribution $f_{Y_{U[i,2]}}$, $i = 1,2,\ldots,6$ for the IGURRSS data (Table 7).

Distributional Model	Observations viewed as a sample of size 6 from the considered distribution	K-S statistic	Tabled Critical Value	Inference (at 5%)
$f_{Y_{U(1,2)}}$	8.129, 7.119, 4.598, 8.492, 5.989, 1.434	0.144	0.563	Model Validated
$f_{Y_{U(2,2)}}$	11.406, 10.705, 5.911, 22.239, 2.289, 16.004	0.201	0.563	
$f_{Y_{U(3,2)}}$	13.799, 20.419, 25.113, 2.130, 44.955, 16.736	0.262	0.563	
$f_{Y_{U(4,2)}}$	20.744, 9.789, 6.217, 11.779, 47.835, 12.590	0.282	0.563	
$f_{Y_{U(5,2)}}$	3.246, 5.284, 29.267, 8.067, 8.258, 11.741	0.380	0.563	
$f_{Y_{U(6,2)}}$	9.673, 23.634, 12.549, 17.859, 10.090, 11.351	0.457	0.563	

REMARK 7. *Since the sample size in Table 10 for each case is small with 6 observations, we have chosen the 5% critical value of the K-S test rather than computing the p - value for validating the model.*

Since the model $f_{Y_{U[i,k]}}$ is validated for all $i = 1,\ldots,6$, we recursively accept the pdf $f_{Y_{U[n,2]}}$ of the concomitant of $Y_{U[n,2]}$ of generalized upper (2) record height of acacia trees as

$$f_{Y_{U[n,2]}}(y) = \frac{e^{-\frac{y-10.8733}{6.4842}}}{6.4842\left[1+e^{-\frac{y-10.8733}{6.4842}}\right]^2}\left\{1+(1-2^{1-n})\left[\frac{1-e^{-\frac{y-10.8733}{6.4842}}}{1+e^{-\frac{y-10.8733}{6.4842}}}\right]\right\}. \tag{56}$$

Based on the data generated from the characteristics of acacia trees, we have modelled the pdf of the marginal distribution $f_X(x;74.4681, 11.2002)$ of the random variable X in (X,Y) as

$$f_X(x;74.4681, 11.2002) = \frac{1}{11.2002}e^{\frac{x-74.4681}{11.2002}}\left\{1+e^{\frac{x-74.4681}{11.2002}}\right\}^{-2} \tag{57}$$

and modelled the pdf of the auxiliary density determined by the pdf given in Eq. (56) as

$$f_{Y_{A(\omega,2)}}(y) = \frac{1}{6.4842}\frac{e^{-\frac{y-10.8733}{6.4842}}}{\left\{1+e^{-\frac{y-10.8733}{6.4842}}\right\}}\left\{1+\left[1-2\left(\frac{2}{3}\right)^{\omega}\right]\frac{1-e^{-\frac{y-10.8733}{6.4842}}}{1+e^{-\frac{y-10.8733}{6.4842}}}\right\}. \tag{58}$$

As a consequence of the results established in Thomas *et al.* (2014), we now state the following theorem and outline its proof.

THEOREM 8. *Let (X,Y) be a bivariate random vector with marginal pdf $f_X(x;74.4681,11.2002)$ of X as given by Eq. (57). Suppose the auxiliary pdf $f_{Y_{A[\omega,2]}}(y)$ determined by the n-th concomitant of generalized upper (2) record value is defined as in Eq. (58). Then, $f_X(x;74.4681,11.2002)$ and $f_{Y_{A[\omega,2]}}(y)$ together determine uniquely the parent bivariate density $f_{X,Y}(x,y)$ as*

$$\begin{aligned} f_{X,Y}(x,y) &= \frac{e^{-\frac{x-74.4681}{11.2002}}}{11.2002\left\{1+e^{-\frac{x-74.4681}{11.2002}}\right\}^2}\frac{e^{-\frac{x-10.8733}{6.48421}}}{6.48421\left\{1+e^{-\frac{x-10.8733}{6.48421}}\right\}^2} \\ &\quad \times\left\{1+\frac{1-e^{-\frac{x-74.4681}{11.2002}}}{1+e^{-\frac{x-74.4681}{11.2002}}}\frac{1-e^{-\frac{x-10.8733}{6.4842}}}{1+e^{-\frac{x-10.8733}{6.4842}}}\right\}. \end{aligned} \tag{59}$$

PROOF. Using Eq. (57) we conveniently write

$$f_X(x)=\frac{e^{-\frac{x-74.4681}{11.2002}}}{11.2002\left\{1+e^{-\frac{x-74.4681}{11.2002}}\right\}^2}, \tag{60}$$

$$F_X(x)=\frac{1}{1+e^{-\frac{x-74.4681}{11.2002}}}. \tag{61}$$

If we also write

$$f_Y(y)=\frac{e^{-\frac{x-10.8733}{6.4842}}}{6.48421\left\{1+e^{-\frac{x-10.8733}{6.4842}}\right\}^2}, \tag{62}$$

$$F_Y(y)=\frac{1}{1+e^{-\frac{x-10.8733}{6.4842}}}, \tag{63}$$

then, instead of Eq. (59), it is enough to prove that

$$f_{X,Y}(x,y)=f_X(x)f_Y(y)\{1+(1-2F_X(x))(1-2F_Y(y))\}. \tag{64}$$

From Eq. (58) we have

$$f_{Y_{A[\omega,2]}}(y)=f_Y(y)\left\{1-\left[1-2\left(\frac{2}{3}\right)^{\omega}\right][1-2F_Y(y)]\right\}, \tag{65}$$

but by definition

$$f_{Y_{A[\omega,2]}}(y) = \frac{2^{\omega}}{\Gamma(\omega)}\int_x [-\log(1-F_X(x))]^{\omega-1}[1-F_X(x)]^{k-1}f_{X,Y}(x,y)\mathrm{d}x.$$

Putting $U = 1 - \log(1 - F_X(x))$ we get

$$f_{Y_{A[\omega,2]}}(y) = \frac{f_Y(y)2^\omega}{\Gamma(\omega)} \int_0^\infty u^{\omega-1} e^{-u} f_{U|Y}(u|y)\mathrm{d}u. \tag{66}$$

Then we have

$$\int_0^\infty u^{\omega-1} e^{-u} f_{U|Y}(u|y)\mathrm{d}u = \frac{f_{Y_{A[\omega,2]}}(y)\Gamma(\omega)}{f_Y(y)2^\omega}. \tag{67}$$

Clearly, the left side integral is the Mellin transform of the function $e^{-u} f_{U|Y}(u|y)$. Hence, from Equations from (65) to (66) we write

$$\int_0^\infty u^{\omega-1} e^{-u} f_{U|Y}(u|y)\mathrm{d}u = \frac{\Gamma(\omega)}{2^\omega} - \left\{\frac{\Gamma(\omega)}{2^\omega} - \frac{2\Gamma(\omega)}{3^\omega}\right\}(1 - 2F_Y(y)). \tag{68}$$

We may write $M^{-1}(h(\omega))$ as the inverse Mellin transform of $h(\omega)$. Now from Bateman (1954, p.312), we use the expressions $M^{-1}\left\{\frac{\Gamma(\omega)}{2^\omega}\right\} = (e^{-u})^2$ and $M^{-1}\left\{\frac{\Gamma(\omega)}{3^\omega}\right\} = (e^{-u})^3$ in Eq. (68) to solve for $f_{U|Y}(u|y)$. It is given by

$$f_{U|Y}(u|y) = e^{-u} - \left\{e^{-u} - 2e^{-2u}\right\}(1 - 2F_Y(y)).$$

As $1 - F_X(x) = e^{-u}$, we have $F_X(x) = 1 - e^{-u}$, and consequently we have

$$f_{X,Y}(x,y) = f_X(x) f_Y(y)\{1 + (1 - 2F_X(x))(1 - 2F_Y(y))\}.$$

From this we obtain Eq. (64). Thus, the theorem is proved. □

As a consequence of the revalidation of the Morgenstern type bivariate logistic distribution, we conclude that the estimated mean timber volume of usable timber of acacia trees in the plantation from which the igurrss data were generated is $\tilde{v}_2$=10.8733. This estimate will be of much use to the authorities of the University of Kerala if they take a decision to dispose this tree wealth.

The IGURRSS strategy of sampling together with the approach adopted in this section can be adopted as such to modelling problems in similar situations for other bivariate populations as well.

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