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THE EXPONENTIATED XGAMMA DISTRIBUTION: A NEW MONOTONE FAILURE RATE MODEL AND ITS APPLICATIONS TO LIFETIME DATA

Abhimanyu Singh Yadav
Department of Statistics, Banaras Hindu University, Varanasi, India

Mahendra Saha
Department of Statistics, Central University of Rajasthan, Ajmer, India

Harsh Tripathi [1]
Department of Statistics, Central University of Rajasthan, Ajmer, India
Department of Mathematics, Lovely Professional University, Punjab, India

Sumit Kumar
Department of Engineering and Physical Sciences, Institute of Advanced Research, Gandhinagar, Gujarat, India

1. Introduction

In reliability analysis, the lifetime of any electronic device or items is varying in nature. Hence, it seems to be logical to model the lifetime of equipments/items with a specific probability distribution. The exponential distribution and it's different generalizations, e.g., Weibull, gamma, exponentiated exponential etc. have been often used to model the data with constant, monotone hazard rate etc. Sometimes, the finite mixtures of two or more probability distributions might be the better alternative choice to analyze any life time data sets, such as Lindley [see, Lindley (1958)], generalized Lindley [see, Nadarajah *et al.* (2011)], xgamma [see, Sen *et al.* (2016)], Akash distribution [see, Shankar (2015)]. In the same era of generalization of statistical distributions, the one parameter xgamma distribution (XGD) is one of them, a special finite mixture of exponential and gamma distributions, proposed by Sen *et al.* (2016). In many situations, finite mixture distributions arising from the standard distributions, play a better role in modelling lifetime phenomena as compared to the standard distributions. Recently, Yadav *et al.* (2019) introduced the inverted version of XGD which possesses the upside-down bathtub-shaped

[1] Corresponding Author. E-mail: rsearchstat21@gmail.com

hazard function. The XGD did not provide enough flexibility for analyzing different types of lifetime data as it is of one parameter. It will be useful to consider further alternatives of XGD to increase the flexibility for modelling purposes. In this article, we propose a two parameter family of distribution which generalizes the XGD, named as the exponentiated xgamma distribution (EXGD) and hence the name proposed. The procedure used is based on certain finite mixtures of exponential and gamma distributions. The shape parameter provides more flexibility for describing different types of data allowing hazard rate modelling. Moreover, we also derived some statistical characteristics, viz., moments, conditional moments, order statistics, reliability curves and indices in this present study.

However, the main objective of this article is three fold: First, we have introduced a new probability distribution and studied its several statistical properties as the generalized version of XGD, introduced by Shankar (2015). Second, we have estimate the model parameters, the survival function and the hazard rate function for a specified mission time by using different classical methods of estimation, viz., method of maximum likelihood (ML), method of ordinary least square and weighted least square (LS and WLS), method of Cramèr-von-Mises (CM) and method of maximum product spacing (MPS), respectively. Third, Monte Carlo simulations study has been carried out to compare the performances of considered methods of estimation of the proposed model. Recently, many authors have contributed in development of new distributions and estimation of model parameters by using different estimation methods, viz., Sen *et al.* (2019), Afify and Mohamed (2020), Nassar *et al.* (2020) and Afify *et al.* (2020) have introduced Quasi Xgamma-Geometric distribution, three parameter exponential distribution, estimation methods of alpha power exponential distribution, heavy tailed exponential distribution, Weibull Marshall-Olkin Lindley distribution, respectively. Here, our aim is to fill up this gap through this present study. To the best of our knowledge thus so far, no attempt has been made to introduced the generalized version of XGD as well as the different methods of estimation of model parameters, the survival function and the hazard rate functions, respectively.

Rest of the article is organized as follows: Section 2 introduced EXGD and it's reliability characteristics. Moments, generating function, mean deviation, conditional moments, order statistics and reliability curve etc. and algorithm of random number generation from EXGD are discussed in Section 3. Section 4 discussed the different methods of estimation, viz., maximum likelihood estimate (MLE), ordinary least square and weighted least square estimate (LSE and WLSE), Cramèr-von-Mises estimate (CME) and maximum product spacing estimate (MPSE) of the parameters (α, θ), survival function and hazard rate function. Monte Carlo simulation study is carried out to compare the performances among the estimators (MLE, LSE, WLSE, CME, MPSE) of the survival and hazard rate functions in terms of their mean squared errors (MSEs) in Section 5. Two data sets are analyzed to illustrate the applicability of the proposed model in real life scenario in Section 6 and finally concluding remarks are made in Section 7.

2. The model and its reliability characteristics

Recently, Sen *et al.* (2016) introduced a finite mixture of exponential (θ) and gamma $(3,\theta)$ distributions with mixing proportion $\pi_1 = \frac{\theta}{1+\theta}$ and $\pi_2 = 1-\pi_1 = \frac{1}{1+\theta}$ to obtain a probability distribution, named as xgamma distribution (XGD), given with the following probability density function (PDF) and cumulative distribution function(CDF)

$$f(x;\theta) \quad = \quad \frac{\theta^2}{(1+\theta)}\left(1+\frac{\theta}{2}x^2\right)e^{-\theta x} \quad ;x>0,\ \theta>0 \tag{1}$$

and

$$F(x;\theta) \quad = \quad 1-\frac{\left(1+\theta+\theta x+\frac{\theta^2x^2}{2}\right)}{(1+\theta)}e^{-\theta x} \quad ;x>0,\ \theta>0, \tag{2}$$

respectively, where θ is a scale parameter. They have also investigated some important mathematical, structural and survival properties and shown that XGD has more flexibility than the exponential as well as the Lindley distributions. In the era of generalization of new distributions by introducing an extra parameter to any baseline distributions, numerous methods are available in literature which possess different shapes of hazard rate. For example, Gupta and Kundu (2001) introduced the exponentiated exponential distributions as an alternative to Weibull and gamma distributions, Nadarajah *et al.* (2011) proposed generalized version of the Lindley distribution and shown the superiority of that model compared to the Lindley distribution, exponentiated Rayleigh distribution [see, Surles and Padgett (2001)], exponentiated gamma distribution [see, Shawky and Bakoban (2006)], exponentiated Weibull distribution [see, Mudholkar and Srivastava (1993)], exponentiated transmuted generalized Rayleigh distribution [see, Afify *et al.* (2015)], exponentiated Weibull-Pareto distribution [see, Afify *et al.* (2016)] and exponentiated Weibull-H family of distributions [see, Cordeiro *et al.* (2017)] etc. All these models are generalized by introducing a shape parameter as power of CDF of the base line model. Obviously, the model with more parameters provides more flexibility but it adds the complexity in the estimation procedure at the same time. Moving on the same path, here we proposed the exponentiated version of XGD, i.e., exponentiated XGD, named as EXGD.

Let X be a continuous random variable with CDF, given in Equation (3), then by introducing shape parameter α as the power of CDF, i.e., $[F(x;\theta)]^\alpha$, where, $\alpha \in \mathscr{R}^+$, provides more flexible shapes than the base line distribution. The CDF and PDF of EXGD are respectively obtained as

$$F(x;\alpha,\theta\quad) = \quad \left[1-\left(1+\theta+\theta x+\frac{\theta^2x^2}{2}\right)\frac{e^{-\theta x}}{1+\theta}\right]^\alpha; \quad x>0,\ \alpha>0,\ \theta>0 \tag{3}$$

and hence the corresponding PDF is obtained as

$$f(x;\alpha,\theta) = \frac{\alpha\theta^2}{1+\theta}\left[1-\left(1+\theta+\theta x+\frac{\theta^2 x^2}{2}\right)\frac{e^{-\theta x}}{1+\theta}\right]^{(\alpha-1)}\left(1+\frac{\theta x^2}{2}\right)e^{-\theta x}. \tag{4}$$

If $\alpha = 1$, then the PDF, given in Equation (4), coincide with the PDF, given in Equation (1), i.e., converted to XGD. Now, the shape of PDF and CDF for different values of α and θ are presented in Figure 1 which indicates that EXGD is right-skewed and unimodel or inverted J-shaped distribution.

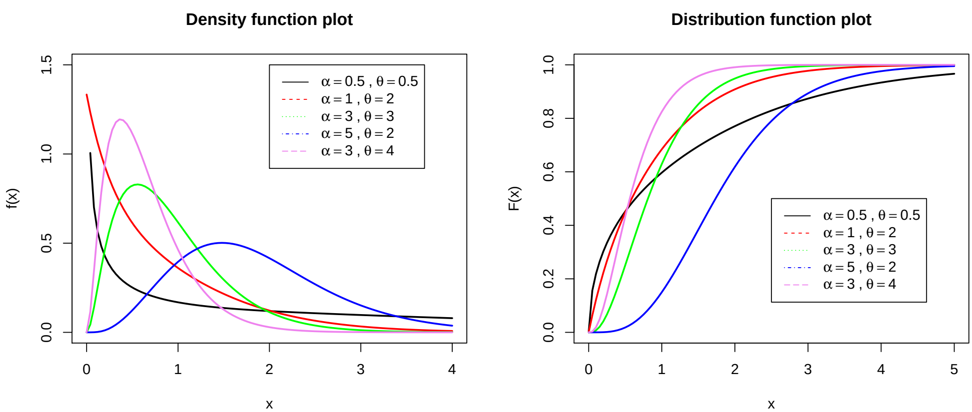

Figure 1 – PDF and CDF plots of EXGD.

It is to be noted that the basic tools for studying the ageing and associated characteristics of any lifetime equipments, may be a living organism or a system of components are the survival and hazard rate functions. The probability that a patient, component or system will survive beyond any specified time ($t > 0$) is called the survival function, where as hazard rate function is the conditional probability that the failure will be in time interval $(t, t+\Delta t)$, where Δt is very small time interval, given that a patient, component or system will survive beyond time ($t > 0$). The survival function and hazard rate function of EXGD(α,θ) for given values of t are

$$S(t;\alpha,\theta) = 1-\left[1-\left(1+\theta+\theta t+\frac{\theta^2 t^2}{2}\right)\frac{e^{-\theta t}}{1+\theta}\right]^{\alpha} \tag{5}$$

and

$$H(t;\alpha,\theta) = \left\{ \frac{\frac{\alpha\theta^2}{1+\theta}\left[1-\left(1+\theta+\theta t+\frac{\theta^2 t^2}{2}\right)\frac{e^{-\theta t}}{1+\theta}\right]^{(\alpha-1)}\left(1+\frac{\theta t^2}{2}\right)e^{-\theta t}}{1-\left[1-\left(1+\theta+\theta t+\frac{\theta^2 t^2}{2}\right)\frac{e^{-\theta t}}{1+\theta}\right]^{\alpha}} \right\}, \quad (6)$$

respectively and typical shapes of the survival and hazard rate functions for EXGD are displayed in Figure 2 for certain choices of α and θ.

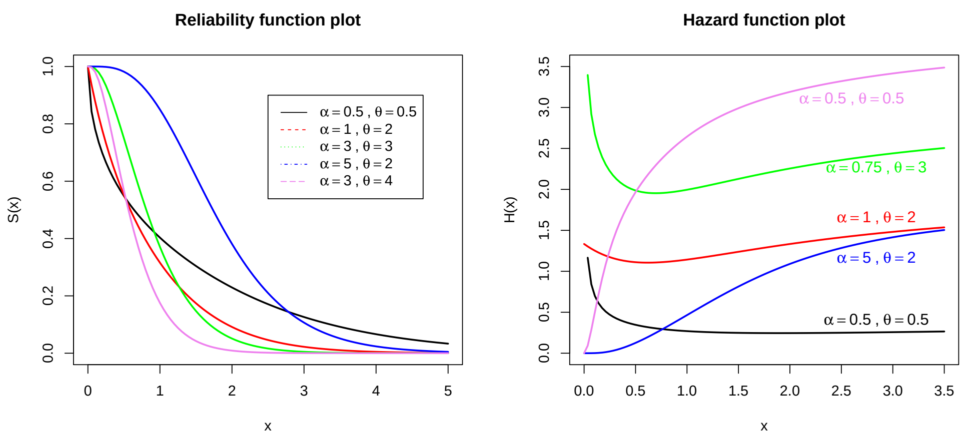

Figure 2 – Survival and hazard rate functions plot EXGD.

It is to be noted that for $\alpha = 1$, hazard rate function coincide with hazard rate function of baseline distribution [see, Sen *et al.* (2016)], while for other choices of shape and scale parameters, viz., $\alpha \geq 1$, $\theta > 1$, it follows the pattern of increasing failure rate (IFR), decreasing failure rate (DFR) when $\alpha < 1$, $\theta < 1$ and the pattern of bathtub shaped hazard rate may be traced for $\alpha < 1, \theta > 1$.

3. Some statistical properties

In this section, we have studied some statistical properties of EXGD such as moments, generating function, mean deviation, quantile function, conditional moments, order statistics etc.

3.1. Moments

Here, we have derived the expression for the moments of the EXGD(α, θ). The c-th order raw moment about origin for EXGD is given as

$$
\begin{aligned}
E(x^c) &= \int_0^\infty x^c f(x) dx \\
&= \int_0^\infty x^c \frac{\alpha\theta^2}{1+\theta} e^{-\theta x} \left[1 - \left(1+\theta+\theta x + \frac{\theta^2 x^2}{2}\right)\frac{e^{-\theta x}}{1+\theta}\right]^{(\alpha-1)} \left(1+\frac{\theta x^2}{2}\right) dx \\
&= \int_0^\infty x^c \frac{\alpha\theta^2}{1+\theta} e^{-\theta x} \left[1 - \left(1+\theta+\theta x + \frac{\theta^2 x^2}{2}\right)\frac{e^{-\theta x}}{1+\theta}\right]^{(\alpha-1)} dx \\
&\quad + \frac{\theta}{2} \int_0^\infty x^{c+2} \frac{\alpha\theta^2}{1+\theta} e^{-\theta x} \left[1 - \left(1+\theta+\theta x + \frac{\theta^2 x^2}{2}\right)\frac{e^{-\theta x}}{1+\theta}\right]^{(\alpha-1)} dx.
\end{aligned}
$$

Moreover, the expression for moments is not in explicit form, thus, the results based on the following Lemma 1, stated below, has been used to calculate the moments.

LEMMA 1. *Let*

$$
\begin{aligned}
K_1(a,b,c,\delta) &= \int_0^\infty x^c e^{-\delta x} \left[1 - \left(1+b+bx+\frac{b^2x^2}{2}\right)\frac{e^{-bx}}{1+b}\right]^{(a-1)} dx \\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\sum_{l=0}^{k} \binom{a-1}{i}\binom{i}{j}\binom{j}{k}\binom{k}{l} \\
&\quad \times \frac{(-1)^i b^j (b/2)^l \Gamma(c+k+l+1)}{(1+b)^i (bi+\delta)^{(c+k+l+1)}}
\end{aligned}
$$

and

$$
\begin{aligned}
K_2(a,b,c,\delta) &= \int_0^\infty x^{c+2} e^{-\delta x} \left[1 - \left(1+b+bx+\frac{b^2x^2}{2}\right)\frac{e^{-bx}}{1+b}\right]^{(a-1)} dx, \\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\sum_{l=0}^{k} \binom{a-1}{i}\binom{i}{j}\binom{j}{k}\binom{k}{l} \\
&\quad \times \frac{(-1)^i b^j (b/2)^l \Gamma(c+2+k+l+1)}{(1+b)^i (bi+\delta)^{(c+2+k+l+1)}}.
\end{aligned}
$$

PROOF.

$$K_1(a,b,c,\delta)=\sum_{i=0}^{\infty}\binom{a-1}{i}\frac{(-1)^i}{(1+b)^i}\int_0^{\infty}x^c e^{-\delta x-ibx}\left(1+b+bx+\frac{b^2x^2}{2}\right)^i dx$$

$$=\sum_{i=0}^{\infty}\binom{a-1}{i}\frac{(-1)^i}{(1+b)^i}\sum_{j=0}^{i}\binom{i}{j}b^j\sum_{k=0}^{j}\binom{j}{k}\int_0^{\infty}x^{(c+k)}\left(1+\frac{bx}{2}\right)^k e^{-\delta x-ibx}dx$$

$$=\sum_{i=0}^{\infty}\binom{a-1}{i}\frac{(-1)^i}{(1+b)^i}\sum_{j=0}^{i}\binom{i}{j}b^j\sum_{k=0}^{j}\binom{j}{k}\sum_{l=0}^{k}\binom{k}{l}(b/2)^l\int_0^{\infty}x^{c+k+l}e^{-\delta x-ibx}dx, \tag{7}$$

by use of gamma function, the above equation is written as

$$\begin{aligned}K_1(a,b,c,\delta) &= \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\sum_{l=0}^{k}\binom{a-1}{i}\binom{i}{j}\binom{j}{k}\binom{k}{l}\\ &\quad\times\frac{(-1)^i b^j(b/2)^l\Gamma(c+k+l+1)}{(1+b)^i(bi+\delta)^{(c+k+l+1)}}.\end{aligned}$$

Similarly solve the $K_2(a,b,c,\delta)$-

$$\begin{aligned}K_2(a,b,c,\delta) &= \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\sum_{l=0}^{k}\binom{a-1}{i}\binom{i}{j}\binom{j}{k}\binom{k}{l}\\ &\quad\times\frac{(-1)^i b^j(b/2)^l\Gamma(c+2+k+l+1)}{(1+b)^i(bi+\delta)^{(c+2+k+l+1)}}.\end{aligned}$$

□

By putting $\alpha=a$, $\theta=b$, $\delta=\theta$, $c=r$ in the above Lemma 1, the expression of r-th raw moment is given as

$$E(x^r)=\frac{\alpha\theta^2}{1+\theta}\left[K_1(\alpha,\theta,r,\theta)+\frac{\theta}{2}K_2(\alpha,\theta,r,\theta)\right]. \tag{8}$$

Hence, the first four raw moments of EXGD are obtained as

$$E(x)=\frac{\alpha\theta^2}{1+\theta}\left[K_1(\alpha,\theta,1,\theta)+\frac{\theta}{2}K_2(\alpha,\theta,1,\theta)\right],$$

$$E(x^2)=\frac{\alpha\theta^2}{1+\theta}\left[K_1(\alpha,\theta,2,\theta)+\frac{\theta}{2}K_2(\alpha,\theta,2,\theta)\right],$$

$$E(x^3) = \frac{\alpha\theta^2}{1+\theta}\left[K_1(\alpha,\theta,3,\theta) + \frac{\theta}{2}K_2(\alpha,\theta,3,\theta)\right],$$
$$E(x^4) = \frac{\alpha\theta^2}{1+\theta}\left[K_1(\alpha,\theta,4,\theta) + \frac{\theta}{2}K_2(\alpha,\theta,4,\theta)\right].$$

Moreover, the first four central moments can be obtained by using the relation between the raw moments and central moments. Hence, the Pearson measures of skewness (SK) and kurtosis (KR) based on second(μ_2), third (μ_3) and fourth (μ_4) central moments are obtained by using the following relation, given below:

$$SK = \frac{\mu_3^2}{\mu_2^3} \text{ and } KR = \frac{\mu_4}{\mu_2^2}.$$

3.2. *Generating functions*

Here, in this subsection, the different generating functions, namely, moment generating function $M_x(t)$, characteristics function $\Phi_x(t)$ and Kumulants generating function $K_x(t)$ are derived and presented in the following equations:

$$\begin{aligned} M_x(t) &= E(e^{tx}) = \int_0^\infty e^{tx} f(x)dx \\ &= \int_0^\infty \frac{\alpha\theta^2}{1+\theta}\left[1 - \left(1+\theta+\theta x+\frac{\theta^2x^2}{2}\right)\frac{e^{-\theta x}}{1+\theta}\right]^{(\alpha-1)}\left(1+\frac{\theta x^2}{2}\right) \\ &\quad \times e^{-x(\theta-t)}dx. \end{aligned} \tag{9}$$

By using the Lemma 1, the moment generating is given as

$$M_x(t) = \frac{\alpha\theta^2}{1+\theta}\left[K_1(\alpha,\theta,0,\theta-t) + \frac{\theta}{2}K_2(\alpha,\theta,0,\theta-t)\right].$$

The characteristic function for EXGD is simply obtained by replacing dummy parameter t by it, where, $i^2 = -1$, given as

$$\Phi_x(t) = \frac{\alpha\theta^2}{1+\theta}\left[K_1(\alpha,\theta,0,\theta-it) + \frac{\theta}{2}K_2(\alpha,\theta,0,\theta-it)\right].$$

The kumulants generating function is the logarithm of the moment generating function and is obtained as

$$K_x(t) = \log\Phi_x(t) = \log\left(\frac{\alpha\theta^2}{1+\theta}\right) + \log\left[K_1(\alpha,\theta,0,\theta-it) + \frac{\theta}{2}K_2(\alpha,\theta,0,\theta-it)\right].$$

3.3. *Mean deviation*

The mean deviation about mean of random variable X, having density function (3) is obtained by;

$$M.D = \int_{0}^{\infty} |(x-\mu)| f(x) dx$$

where, μ=E(x)

$$M.D = \int_{o}^{\mu} (\mu - x) f(x) dx + \int_{\mu}^{\infty} (x-\mu) f(x) dx,$$

$$M.D = \mu F(\mu) - \int_{0}^{\mu} x f(x) dx + \int_{\mu}^{\infty} x f(x) dx - \mu + \mu F(\mu),$$

$$M.D = 2\mu F(\mu) - 2\mu + 2\int_{\mu}^{\infty} x f(x) dx,$$

where, $F(\mu)$ stands for CDF of X upto point μ and

$$\int_{\mu}^{\infty} x f(x) dx = \frac{\alpha\theta^2}{1+\theta}\Big[L_1(\alpha,\theta,1,\theta,x) + \frac{\theta}{2} L_2(\alpha,\theta,1,\theta,x)\Big].$$

Using the value from the above integral one can evaluate mean deviation about mean.

3.4. *Quantile Function*

If $Q(p)$ be the quantile of order p of the EXGD random variable X, then the quantile function will be the solution of the following equation

$$p \quad = \quad \left[1 - \left(1 + \theta + \theta Q(p) + \frac{\theta^2 Q(p)^2}{2}\right) \frac{e^{-\theta Q(p)}}{1+\theta}\right]^{\alpha}. \tag{10}$$

The skewness and kurtosis are the two important measures to study the symmetry and convexity of the curve. The Bowley measure of skewness [see, Bowley (1920)] and Moors measure of kurtosis [see, Moors (1988)] based on quantile can be used and are given as follows:

$$SK \quad = \quad \frac{Q(\frac{3}{4}) - 2Q(\frac{1}{2}) + Q(\frac{1}{4})}{Q(\frac{3}{4}) - Q(\frac{1}{4})}$$

and

$$KR = \frac{Q(\frac{7}{8}) - Q(\frac{5}{8}) + Q(\frac{3}{8}) - Q(\frac{1}{8})}{Q(\frac{6}{8}) - Q(\frac{2}{8})}.$$

3.5. Conditional moments

The conditional moments about origin is defined as:

$$E(X^n|X > x) = \int_x^\infty x^n \frac{f(x)}{1 - F(x)} dx$$

where, F(x) is CDF of EXGD, given in Equation (3).

$$E(X^n|X > x) = \frac{1}{1 - F(x)} \frac{\alpha\theta^2}{1 + \theta} \left[\int_x^\infty x^n \eta(x_i, \theta) dx + \int_x^\infty \frac{\theta}{2} x^{n+2} \eta(x_i, \theta) dx \right], \qquad (11)$$

where $\eta(x_i, \theta) = e^{-\theta x}\left[1 - \left(1 + \theta + \theta x + \frac{\theta^2 x^2}{2}\right)\frac{e^{-\theta x}}{1+\theta}\right]^{(\alpha-1)}$. The above Equation (11) involves two integral which can not be easily tractable. Thus, the following Lemma is used to evaluate the integral.

LEMMA 2. *Let*

$$\begin{aligned} L_1(a, b, c, \delta, t) &= \int_t^\infty x^c e^{-\delta x} \left[1 - \left(1 + b + bx + \frac{b^2 x^2}{2}\right) \frac{e^{-bx}}{1 + b}\right]^{(a-1)} dx \\ &= \sum_{i=0}^{\infty} \sum_{j=0}^{i} \sum_{k=0}^{j} \sum_{l=0}^{k} \binom{a-1}{i} \binom{i}{j} \binom{j}{k} \binom{k}{l} \frac{(-1)^i b^j (b/2)^l}{(1+b)^i} \\ &\quad \times \frac{\Gamma c + k + l + 1, t(bi + \delta)}{(bi + \delta)^{(c+k+l+1)}} \end{aligned}$$

and

$$\begin{aligned} L_2(a, b, c, \delta, t \quad) &= \int_t^\infty x^{c+2} e^{-\delta x} \left[1 - \left(1 + b + bx + \frac{b^2 x^2}{2}\right) \frac{e^{-bx}}{1 + b}\right]^{(a-1)} dx \\ &= \sum_{i=0}^{\infty} \sum_{j=0}^{i} \sum_{k=0}^{j} \sum_{l=0}^{k} \binom{a-1}{i} \binom{i}{j} \binom{j}{k} \binom{k}{l} \frac{(-1)^i b^j (b/2)^l}{(1+b)^i} \\ &\quad \times \frac{\Gamma c + 2 + k + l + 1, t(bi + \delta)}{(bi + \delta)^{(c+2+k+l+1)}}. \end{aligned}$$

PROOF. The proof of the above lemmas are straight forward as the previous one.

$$
\begin{aligned}
L_1(a,b,c,\delta,t) &= \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\sum_{l=0}^{k}\binom{a-1}{i}\binom{i}{j}\binom{j}{k}\binom{k}{l}\frac{(-1)^i b^j (b/2)^l}{(1+b)^i} \\
&\quad \times \int_t^{\infty} x^{c+k+l} e^{-x(bi+\delta)} dx,
\end{aligned}
$$

On simplifications,

$$
\begin{aligned}
L_1(a,b,c,\delta,t) &= \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\sum_{l=0}^{k}\binom{a-1}{i}\binom{i}{j}\binom{j}{k}\binom{k}{l}\frac{(-1)^i b^j (b/2)^l}{(1+b)^i} \\
&\quad \times \frac{\Gamma c+k+l+1, t(bi+\delta)}{(bi+\delta)^{(c+k+l+1)}}.
\end{aligned}
$$

and in similar way

$$
\begin{aligned}
L_2(a,b,c,\delta,t) &= \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\sum_{l=0}^{k}\binom{a-1}{i}\binom{i}{j}\binom{j}{k}\binom{k}{l}\frac{(-1)^i b^j (b/2)^l}{(1+b)^i} \\
&\quad \times \frac{\Gamma c+2+k+l+1, t(bi+\delta)}{(bi+\delta)^{(c+2+k+l+1)}}.
\end{aligned}
$$

□

Hence, the expression of $E(X^n|X > x)$ is given as

$$
E(X^n|X>x) = \frac{1}{1-F(x)}\frac{\alpha\theta^2}{1+\theta}\left[L_1(\alpha,\theta,n,\theta,x)+\frac{\theta}{2}L_2(\alpha,\theta,n,\theta,x)\right].
$$

Using Lemma 2, the conditional moments are given as

$$
E(X|X>x) = \frac{1}{1-F(x)}\frac{\alpha\theta^2}{1+\theta}\left[L_1(\alpha,\theta,1,\theta,x)+\frac{\theta}{2}L_2(\alpha,\theta,1,\theta,x)\right],
$$

$$
E(X^2|X>x) = \frac{1}{1-F(x)}\frac{\alpha\theta^2}{1+\theta}\left[L_1(\alpha,\theta,2,\theta,x)+\frac{\theta}{2}L_2(\alpha,\theta,2,\theta,x)\right],
$$

$$
E(X^3|X>x) = \frac{1}{1-F(x)}\frac{\alpha\theta^2}{1+\theta}\left[L_1(\alpha,\theta,3,\theta,x)+\frac{\theta}{2}L_2(\alpha,\theta,3,\theta,x)\right]
$$

and

$$
E(X^4|xX>x) = \frac{1}{1-F(x)}\frac{\alpha\theta^2}{1+\theta}\left[L_1(\alpha,\theta,4,\theta,x)+\frac{\theta}{2}L_2(\alpha,\theta,4,\theta,x)\right].
$$

3.6. Order statistics

Let X_1, X_2, X_3,..., X_n is a random sample of size from EXGD(α, θ). Then, the ordered observations $X_{(1)} < X_{(2)} < X_{(3)} < ..... < X_{(n)}$ constitute the order statistic. Let $X_{(k;n)}$ denotes the k-th order statistic, then the CDF and PDF of k-th order statistic are computed as

$$F(X_{(k;n)} = t) = \sum_{j=k}^{n} \sum_{l=0}^{n-j} \binom{n}{j} \binom{n-j}{l} (-1)^l F^{(j+l)}(t)$$

and

$$f(X_{(k;n)} = t) = \frac{n!}{(n-k)!(k-1)!} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l \, [F(t)]^l \, F^{k-1}(t) \, f(t),$$

respectively. Using Equations (3) and (4), the CDF and PDF of k-th order statistic are

$$F(X_{(k;n)} = t) = \sum_{j=k}^{n} \sum_{l=0}^{n-j} \binom{n}{j} \binom{n-j}{l} (-1)^l \left[1 - \left(1 + \theta + \theta t + \frac{\theta^2 t^2}{2}\right) \frac{e^{-\theta t}}{1+\theta}\right]^{\alpha(j+l)} \tag{12}$$

and

$$\begin{aligned} f(X_{(k;n)} = t) \quad = \quad & \frac{\alpha\theta^2}{1+\theta} \frac{n!}{(n-k)!(k-1)!} e^{-\theta t} (1 + \frac{\theta t^2}{2}) \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l \\ & \times \left[1 - \left(1 + \theta + \theta t + \frac{\theta^2 t^2}{2}\right) \frac{e^{-\theta t}}{1+\theta}\right]^{\alpha(k+l)-1}, \end{aligned} \tag{13}$$

respectively. The distribution $X_{(1)} = \min(X_{(1)} < X_{(2)} < X_{(3)} < ..... < X_{(n)})$ and $X_{(n)} = \max(X_{(1)} < X_{(2)} < X_{(3)} < ..... < X_{(n)})$ can be computed with help of above Equations (12) and (13) by putting $k = 1$ and $k = n$ respectively.

3.7. Reliability curves and indices

Bonferroni and Lorenz curves are very important tools in actuarial and population science to study the income and poverty level. Besides these filed, the reliability curve also evaluated based on specific probability distributions. Let X be a random variable with PDF $f(x)$, defined in Equation (4) then Bonferroni curve $B(p)$ and Lorenz curve $L(p)$ are defined by the following Equations (14) and (15)

$$B(p) = \frac{1}{p\mu} \int_0^q x f(x) dx,$$

$$B(p)=\frac{1}{p\mu}\left[\mu-\int_{q}^{\infty}xf(x)dx\right],$$

$$B(p)=\frac{1}{p\mu}\left[\mu-\frac{\alpha\theta^2}{1+\theta}\left(L_1(\alpha,\theta,1,\theta,q)+\frac{\theta}{2}L_2(\alpha,\theta,1,\theta,q)\right)\right] \tag{14}$$

and

$$L(p)=\frac{1}{\mu}\int_{0}^{q}xf(x)dx,$$

$$L(p)=\frac{1}{\mu}\left[\mu-\int_{q}^{\infty}xf(x)dx\right],$$

$$L(p)=\frac{1}{\mu}\left[\mu-\frac{\alpha\theta^2}{1+\theta}\left(L_1(\alpha,\theta,1,\theta,q)+\frac{\theta}{2}L_2(\alpha,\theta,1,\theta,q)\right)\right], \tag{15}$$

where $\mu=E(x)$ and the indices based on these two curves are given as

$$B=1-\int_{0}^{1}B(p)dp$$

and

$$G=1-2\int_{0}^{1}L(p)dp.$$

3.8. Random number generation

To generate random number from EXGD (α,θ). The following steps may be used.

1. Specify the values of α, θ and n.
2. Generate U_i from uniform(0,1) distribution $(i=1,2,...,n)$.
3. Generate V_i from $gamma(\alpha,\theta)$ distribution $(i=1,2,...,n)$.
4. Generate W_i from $gamma(\alpha+2,\theta)$ distribution $(i=1,2,...,n)$.

5. If $U_i \leq \frac{\theta}{\theta+1}$, set $X_i = V_i$, otherwise set $X_i = W_i$.

If we take $\alpha = 1$, then we get the random variates from XGD(θ).

4. Different methods of estimation

In this section, we have used five methods of estimation to estimate the unknown parameters as well as survival function $S(t)$ and hazard rate function $H(t)$, namely, maximum likelihood estimation (MLE), least squares estimation (LSE), weighted least squares estimation (WLSE), Cramèr-von-Mises estimator estimation (CME) and maximum product of spacings estimation (MPSE) respectively for the EXGD(α, θ).

4.1. *Maximum likelihood estimator*

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from Equation (4). Then, the log-likelihood function for the observed random sample $x_1, x_2, \ldots, x_n$ is given as

$$\begin{aligned}\ell(\alpha,\theta) &= n\log\alpha + 2n\log\theta - n\log(1+\theta) - \theta\sum_{i=1}^{n} x_i + (\alpha-1)\sum_{i=1}^{n}\log U(x_i)\\ &\quad + \sum_{i=1}^{n}\log\left(1+\frac{\theta x_i^2}{2}\right)\end{aligned}$$

where,

$$U(x_i) = \left[1-\left(1+\theta+\theta x_i+\frac{\theta^2 x_i^2}{2}\right)\frac{e^{-\theta x_i}}{1+\theta}\right].$$

The resulting partial derivatives of the log-likelihood function are

$$\frac{\partial\,\ell(\alpha,\theta)}{\partial\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\log U(x_i) = 0 \tag{16}$$

and

$$\begin{aligned}\frac{\partial\,\ell(\alpha,\theta)}{\partial\theta} &= -\frac{n(\theta+2)}{\theta(1+\theta)} - \sum_{i=1}^{n}\frac{(x_i^2/2)}{(1+(\theta x_i^2/2))} + \sum_{i=1}^{n} x_i +\\ &\quad \frac{(1-\alpha)}{(1+\theta)^2}\sum_{i=1}^{n}\frac{e^{-\theta x_i}(2\theta x_i + \theta^2 x_i + (\theta^2 x_i^2/2) + (\theta^2 x_i^3/2) + (\theta^3 x_i^3/2))}{U(X_i)}.\end{aligned} \tag{17}$$

Equating these partial derivatives to zero do not yield closed-form solutions for the MLEs and thus a numerical method is used for solving these equations simultaneously.

Substituting the MLEs $(\hat{\alpha}_{mle}, \hat{\theta}_{mle})$ of (α, θ) and using the invariance properties of MLES, we can get the estimators of $S(t)$ and $H(t)$ as

$$\hat{S}(t)_{mle} = 1 - \left[1 - \left(1 + \hat{\theta}_{mle} + \hat{\theta}_{mle} t + \frac{\hat{\theta}^2_{mle} t^2}{2}\right) \frac{e^{-\hat{\theta}_{mle} t}}{1 + \hat{\theta}_{mle}}\right]^{\hat{\alpha}_{mle}} \tag{18}$$

and

$$\hat{H}(t)_{mle} = \left\{ \frac{\frac{\alpha \hat{\theta}^2_{mle}}{1+\hat{\theta}_{mle}} [b_1]^{(\hat{\alpha}_{mle}-1)} \times a_1}{1 - [b_1]^{\hat{\alpha}_{mle}}} \right\}, \tag{19}$$

where

$$a_1 = \left(1 + \frac{\hat{\theta}_{mle} t^2}{2}\right) e^{-\hat{\theta}_{mle} t}$$

and

$$b_1 = 1 - \left(1 + \hat{\theta}_{mle} + \hat{\theta}_{mle} t + \frac{\hat{\theta}^2_{mle} t^2}{2}\right) \frac{e^{-\hat{\theta}_{mle} t}}{1 + \hat{\theta}_{mle}},$$

respectively for the given value of t.

4.2. *Ordinary least square and weighted least square estimator*

The least square estimator (LSE) and the weighted least square estimator (WLSE) were proposed by Swain *et al.* (1988) to estimate the parameters of the Beta distribution. Suppose $F(x_{(j)})$ denotes the CDF of the ordered random variables $x_{(1)} < x_{(2)} < \cdots < x_{(n)}$, where, $\{x_1, x_2, \cdots, x_n\}$ is a random sample of size n from a distribution function $F(\cdot)$. Therefore, in this case, the LSEs of (α, θ), say, $(\hat{\alpha}_{lse}, \hat{\theta}_{lse})$ can be obtained by minimizing

$$\mathscr{L}(\alpha, \theta) = \sum_{i=1}^{n} \left[F(x_{i:n} | \alpha, \theta) - \frac{i}{n+1}\right]^2$$

with respect to α and θ, where, $F(\cdot)$ is the CDF, given in Equation (3). Equivalently, it can be obtained by solving

$$\frac{\partial \mathscr{L}(\alpha, \theta)}{\partial \alpha} = \sum_{i=1}^{n} \left[F(x_{i:n} | \alpha, \theta) - \frac{i}{n+1}\right] \frac{\partial F(x_{i:n} | \alpha, \theta)}{\partial \alpha} = 0$$

and

$$\frac{\partial \mathscr{L}(\alpha,\theta)}{\partial \theta}=\sum_{i=1}^{n}\left[F(x_{i:n}|\alpha,\theta)-\frac{i}{n+1}\right]\frac{\partial F(x_{i:n}|\alpha,\theta)}{\partial \theta}=0,$$

where

$$\begin{aligned}\frac{\partial F(x_{i:n}|\alpha,\theta)}{\partial \alpha} &= \left[1-\left(1+\theta+\theta x+\frac{\theta^2x^2}{2}\right)\frac{e^{-\theta x}}{1+\theta}\right]^{\alpha}\times \\ &\quad \log\left[1-\left(1+\theta+\theta x+\frac{\theta^2x^2}{2}\right)\frac{e^{-\theta x}}{1+\theta}\right]\end{aligned} \tag{20}$$

and

$$\frac{\partial F(x_{i:n}|\alpha,\theta)}{\partial \theta} = \alpha\left[1-\left(1+\theta+\theta x+\frac{\theta^2x^2}{2}\right)\frac{e^{-\theta x}}{1+\theta}\right]^{\alpha-1}\times a_2, \tag{21}$$

where $a_2=\left[\frac{e^{-\theta x}}{(1+\theta)^2}\left(1+\theta+\theta x+\frac{\theta^2x^2}{2}\right)(1+x+\theta x)-(1+x+\theta x^2)\frac{e^{-\theta x}}{(1+\theta)}\right]$.

Hence, substituting the LSEs, we can get the estimators of survival and hazard rate functions as

$$\hat{S}(t)_{lse} = 1-\left[1-\left(1+\hat{\theta}_{lse}+\hat{\theta}_{lse}t+\frac{\hat{\theta}^2_{lse}t^2}{2}\right)\frac{e^{-\hat{\theta}_{lse}t}}{1+\hat{\theta}_{lse}}\right]^{\hat{\alpha}_{lse}} \tag{22}$$

and

$$\hat{H}(t)_{lse}=\left\{\frac{\frac{\alpha\hat{\theta}^2_{lse}}{1+\hat{\theta}_{lse}}[b_2]^{(\hat{\alpha}_{lse}-1)}\times a_3}{1-[b_2]^{\hat{\alpha}_{lse}}}\right\}, \tag{23}$$

respectively. Where,

$$a_3=\left(1+\frac{\hat{\theta}_{lse}t^2}{2}\right)e^{-\hat{\theta}_{lse}t}$$

and

$$b_2=1-\left(1+\hat{\theta}_{lse}+\hat{\theta}_{lse}t+\frac{\hat{\theta}^2_{lse}t^2}{2}\right)\frac{e^{-\hat{\theta}_{lse}t}}{1+\hat{\theta}_{lse}}$$

.

WLSE proposed by Swain *et al.* (1988). The WLSEs of (α,θ), say, $(\hat{\alpha}_{wlse},\hat{\theta}_{wlse})$ can be obtained by minimizing the function $\mathscr{W}(\alpha,\theta)$

$$\mathscr{W}(\alpha,\theta)=\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F(x_{i:n}|\alpha,\theta)-\frac{i}{n+1}\right]^2.$$

These estimators can also be obtained by partial derivatives of $\mathscr{W}(\alpha,\theta)$ with respect to α and θ and by equating both the equations to zero

$$\frac{\partial\mathscr{W}(\alpha,\theta)}{\partial\alpha}=\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F(x_{i:n}|\alpha,\theta)-\frac{i}{n+1}\right]\frac{\partial F(x_{i:n}|\alpha,\theta)}{\partial\alpha}=0$$

and

$$\frac{\partial\mathscr{W}(\alpha,\theta)}{\partial\theta}=\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F(x_{i:n}|\alpha,\theta)-\frac{i}{n+1}\right]\frac{\partial F(x_{i:n}|\alpha,\theta)}{\partial\theta}=0,$$

where $\frac{\partial F(x_{i:n}|\alpha,\theta)}{\partial\alpha}$ and $\frac{\partial F(x_{i:n}|\alpha,\theta)}{\partial\theta}$ are already defined in Equations (20) and (21) respectively. Hence, the estimators of survival and hazard rate functions are obtained as

$$\hat{S}(t)_{wlse} = 1-\left[1-\left(1+\hat{\theta}_{wlse}+\hat{\theta}_{wlse}t+\frac{\hat{\theta}^2_{wlse}t^2}{2}\right)\frac{e^{-\hat{\theta}_{wlse}t}}{1+\hat{\theta}_{wlse}}\right]^{\hat{\alpha}_{wlse}} \tag{24}$$

and

$$\hat{H}(t)_{wlse} = \left\{\frac{\frac{\alpha\hat{\theta}^2_{wlse}}{1+\hat{\theta}_{wlse}}[b_3]^{(\hat{\alpha}_{wlse}-1)}\times a_4}{1-[b_3]^{\hat{\alpha}_{wlse}}}\right\}. \tag{25}$$

Where

$$a_4=\left(1+\frac{\hat{\theta}_{wlse}t^2}{2}\right)e^{-\hat{\theta}_{wlse}t}$$

and

$$b_3=1-\left(1+\hat{\theta}_{wlse}+\hat{\theta}_{wlse}t+\frac{\hat{\theta}^2_{wlse}t^2}{2}\right)\frac{e^{-\hat{\theta}_{wlse}t}}{1+\hat{\theta}_{wlse}}.$$

4.3. Cramèr-von-Mises estimator

To motivate our choice of Cramèr-von-Mises type minimum distance estimators, Shawky and Bakoban (2006) provided empirical evidence that the bias of the estimator is smaller than the other minimum distance estimators. Thus, the Cramer-von-Mises estimators of (α,θ), say, $(\hat{\alpha}_{cme},\hat{\theta}_{cme})$ can be obtained by minimizing the function

$$\mathscr{C}(\alpha,\theta)=\frac{1}{12n}+\sum_{i=1}^{n}\left(F(x_{i:n}|\alpha,\theta)-\frac{2i-1}{2n}\right)^2$$

with respect to α and θ. The estimators can also be obtained by solving the non-linear equations

$$\frac{\partial\mathscr{C}(\alpha,\theta)}{\partial\alpha}=\sum_{i=1}^{n}\left(F(x_{i:n}|\alpha,\theta)-\frac{2i-1}{2n}\right)\frac{\partial F(x_{i:n}|\alpha,\theta)}{\partial\alpha}=0$$

and

$$\frac{\partial\mathscr{C}(\alpha,\theta)}{\partial\theta}=\sum_{i=1}^{n}\left(F(x_{i:n}|\alpha,\theta)-\frac{2i-1}{2n}\right)\frac{\partial F(x_{i:n}|\alpha,\theta)}{\partial\theta}=0,$$

where $\frac{\partial F(x_{i:n}|\alpha,\theta)}{\partial\alpha}$ and $\frac{\partial F(x_{i:n}|\alpha,\theta)}{\partial\theta}$ are defined in Equations (20) and (21) respectively. Hence, substituting the CMEs, we can get the estimators of survival and hazard rate functions as

$$\hat{S}(t)_{cme} = 1-\left[1-\left(1+\hat{\theta}_{cme}+\hat{\theta}_{cme}t+\frac{\hat{\theta}^2_{cme}t^2}{2}\right)\frac{e^{-\hat{\theta}_{cme}t}}{1+\hat{\theta}_{cme}}\right]^{\hat{\alpha}_{cme}} \quad (26)$$

and

$$\hat{H}(t)_{cme} = \left\{\frac{\frac{\alpha\hat{\theta}^2_{cme}}{1+\hat{\theta}_{cme}}[b_4]^{(\hat{\alpha}_{cme}-1)}\times a_5}{1-[b_4]^{\hat{\alpha}_{cme}}}\right\}, \quad (27)$$

where

$$a_5=\left(1+\frac{\hat{\theta}_{cme}t^2}{2}\right)e^{-\hat{\theta}_{cme}t}$$

and

$$b_4=1-\left(1+\hat{\theta}_{cme}+\hat{\theta}_{cme}t+\frac{\hat{\theta}^2_{cme}t^2}{2}\right)\frac{e^{-\hat{\theta}_{cme}t}}{1+\hat{\theta}_{cme}}.$$

4.4. *Maximum product spacings estimator*

The maximum product spacing method has been introduced by H and K (1979), Cheng and Amin (1983) as an alternative to MLE for the estimation of the unknown parameters of continuous univariate distributions. This method was also derived independently by Ranneby (1984) as an approximation to the Kullback-Leibler measure of information. To motivate our choice, Cheng and Amin (1983) proved that this method is as efficient as the MLE and consistent under more general conditions. Let us define

$$\mathscr{D}_i(\alpha,\theta) = F\left(x_{i:n} \mid \alpha,\theta\right) - F\left(x_{i-1:n} \mid \alpha,\theta\right) \qquad i = 1,2,\ldots,n,$$

where $F(x_{0:n} \mid \alpha,\theta) = 0$, $F(x_{n+1:n} \mid \theta) = 1 - F(x_n \mid \theta)$ and clearly $\sum_{i=1}^{n+1} \mathscr{D}_i(\theta) = 1$. The MPSEs of the parameters (α,θ), say, $(\hat{\alpha}_{mpse}, \hat{\theta}_{mpse})$ are obtained by maximizing the geometric mean of the spacings with respect to α and θ as

$$GM = \left[\prod_{i=1}^{n+1} \mathscr{D}_i(\alpha,\theta)\right]^{\frac{1}{n+1}},$$

or, equivalently, by maximizing the function

$$H = \log GM = \frac{1}{n+1}\sum_{i=1}^{n+1} \log \mathscr{D}_i(\alpha,\theta)$$

with respect to α and θ. The estimates of α and θ are obtained by solving the following non-linear equations

$$\frac{\partial H}{\partial \alpha} = \frac{1}{n+1}\sum_{i=1}^{n+1} \frac{1}{D_i(\alpha,\theta)} \frac{\partial D_i(\alpha,\theta)}{\partial \alpha} = 0$$

and

$$\frac{\partial H}{\partial \theta} = \frac{1}{n+1}\sum_{i=1}^{n+1} \frac{1}{D_i(\alpha,\theta)} \frac{\partial D_i(\alpha,\theta)}{\partial \theta} = 0,$$

where

$$\frac{\partial D_i(\alpha,\theta)}{\partial \alpha} = \frac{\partial F\left(x_{i:n} \mid \alpha,\theta\right)}{\partial \alpha} - \frac{\partial F\left(x_{i-1:n} \mid \alpha,\theta\right)}{\partial \alpha}$$

and

$$\frac{\partial D_i(\alpha,\theta)}{\partial \theta} = \frac{\partial F\left(x_{i:n} \mid \alpha,\theta\right)}{\partial \theta} - \frac{\partial F\left(x_{i-1:n} \mid \alpha,\theta\right)}{\partial \theta}$$

can be computed from Equations (21) and (22) respectively. Hence, by using $(\hat{\alpha}_{mps}, \hat{\theta}_{mps})$ the estimators of survival and hazard rate functions are obtained as

$$\hat{S}(t)_{mps} \quad = \quad 1 - \left[1 - \left(1 + \hat{\theta}_{mps} + \hat{\theta}_{mps} t + \frac{\hat{\theta}^2_{mps} t^2}{2}\right) \frac{e^{-\hat{\theta}_{mps} t}}{1 + \hat{\theta}_{mps}}\right]^{\hat{\alpha}_{mps}} \tag{28}$$

and

$$\hat{H}(t)_{mps} = \left\{ \frac{\frac{\alpha\hat{\theta}^2_{mps}}{1+\hat{\theta}_{mps}}[b_5]^{(\hat{\alpha}_{mps}-1)} \times a_6}{1-[b_5]^{\hat{\alpha}_{mps}}} \right\} \tag{29}$$

where $a_6 = \left(1 + \frac{\hat{\theta}_{mps} t^2}{2}\right) e^{-\hat{\theta}_{mps} t}$ and

$$b_5 = 1 - \left(1 + \hat{\theta}_{mps} + \hat{\theta}_{mps} t + \frac{\hat{\theta}^2_{mps} t^2}{2}\right) \frac{e^{-\hat{\theta}_{mps} t}}{1+\hat{\theta}_{mps}}.$$

5. SIMULATION STUDY

In this section, the numerical comparisons have been made through Monte Carlo simulation study to assess the performances of the proposed estimators (MLE, LSE, WLSE, CME and MPSE) of the parameters α ,θ, survival function $S(t)$ and hazard rate function $H(t)$ for EXGD, discussed in Section 5. For this purpose, we considered different choices of the parameters values, such as, (α,θ)=(0.5, 0.75), (0.75, 0.75), (1.5, 0.75), (1.5, 1.5), specified time points $t = 2,3,4,2,1$ and different sample sizes $n =$ 10, 20, 30, 50 and 100 respectively. For each design, sample with each of size n are drawn from the original sample and replicated 3,000 times. It is to be noted that the estimates of the parameters are not explicitly available. Therefore, we may use any numerical methods of solution for so. The estimates of α and θ are obtained by using non-linear minimization (NLM) [see, Dennis and Schnabel (1983)] method. Using NLM method, we have to iterate the negative log-likelihood function with some initial guess value for the estimator say $\alpha_0 = 0.01$ and $\theta_0 = 0.01$, and get the estimates of α and θ as $\hat{\alpha}$ and $\hat{\theta}$ respectively. For the specified time $t > 0$, $S(t)$ and $H(t)$ are nothing but the functions of the parameters, the estimates of $S(t)$ and $H(t)$ are computed by plug-in principle. First, we have calculated the average estimates and corresponding mean squared errors (MSEs) of the parameters (α,θ), $S(t)$ and $H(t)$ using MLE, LSE, WLSE, CME, MPSE. The results are reported in Tables 1 2, 3, and 4 respectively.

From Tables 1, 2, 3 and 4 it has been observed that as the sample sizes increases, the MSEs of all the estimators decreases, which actually verified the consistency of the estimators that we have considered. Also from Tables 1, 2, it is observed that MLE is the best as it produces the least MSEs for most of the considered set ups in our studies for α, and θ. From Table 3, 4, the similar patterns are observed in case of $S(t)$ and $H(t)$ respectively. The overall positions of the estimators in terms of MSEs are followed the order $MLE < WLSE < LSE < CME < MPSE$ for α, θ, $S(t)$ and $H(t)$ respectively.

TABLE 1

True value of (α, θ) and its estimates by using different methods of estimation along with their corresponding MSEs.

			Estimates and corresponding MSEs of α					Estimates and corresponding MSEs of θ				
n	α	θ	MLE MSE	LSE MSE	WLSE MSE	CME MSE	MPSE MSE	MLE MSE	LSE MSE	WLSE MSE	CME MSE	MPSE MSE
10	0.5	0.75	0.858103	0.899127	0.838471	1.370647	1.486183	0.861804	0.736006	0.734845	0.891224	1.094162
			0.616217	0.190686	0.168647	0.342776	0.543987	0.104068	0.135183	0.113724	0.209664	0.309406
20			0.687265	0.674997	0.672025	0.765801	0.839007	0.779506	0.715396	0.724732	0.787123	0.891469
			0.096622	0.086871	0.086844	0.131625	0.195894	0.031792	0.041674	0.036979	0.050564	0.062660
30			0.649013	0.658538	0.656208	0.713501	0.744831	0.759343	0.715654	0.726309	0.762084	0.836837
			0.053669	0.072251	0.074118	0.097786	0.137928	0.019812	0.026924	0.023464	0.029244	0.032009
50			0.624580	0.646945	0.642885	0.677630	0.683910	0.741225	0.714757	0.724282	0.742032	0.792090
			0.032337	0.061171	0.063110	0.074756	0.097640	0.010642	0.016292	0.013469	0.016240	0.014095
100			0.609841	0.639528	0.634343	0.654150	0.641968	0.728270	0.713199	0.721259	0.726596	0.757134
			0.019565	0.052268	0.054793	0.058405	0.071318	0.005235	0.008169	0.006664	0.007607	0.005251
10	0.75	0.75	1.168497	1.093585	1.086024	1.764682	1.921184	0.887667	0.762535	0.771820	0.913919	1.112509
			1.702480	0.108318	0.116774	0.188454	0.289401	0.104995	0.108318	0.116774	0.188454	0.289401
20			0.943057	0.886851	0.887521	1.016612	1.159740	0.813603	0.753190	0.760695	0.822166	0.921410
			0.162589	0.040466	0.036392	0.053729	0.072392	0.036197	0.040466	0.036392	0.053729	0.072392
30			0.880700	0.846278	0.848720	0.920860	1.016204	0.787237	0.746076	0.753716	0.790603	0.862650
			0.075333	0.023766	0.020923	0.028372	0.035516	0.019904	0.023766	0.020923	0.028372	0.035516
50			0.852951	0.845306	0.844469	0.888061	0.937417	0.769180	0.749023	0.753602	0.775309	0.819161
			0.039886	0.013416	0.011324	0.015020	0.016171	0.010278	0.013416	0.011324	0.015020	0.016171
100			0.829329	0.832868	0.832206	0.853041	0.874911	0.758925	0.747337	0.751735	0.760205	0.787224
			0.019249	0.006749	0.005587	0.007080	0.006704	0.004951	0.006749	0.005587	0.007080	0.006704
10	1.5	0.75	1.972062	1.973538	1.786945	3.571968	3.817322	0.823725	0.711375	0.713490	0.837141	1.004425
			3.486727	0.699307	0.686847	0.651645	0.656050	0.067284	0.078870	0.069582	0.119857	0.175188
20			1.589454	1.386714	1.407137	1.637962	1.997092	0.776430	0.718191	0.726669	0.776971	0.866473
			0.371656	0.642130	0.625459	0.620319	0.602912	0.024328	0.031917	0.027963	0.038275	0.045122
30			1.488592	1.338427	1.367819	1.475458	1.740939	0.756870	0.714173	0.723774	0.751693	0.820000
			0.179638	0.635307	0.618250	0.599827	0.579627	0.014300	0.019066	0.016411	0.019867	0.022128
50			1.430244	1.328887	1.357467	1.405899	1.583591	0.743564	0.717294	0.725857	0.739432	0.785663
			0.092378	0.623144	0.608371	0.589666	0.519365	0.008028	0.011585	0.009657	0.011314	0.010359
100			1.388906	1.312627	1.342134	1.349045	1.470795	0.734675	0.718115	0.725374	0.729052	0.758378
			0.050500	0.616432	0.604382	0.579610	0.515416	0.004108	0.006105	0.004944	0.005687	0.004231

TABLE 2
True value of (α, θ) and its estimates by using different methods of estimation along with their corresponding MSEs.

n	α	θ	Estimates and corresponding MSEs of α					Estimates and corresponding MSEs of θ				
			MLE MSE	LSE MSE	WLSE MSE	CME MSE	MPSE MSE	MLE MSE	LSE MSE	WLSE MSE	CME MSE	MPSE MSE
10	1.5	1.5	2.184685	1.924997	1.927290	3.247726	4.306946	1.715494	1.491082	1.523281	1.775920	2.127722
			5.698922	0.401934	0.423108	0.692975	0.950378	0.353216	0.401934	0.423108	0.692975	0.950378
20			1.728760	1.540518	1.640325	1.803127	2.179444	1.592667	1.491895	1.537462	1.620707	1.798670
			0.753906	0.153095	0.166985	0.199500	0.240382	0.120898	0.153095	0.166985	0.199500	0.240382
30			1.608230	1.471991	1.659420	1.622261	1.884244	1.542194	1.467905	1.534397	1.549800	1.682135
			0.236033	0.090937	0.122130	0.103317	0.119059	0.070367	0.090937	0.122130	0.103317	0.119059
50			1.541791	1.453209	1.672222	1.536734	1.711651	1.513805	1.471887	1.546077	1.520022	1.606875
			0.112280	0.051412	0.115556	0.054595	0.055783	0.038958	0.051412	0.115556	0.054595	0.055783
100			1.496888	1.448993	1.681995	1.489394	1.588190	1.492056	1.474685	1.545019	1.498509	1.544732
			0.048350	0.025602	0.097288	0.025803	0.021653	0.018332	0.025602	0.097288	0.025803	0.021653
10	2.5	3	4.762552	5.790492	4.546887	4.986600	4.657430	3.110367	2.707796	2.711887	3.188887	3.829733
			9.304540	1.163874	1.022612	2.152210	3.458128	0.899018	1.206078	1.060726	1.713322	2.378394
20			3.279310	2.950881	2.975397	3.580214	4.393641	2.873421	2.688847	2.714942	2.907140	3.227455
			3.074617	0.463685	0.432908	0.676848	0.966691	0.333062	0.524838	0.467966	0.519707	0.489236
30			3.062675	2.819161	2.849529	3.172138	3.722313	2.819203	2.694522	2.718315	2.836954	3.067424
			1.527878	0.309202	0.288670	0.416541	0.580386	0.240936	0.364681	0.320355	0.329587	0.262962
50			2.877012	2.741952	2.768424	2.933081	3.266679	2.758044	2.690167	2.707660	2.773928	2.921878
			0.649278	0.193189	0.179192	0.242458	0.313645	0.174678	0.253022	0.221532	0.218531	0.141767
100			2.768558	2.717331	2.732587	2.807749	2.977923	2.725037	2.704932	2.712260	2.746434	2.820043
			0.283257	0.115785	0.107557	0.136939	0.160536	0.129146	0.160853	0.145297	0.140505	0.090493

TABLE 3

True value of $S(t)$ and $H(t)$ and their estimates using different methods of estimation along with their corresponding MSEs.

				Estimates and corresponding MSEs of $S(t)$					Estimates and corresponding MSEs of $H(t)$				
n	S(t)	H(t)	t	MLE MSE	LSE MSE	WLSE MSE	CVM MSE	MPS MSE	MLE MSE	LSE MSE	WLSE MSE	CVM MSE	MPS MSE
10	0.33503	0.40240	2	0.403013 0.021538	0.430824 0.024324	0.426302 0.023197	0.423072 0.027914	0.414464 0.018229	0.438652 0.037863	0.364962 0.048998	0.368017 0.044230	0.445785 0.086011	0.334709 0.021926
20				0.403385 0.012189	0.429153 0.016683	0.418722 0.017445	0.425107 0.017243	0.410056 0.011637	0.398460 0.010502	0.359227 0.013899	0.440347 0.069946	0.394786 0.016244	0.339972 0.010702
30				0.401571 0.009332	0.426333 0.013436	0.419664 0.012155	0.423571 0.013490	0.406458 0.009269	0.387889 0.007147	0.356576 0.009535	0.364420 0.008397	0.379065 0.009423	0.345460 0.008367
50				0.401882 0.007138	0.426153 0.011346	0.419082 0.009988	0.424493 0.011240	0.405062 0.007282	0.377926 0.003958	0.356429 0.006000	0.363772 0.005098	0.369541 0.005395	0.349492 0.005522
100				0.402934 0.005990	0.425630 0.009743	0.418698 0.008481	0.424777 0.009637	0.404606 0.006134	0.370707 0.002538	0.356519 0.004013	0.362264 0.003348	0.362969 0.003563	0.354445 0.003661
10	0.31262	0.40291	3	0.325082 0.015375	0.352741 0.015389	0.349842 0.015029	0.332612 0.017722	0.353879 0.013046	0.489331 0.055758	0.402821 0.047612	0.411399 0.084501	0.497418 0.090799	0.366032 0.026334
20				0.328780 0.007786	0.345736 0.008685	0.342608 0.008270	0.334888 0.009111	0.347057 0.007393	0.436451 0.017472	0.396490 0.017078	0.401703 0.015913	0.438743 0.023489	0.366626 0.012204
30				0.331761 0.005149	0.345800 0.006207	0.338471 0.006461	0.338534 0.006244	0.345380 0.005203	0.421905 0.009108	0.396535 0.010162	0.432652 0.013132	0.423675 0.012451	0.371192 0.007478
50				0.333063 0.003094	0.343098 0.003847	0.339975 0.003516	0.338645 0.003756	0.342495 0.003313	0.408748 0.004797	0.392217 0.006045	0.397390 0.005536	0.407922 0.006606	0.374798 0.004668
100				0.332785 0.001792	0.340737 0.002369	0.337992 0.002119	0.338478 0.002290	0.338298 0.001963	0.402731 0.002321	0.394604 0.003128	0.398032 0.002884	0.402365 0.003220	0.382971 0.002454
10	0.36750	0.39248	4	0.348998 0.017136	0.366588 0.015452	0.366138 0.015080	0.347039 0.020017	0.378447 0.012482	0.460832 0.045915	0.389332 0.044117	0.389845 0.037919	0.476102 0.080885	0.348926 0.024278
20				0.355312 0.008550	0.363199 0.008595	0.362060 0.008426	0.352943 0.009919	0.374293 0.006927	0.415521 0.015217	0.380523 0.016055	0.386365 0.015295	0.418267 0.020984	0.351690 0.011675
30				0.355121 0.005470	0.359065 0.005718	0.358222 0.005536	0.351977 0.006386	0.369815 0.004577	0.407056 0.008625	0.382728 0.009773	0.388248 0.008966	0.407231 0.011618	0.359948 0.007396
50				0.357460 0.003128	0.358262 0.003379	0.357696 0.003280	0.353956 0.003652	0.367637 0.002727	0.395087 0.004514	0.380022 0.005679	0.385612 0.005222	0.394239 0.006079	0.363501 0.004547
100				0.358796 0.001724	0.357611 0.001910	0.357313 0.001889	0.355445 0.002006	0.364784 0.001557	0.389066 0.002196	0.379170 0.002787	0.385002 0.002904	0.386119 0.002776	0.370790 0.002416

TABLE 4
True value of $S(t)$ and $H(t)$ and their estimates using different methods of estimation along with their corresponding MSEs.

				Estimates and corresponding MSEs of $S(t)$					Estimates and corresponding MSEs of $H(t)$				
n	S(t)	H(t)	t	MLE MSE	LSE MSE	WLSE MSE	CVM MSE	MPS MSE	MLE MSE	LSE MSE	WLSE MSE	CVM MSE	MPS MSE
10	0.28331	0.84921	2	0.271868 0.015310	0.296174 0.014976	0.295293 0.014592	0.270386 0.017625	0.310497 0.012865	1.049627 0.289377	0.893131 0.306622	0.896609 0.296678	1.103364 0.544354	0.778464 0.138532
20				0.279342 0.007282	0.289177 0.007821	0.287945 0.007552	0.275056 0.008625	0.304862 0.006682	0.921706 0.076568	0.849855 0.090570	0.859651 0.083327	0.941882 0.124689	0.768845 0.054504
30				0.281957 0.004702	0.287333 0.005134	0.286369 0.004902	0.277702 0.005501	0.301298 0.004512	0.883529 0.045595	0.838095 0.053816	0.846844 0.048364	0.896782 0.065621	0.772641 0.039191
50				0.285078 0.002943	0.286636 0.003327	0.286300 0.003127	0.280773 0.003465	0.298435 0.002933	0.864992 0.023981	0.840389 0.029877	0.846701 0.026453	0.874912 0.033573	0.789806 0.022907
100				0.284581 0.001406	0.283505 0.001599	0.283750 0.001489	0.280534 0.001641	0.292501 0.001423	0.846417 0.010749	0.837091 0.014387	0.840746 0.012321	0.853967 0.014979	0.802834 0.011682
10	0.32036	1.73342	1	0.407736 0.025291	0.418699 0.025164	0.418642 0.024922	0.405615 0.027553	0.428332 0.024045	1.722624 0.545891	1.483976 0.657963	1.485237 0.614588	1.801969 1.051728	1.315671 0.463718
20				0.413095 0.017028	0.418067 0.017977	0.417550 0.017629	0.411390 0.018023	0.426320 0.017904	1.557050 0.228013	1.448767 0.315897	1.460423 0.287453	1.587018 0.323967	1.325180 0.299034
30				0.417170 0.015349	0.419832 0.015945	0.419567 0.015792	0.415420 0.015695	0.427007 0.016385	1.500063 0.175238	1.427147 0.229519	1.440818 0.214170	1.514267 0.207198	1.331370 0.250380
50				0.416309 0.012433	0.416978 0.012800	0.416891 0.012651	0.414230 0.012486	0.423311 0.013450	1.469416 0.131785	1.428586 0.168420	1.440390 0.154152	1.479716 0.147497	1.355142 0.193786
100				0.418175 0.011214	0.416920 0.011111	0.417395 0.011121	0.415539 0.010899	0.422357 0.011935	1.443538 0.113801	1.428760 0.131036	1.434238 0.122886	1.453913 0.118222	1.376747 0.153576

All the computations are performed using programs written in the open source statistical package R [see, Ihaka and Gentleman (1996)].

6. Real data analysis

In this Section, we considered two real data sets for illustrative purpose to show the applicability of the proposed study and calculated $S(t)$ and $H(t)$ using considered methods of estimation, namely, MLE, LSE, WLSE, CME and MPSE for EXGD. At first, we checked whether the considered data sets actually come from EXGD or not by goodness-of-fit test and compared it with the following lifetime distributions:

Exponential distribution (ED)

$$F(x;\theta)=1-e^{-\theta x};$$

Lindley distribution (LD)

$$F(x;\theta)=1-\frac{1+\theta+x\theta}{(1+\theta)}e^{-\theta x};$$

Xgamma distribution (XGD)

$$F(x;\theta)=1-\frac{\left(1+\theta+\theta x+\frac{\theta^2x^2}{2}\right)}{(1+\theta)}e^{-\theta x};$$

Inverted exponential distribution (IED)

$$F(x;\theta)=e^{-1/\theta x};$$

Akash distribution (AKD)

$$F(x;\theta)=1-\left[1+\frac{\theta x(\theta x+2)}{\theta^2+2}\right]e^{-\theta x};$$

Generalized exponential distribution (GED)

$$F(x;\theta,\alpha)=(1-e^{-\theta x})^{\alpha};$$

Weibull distribution (WD)

$$F(x;\theta,\alpha)=1-e^{-\theta x^{\alpha}};$$

Exponential power distribution (EPD)

$$F(x;\alpha,\theta)=1-e^{1-e^{(x/\theta)^{\alpha}}};$$

Pareto type II (Pty2)

$$F(x;\theta,\alpha)=1-\left(1+\frac{x}{\theta}\right)^{-\alpha};$$

Frechet distribution (FD)

$$F(x;\theta,\alpha)=e^{-(\alpha/x)^{\theta}};$$

Transmuted burr XII distribution (TBXIID)

$$F(x;\alpha,\theta,\nu)=1+[(\nu-1)(1+x^{\theta})^{-\alpha}-\lambda(1+x^{\theta})^{-2\alpha}]\ \ |\nu|\leq 1;$$

Generalized inverted Kumaraswamy distribution (GIKD)

$$F(x;\alpha,\theta,\lambda)=[1-(1+x^{\lambda})^{-\alpha}]^{\theta};$$

Exponentiated exponential weibull distribution (EEWD)

$$F(x;a,\alpha,\lambda,\theta)=[1-e^{-\alpha(e^{\lambda x^{\theta}}-1)}]^{a};$$

Exponentiated Rayleigh weibull distribution (ERWD) :

$$F(x;a,\alpha,\lambda,\theta)=[1-e^{-\alpha((e^{\lambda x^{\theta}}-1)^2)}]^{a};$$

where, $x\in\mathscr{R}^{+}$ and $\Theta=(a,\alpha,\lambda,\theta)\in\mathscr{R}^{+}$, $|\nu|\leq 1$.

This procedure is based on the Kolmogorov-Smirnov (KS) statistic which compare an empirical and a theoretical CDFs and is defined as $D_n = Sup_x|F_n(x)-F(x;\alpha,\theta)|$, where Sup_x is the supremum of the set of distances, $F_n(x)$ is the empirical distribution function and $F(x;\alpha,\theta)$ is the theoretical CDF.

Note that, KS statistic to be used only to verify the goodness-of-fit and not as a discrimination criteria. Therefore, we considered four discrimination criteria based on the log-likelihood function evaluated at the MLEs. The criteria are: AIC (Akaike Information Criterion), CAIC (Consistent Akaike Information Criterion), BIC (Bayesian Information Criterion) HQIC (Hannan-Quinn Information Criterion). These statistics are given by $AIC = -2l(\hat{\Theta}) + 2p$, $BIC = -2l(\hat{\Theta}) + 2\ln(n)$, $CAIC = -2l(\hat{\Theta}) + p\ln(n) + 1$, $HQIC = -2l(\hat{\Theta}) + 2p\ln(\ln(n))$, where $l(\hat{\Theta})$, $\hat{\Theta} = (\hat{\alpha}, \hat{\theta})$, denotes the log-likelihood function evaluated at the MLEs, p is the number of model parameters and n is the sample size. Information-theoretic criteria are used because they are valid even for non-nested models [see, Burnham and Anderson (2002)]. The model with lowest values for these statistics could be chosen as the best model to fit the data.

The values of MLEs of the parameters, $l(\hat{\Theta})$, AIC, CAIC, HQIC, BIC, KS statistic and p-value are displayed in Table 5 which indicated that EXGD is best choices among the popular one parameter, two parameters, three parameters and four parameters probability distributions. Hence, EXGD may be chosen as an alternative model. Further, the fitted density for all the considered distributions with the histogram of the data sets and empirical CDFs plots are presented in Figures 3 and 4 respectively which indicated that the proposed model provides adequate fitting to the considered data sets. The details of the summary statistics for the considered data sets are given in Table 6. It is to be noted that for both the considered data sets coefficient of skewness (CS) and coefficient of kurtosis (CK) are greater than 0 and 3 respectively. Hence, both the data sets are positively skewed having higher pick than mesokurtic curve. Thus, we may conclude that the considered data sets may fit our proposed model. Again the estimates of (α, θ), $S(t)$ and $H(t)$ are obtained for the specified value of t using different methods of estimation, mentioned in Table 7.

Data Set I: The data set is considered by Hinkley (1977) and consists of thirty successive values of March precipitation in Minneapolis/St Paul. The details description of the data set are also available in Barreto-Souza and Cribari-Neto (2009) in fitting the generalized exponential-Poisson distribution.

$$0.77, 1.74, 0.81, 1.2, 1.95, 1.2, 0.47, 1.43, 3.37, 2.2, 3, 3.09, 1.51, 2.1, 0.52$$
$$1.62, 1.31, 0.32, 0.59, 0.81, 2.81, 1.87, 1.18, 1.35, 4.75, 2.48, 0.96, 1.89, 0.9, 2.05.$$

Data Set II: Following observations represents the new cases of Covid-19 in Italy during 31st May 2020 to 30th June 2020 [see, https://www.worldometers.info/coronavirus/country/italy/] and the observations are:

$$334, 200, 319, 322, 177, 519, 270, 197, 280, 283, 202, 380, 163, 347, 337,$$
$$301, 210, 329, 332, 251, 264, 224, 221, 113, 190, 296, 255, 175, 174, 126, 142.$$

7. Concluding remarks

In this article, we have proposed a new positively skewed probability distribution, namely, EXGD by considering the generalization of XGD, introduced by Sen *et al.* (2016). Different statistical properties have been discussed. Different methods of estimation, viz.,

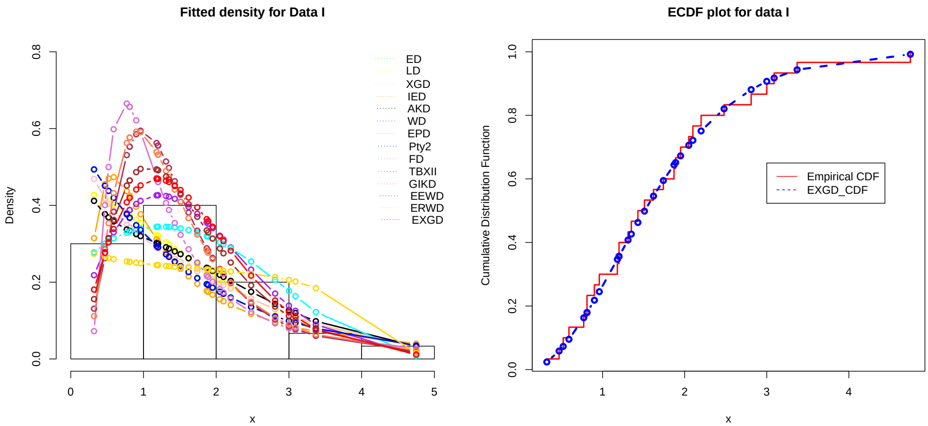

Figure 3 – Fitted PDF and ECDF plot of EXGD for data set-I.

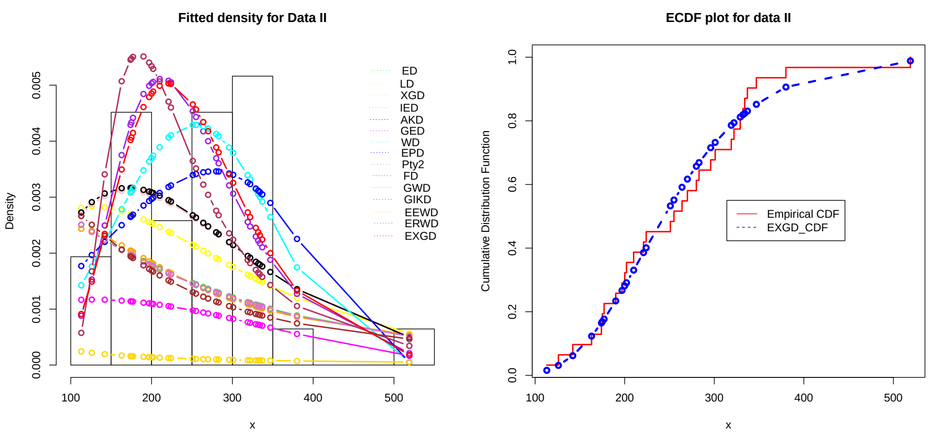

Figure 4 – Fitted PDF and ECDF plot of EXGD for data set-II.

TABLE 5
The model fitting summary for the considered data sets.

Data-I								
Model	MLE	-LogL	AIC	CAIC	HQIC	BIC	KS	p-value
ED	0.5970	45.4743	92.9487	94.3499	93.3970	95.3499	0.2352	0.0723
LD	0.9096	43.1437	88.2874	89.6886	88.7357	90.6886	0.1882	0.2383
XGD	1.1899	44.5735	91.1470	92.5482	91.5953	93.5482	0.2226	0.1022
IED	0.8760	46.2726	94.5452	95.9464	94.9934	96.9464	0.2539	0.0417
AKD	1.2618	43.4281	88.8562	90.2574	89.3044	91.2574	0.1839	0.2618
WD	[1.8086, 1.8923]	38.6432	81.2865	84.0889	82.1830	85.0889	0.0689	0.9988
EPD	[1.2187, 2.7383]	40.4768	84.9537	87.7561	85.8502	88.7561	0.1163	0.8113
Pty2	[217.6379, 130.3267]	45.5498	95.0996	97.9020	95.9961	98.9020	0.2355	0.0716
FD	[1.5497, 1.0160]	41.91701	87.8340	90.6364	88.7305	91.6364	0.1524	0.4882
TBXIID	[0.9786, 2.4267, -0.7736]	39.4976	84.9952	89.1988	86.3399	90.1988	0.1042	0.9004
GIKD	[1.9507, 3.9386, 1.422397]	39.3173	84.6346	88.8381	85.9793	89.8381	0.1094	0.8649
EEWD	[3.593647e+00,4.013854e+02, 2.975671e-03,9.801200e-01]	38.3911	84.7822	90.3870	86.5752	91.3870	0.0645	0.9996
ERWD	[1.178358e-01, 6.708125e+01, 4.865773e-04, 3.876810e+00]	45.8590	99.7180	105.3228	101.5111	106.3228	0.5256	1.261e-07
EXGD	[3.1190, 1.8712]	38.1657	80.3315	83.1339	81.2280	84.1339	0.0549	0.9999
Data-II								
Model	MLE	-LogL	AIC	CAIC	HQIC	BIC	KS	p-value
ED	0.0039	202.8888	407.7776	409.2116	408.2450	410.2116	0.3745	0.0002
LD	0.0077	192.7683	387.5367	388.9706	388.0040	389.9706	0.2685	0.0183
XGD	0.0116	187.9498	377.8997	379.3336	378.3670	380.3336	0.2086	0.1162
IED	0.0045	202.9955	407.9909	409.4249	408.4584	410.4250	0.4082	3.474e-05
AKD	0.0117	187.7058	377.4116	378.8455	377.8790	379.8456	0.2051	0.1275
GED	[17.6204, 74.0342]	180.8895	365.7790	368.6470	366.7139	369.6470	0.0950	0.9174
WD	[3.1477, 285.6479]	181.8270	367.6539	370.5219	368.5889	371.5220	0.0935	0.9258
EPD	[1.9917, 357.3169]	184.8707	373.7413	376.6093	374.6763	377.6094	0.1438	0.4977
Pty2	[18161.8239, 71.1638]	203.0816	410.1632	413.0312	411.0981	414.0312	0.3737	0.0002
FD	[2.9061, 203.2252]	183.7472	371.4944	374.3624	372.4293	9375.3624	0.1303	0.6217
GWD	[0.5684, 0.5325, 0.0020]	231.0539	468.1077	472.4097	469.5101	473.4098	0.5652	7.043e-10
GIKD	[1.4051, 141.6643, 0.6726]	205.8233	417.6467	421.9486	419.0489	422.9486	0.3637	0.0003
EEWD	[0.2112,0.0777, 0.0140,0.4207]	281.1283	570.2566	575.9925	572.1264	576.9925	0.5791	2.147e-10
ERWD	[0.6880, 1.1898, 0.0022, 0.9552]	218.0393	444.0786	449.8146	445.9484	450.8145	1.0000	2.2e-16
EXGD	[4.3109, 0.0193]	180.7038	365.4075	368.2755	366.3425	369.2756	0.0852	0.9637

TABLE 6
Descriptive statistics of the considered data sets I and II.

Data	n	Minimum	1st Quartile	Median	3rd Quartile	Maximum	Mean	Standard deviation	CS	CK
I	30	0.320	0.915	1.470	2.087	4.750	1.675	1.000616	1.086682	4.206884
II	31	113.0	193.5	255.0	320.5	519.0	255.9	86.96565	0.6954319	3.859879

TABLE 7
Estimates of α, θ, $S(t)$ and $H(t)$ using different methods of estimation.

Data	Methods	$\hat{\alpha}$	$\hat{\theta}$	t	S(t)	H(t)
I	MLE	3.11950	1.87116	0.50	0.9332	0.3138
	LSE	2.60052	1.74781	1.0	0.7136	0.6124
	WLSE	2.60025	1.74335	1.5	0.5001	0.8143
	CVM	2.89159	1.83137	2.0	0.3082	1.0448
	MPS	8.72348	48.62405	0.15	0.0099	43.6020
II	MLE	4.30925	0.01936	100	0.9929	0.0005
	LSE	3.20334	0.01744	105	0.9815	0.0009
	WLSE	3.54571	0.01819	110	0.9796	0.0010
	CVM	3.62415	0.01815	115	0.9762	0.0012
	MPS	7.23657	22.08711	0.50	0.0004	19.28633

MLE, LSE, WLSE, CME and MPSE have been discussed for estimating the unknown parameters as well as reliability characteristics of the proposed model. Monte Carlo simulation study has been carried out to compare the performances of different methods of estimation of the parameters as well as survival and hazard rate functions in terms of their corresponding MSEs. Finally, two real data sets have been analyzed for illustration purposes of the proposed study. The Bayesian estimation of the parameters and the reliability characteristics may be further studied under different types of censoring scheme with suitable priors and loss function.

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SUMMARY

In this article, the exponentiated version of xgamma distribution (XGD) has been introduced, named as exponentiated xgamma distribution (EXGD). The proposed model is positively skewed and possess some interesting shapes of hazard rate, i.e., increasing, decreasing and bathtub. Different distributional properties of proposed model, viz., moments, generating functions, mean deviation, quantile function, order statistics, reliability curves and indices etc. have been derived. The estimation of the parameters, survival function and hazard function of EXGD have been approached by different methods of estimation. A Simulation study is carried out to compare the performances of the different estimators obtained via different methods of estimation. Two real data sets have been analyzed to illustrate the applicability of the proposed model.

Keywords: Xgamma distribution; Moments; Generating function; Conditional moments; Reliability curve; Different methods of estimation.