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ASYMPTOTIC PROPERTIES OF THE SEMI-PARAMETRIC ESTIMATORS OF THE CONDITIONAL DENSITY FOR FUNCTIONAL DATA IN THE SINGLE INDEX MODEL WITH MISSING DATA AT RANDOM

Sanaà Dounya Mekki
Department of Mathematics and Computer Science, University center Salhi Ahmed of Naâma, Algeria
Nadia Kadiri
University Djillali LIABES of Sidi Bel Abbes, Algeria
Abbes Rabhi[1]
Laboratory of Mathematics, University Djillali LIABES of Sidi Bel Abbes, Algeria

1. Introduction

Statistical analysis of functional variables has considerably grown over the last two decades. Indeed, an immense innovation on measuring devices has emerged and permitting to monitor several objects in a continuous way, such as stock market index, pollution, climatology, satellite images, etc. Thus, a new branch of statistics, called functional statistics, has developed to treat observations as functional random elements. The study of statistical models for functional data has been a subject of several recent works and developments. The first results on the conditional models were obtained by Ferraty *et al.* (2006), where these authors established the almost complete convergence rate of the kernel estimators for the conditional distribution function, the conditional density and its derivatives, the conditional mode and the conditional quantiles. As a conditional nonparametric model, regression was one of the first predictive analysis tools. Conditional mode estimation is useful in prediction setting, it provides an alternative approach to classical regression estimation. For more recent advances in the topic (see Ezzahrioui and Ould Saïd (2010)). In functional statistics, this model was introduced by Cardot *et al.* (2004). The nonparametric study of this model has been considered by Ferraty and Vieu (2006).

Mode regression is a common way to describe the dependence structure between a response variable Y and some covariate X. Unlike the regression function (which is

[1] Corresponding Author. E-mail: rabhi_abbes@yahoo.fr

defined as the conditional mean) that relies only on the central tendency of the data, the conditional mode function allows the analysts to estimate the functional dependence between variables for all portions of the conditional distribution of the response variable. On the other hand, compared with the standard approach based on functional conditional mean prediction that is sensitive to outliers, functional condition mode prediction could be seen as a reasonable alternative to conditional mean because of its robustness. Moreover, quantiles are well known for their robustness to heavy-tailed error distributions and outliers which allow to consider them as a useful alternative to the regression function (see Chaouch and Khardani (2015)). Conditional quantiles and conditional mode are used in finance and/or insurance to model the risks of extreme values. Conditional quantiles and conditional mode are used in finance and/or insurance to model the risks of extreme values. Furthermore, conditional quantiles can be used to detect outliers in the data as well as establishing probabilistic forecasts. For the above mentioned theoretical and application reasons, the statistical community has placed great interest in estimating conditional quantiles, specifically the conditional median function, is an interesting alternative predictor to the conditional mean thanks to its robustness to the presence of outliers (see Chaudhuri *et al.* (1997)). Estimation of the conditional mode of a scalar response given a functional covariate has attracted the attention of many researchers. Ferraty *et al.* (2005) introduced a nonparametric estimator of the conditional mode when data are dependent. They stated its rate of almost complete consistency. Ezzahrioui and Ould Saïd (2010) established the asymptotic normality of the kernel conditional mode estimator under an α-mixing assumption. In the censored case, Ould Saïd and Cai (2005) stated the uniform strong consistency with rates of the kernel estimator of the conditional mode function, in this context, we refer to Lemdani *et al.* (2009) for the estimation of conditional quantiles. Other authors have been interested in the estimation of conditional models when the observations are censored or truncated (see for instance, Liang and de Uña Álvarez (2010), Khardani *et al.* (2010, 2011, 2012), Ould Saïd and Djabrane (2011) or Tatachak and Ould Saïd (2011), etc.

The ergodic theory has appeared in statistical mechanics, notably in Maxwell's and Gibbs's theories. It is necessary to make a sort of logical transition between the average behavior of the set of dynamic systems and the temporal average of the behaviors of a single dynamic system. It is derived from an ingenious hypothesis used for a long time without justifying it, and in various forms. In the context of the ergodic functional case with censored observations the literature is very restricted. We refer to Chaouch and Khardani (2015), studied the asymptotic properties of the kernel-type estimator of the conditional quantiles when the response variable is right-censored and the data are sampled from an underlying stationary ergodic process. The single-index model represents one of the well-known semi-parametric models, who is very popular in the economics community as which allow to reduce the dimensionality of the covariate space while offering a flexibility in describing the relationship between the response and the covariate, through an unknown link function. The statistical study of these models, in the context of vectorial explanatory random variables, was initiated by Härdle and Marron (1985). Hristache *et al.* (2001) provide both new theoretical and bibliographic elements. Several authors have worked on simple functional index models, we can cite Ferraty *et al.* (2003), Ait Saïdi *et al.* (2008), Attaoui (2014) and Bouchentouf *et al.* (2014).

However, in many practical works such as pharmaceutical tracing test and reliability test and so on, some pairs of observations may be incomplete, in this case we call them missing data. Many examples of missing data and its statistical inferences for regression model can be found in statistical literature when explanatory variables are of finite dimensionality (Cheng (1994), Little and Rubin (2002), Nittner (2003), Tsiatis (2006), Liang *et al.* (2007), Efromovich (2011a,b)) and references therein for details. When the explanatory variable is infinite dimensional or it is of functional nature, only very few literature was reported to investigate the statistical properties of functional nonparametric regression model for missing data. Recently, Ferraty *et al.* (2013) first proposed to estimate the mean of a scalar response based on an i.i.d. functional sample in which explanatory variables are observed for every subject, while part of the responses are missing at random (MAR) for some of them. It generalized the results in Cheng (1994) to the case where the explanatory variables are of functional nature.

The main contribution of this work is to generalize the result of Ling *et al.* (2015, 2016) in the case where a functional parameter θ is present in the model. Our results can be used to construct prediction intervals, for instance in electricity when one wants to construct a maximum interval of demand (or needs) of electricity in the presence of missing data.

In practice, this study has great importance, because, it permits us to construct a prediction method based on the conditional mode estimator. Moreover, in the case where the functional single-index is unknown, our estimate can be used to estimate this parameter via the pseudo-maximum likelihood estimation method. To the best of our knowledge, the estimation of the nonparametric conditional density, in the functional single index structure combining missing data and stationary ergodic processes with functional nature has not been studied in the statistical literature. So, in the present work, we investigate conditional density estimation when the data are both MAR and ergodic. At first, an estimator of the regression operator in the functional single index, and of a scalar response and the functional covariate which are assumed to be sampled from a stationary and ergodic process is constructed with single-index structure. Our aim is to develop a functional methodology for dealing with MAR samples in non-parametric problems (namely in conditional mode estimation). Then, the asymptotic properties of the estimator are obtained under some mild conditions.

Here, we consider a model in which the response variable is missing. Besides the infinite dimensional character of the data, we avoid here the widely used strong mixing condition and its variants to measure the dependency and the very involved probabilistic calculations that it implies. Therefore, we consider, in our setting, the ergodic property to allow the maximum possible generality with regard to the dependence setting. Further motivations to consider ergodic data are discussed in Laib and Laib and Louani (2010) where details defining the ergodic property of processes are also given. As far as we know, the estimation of conditional quantile combining censored data, ergodic theory, and functional data with single-index structure has not been studied in the statistical literature. This work extends, to the functional single index model case, the work of Ling *et al.* (2015, 2016).

This work is organized as follows: in Section 2 we present the model and the hypotheses intervening in the main result, the pointwise almost complete convergence and the uniform almost complete convergence of our estimators are given in Section 3. As application, the conditional mode in functional single index model as well as a confidence interval of the resulted estimator is presented in Section 4. Finally, the proofs of the results are postponed to the last Section.

2. The model and the estimates

Consider a random pair (X,Y) where Y takes values in $\mathbb{R}$ and X takes its values in a separable Hilbert space $\mathscr{H}$ with the norm $||\cdot||$ generated by an inner product $<\cdot,\cdot>$. Let $(X_i,Y_i)_{i=1,\dots,n}$ be a sequence of stationary and ergodic functional samples. Assume that the conditional expectation of Y given X is done through a fixed functional index θ in $\mathscr{H}$, such that

$$\mathbb{E}[Y|X]=\mathbb{E}[Y|<\theta,X>].$$

This model was introduced by Ferraty and Vieu (2003) and we can refer to Attaoui *et al.* (2011) for details. Moreover, we consider $d_\theta(\cdot,\cdot)$ a semi-metric associated with the single index $\theta\in\mathscr{H}$ defined by $d_\theta(x_1,x_2):=|<x_1-x_2,\theta>|$, for x_1 and x_2 in $\mathscr{H}$.

We define the estimator of the conditional density $f(\theta,\cdot,x)$ in complete data of Y given $<X,\theta>=<x,\theta>$ for $x\in\mathscr{H}$ by,

$$\widetilde{f}_n(\theta,y,x)=\frac{h_H^{-1}\sum_{i=1}^{n}K\left(h_K^{-1}(|<x-X_i,\theta>|)\right)H\left(h_H^{-1}(y-Y_i)\right)}{\sum_{i=1}^{n}K\left(h_K^{-1}(|<x-X_i,\theta>|)\right)},$$

where K and H are Kernel functions, and $h_K=h_{n,K}$ (resp. $h_H=h_{n,H}$) is a sequence of smoothing parameters decreasing to zero as n goes to infinity.

Meanwhile, in incomplete case with missing at random for the response variable, we observe $(X_i,Y_i,\delta_i)_{1\le i\le n}$ where X_i is observed completely, and $\delta_i=1$ if Y_i is observed and $\delta_i=0$ otherwise. We define the Bernoulli random variable δ by

$$P(\delta=1|<X,\theta>=<x,\theta>,Y=y)=P(\delta=1|<X,\theta>=<x,\theta>)=p(\theta,x),$$

where $p(\theta,x)$ is a functional operator which is conditionally only on X. Therefore, the estimator of $f(\theta,y,x)$ in the single index model with response MAR is given by

$$\widehat{f}_n(\theta,y,x)=\frac{h_H^{-1}\sum_{i=1}^{n}\delta_i K\left(h_K^{-1}(|<x-X_i,\theta>|)\right)H\left(h_H^{-1}(y-Y_i)\right)}{\sum_{i=1}^{n}\delta_i K\left(h_K^{-1}(|<x-X_i,\theta>|)\right)}=\frac{\widehat{f}_N(\theta,y,x)}{\widehat{F}_D(\theta,x)}$$

where

$$\widehat{f}_N(\theta,y,x)=\frac{1}{n\,h_H\,\mathbb{E}(K_1(\theta,x))}\sum_{i=1}^{n}\delta_i K_i(\theta,x)H_i(\theta,y,x)$$

and

$$\widehat{F}_D(\theta,x)=\frac{1}{n\,\mathbb{E}(K_1(\theta,x))}\sum_{i=1}^{n}\delta_i K_i(\theta,x),$$

with $H_i(\theta,y,x)=H\left(h_H^{-1}(y-Y_i)\right)$, and $K_i(\theta,x)=K\left(h_K^{-1}(|<x-X_i,\theta>|)\right)$.

Finally, the estimator of conditional mode $M_\theta(x)$ is defined as,

$$\widehat{M}_\theta(x)=\arg\sup_{y\in\mathscr{S}_\mathbb{R}}\widetilde{f}_n(\theta,y,x)$$

where $M_\theta(x)=\arg\sup_{y\in\mathscr{S}_\mathbb{R}} f(\theta,y,x)$, $\mathscr{S}_\mathbb{R}$ is a fixed compact subset of $\mathbb{R}$.

Let $\mathscr{F}_i$ be the σ-fields generated by $((<X_1,\theta>,Y_1),\dots,(<X_i,\theta>,Y_i))$ and $\mathscr{G}_i$ be the σ-fields generated by $((<X_1,\theta>,Y_1),\dots,(<X_i,\theta>,Y_i),<X_{i+1},\theta>)$ respectively, and write $B_\theta(x,h)=\{\chi\in\mathscr{H}:|<x-\chi,\theta>|\le h\}$ the ball of center x and radius h and $d_\theta(x,X_i)=|<x-X_i,\theta>|$ denote a random variable such that its cumulative distribution function is given by $F_{\theta,x}(u)=\mathbb{P}(d_\theta(x,X_i)\le u)=\mathbb{P}(X_i\in B_\theta(x,u))$, and the conditional cumulative distribution function of $d_\theta(x,X_i)$ is defined by $F_{\theta,x}^{\mathscr{F}_{i-1}}(u)=\mathbb{P}\left(d_\theta(x,X_i)\le u\mid\mathscr{F}_{i-1}\right)=\mathbb{P}\left(X_i\in B_\theta(x,u)\mid\mathscr{F}_{i-1}\right)$.

(A1) K is a nonnegative bounded kernel function over its support $[0,1]$ with $K(1)>0$, and the derivative K' exists on $[0,1]$ with $K'(t)<0$ for all $t\in[0,1]$ and $\int_0^1 (K^j)'(t)dt<\infty$, for $j=1,2$

(A2) For $x\in\mathscr{H}$, there exist a sequence of nonnegative bounded random functions $(f_{i,1})_{i\ge1}$, a sequence of random functions $(g_{i,\theta,x})_{i\ge1}$, a deterministic nonnegative bounded function f_1 and a nonnegative real function $\phi_\theta(\cdot)$ tending to zero, as its argument tends to zero, such that

(i) $F_{\theta,x}(h)=\phi_\theta(h)f_1(\theta,x)+o(\phi_\theta(h))$ as $h\to0$.

(ii) For any $i\in\mathbb{N}, F_{\theta,x}^{\mathscr{F}_{i-1}}(h)=\phi_\theta(h)f_{i,1}(\theta,x)+g_{i,\theta,x}(h)$ with $g_{i,\theta,x}=o_{a.s}(\phi(t))$ as $h\to0$, $g_{i,\theta,x}(h)/\phi_\theta(h)$ a.s. bounded and $n^{-1}\sum_{i=1}^{n} g_{i,\theta,x}^{j}(h)=o_{a.s}(\phi_\theta^{j}(h))$ as $n\to\infty$ for $j=1,2$.

(iii) $n^{-1}\sum_{i=1}^{n} f_{i,1}^{j}(\theta,x) \to f_1^{j}(\theta,x)$ almost surely as $n\to\infty$ for $j=1,2$.

(iv) There exists a nondecreasing bounded function τ_0 such that, uniformly in $t\in[0,1], \phi_\theta(ht)/\phi_\theta(h)=\tau_0+o(1), \quad as \quad h\downarrow 0$ and $\int_0^1 (K^j)'\tau_0(t)dt < \infty \quad for \quad j\geq 1$.

(v) $n^{-1}\sum_{i=1}^{n} b_i(\theta,x)\to D(\theta,x)<\infty \quad as \quad n\to\infty.$

(A3) The conditional density $f(\theta,y,x)$ satisfies the Hölder condition, i.e., $\forall(x_1,x_2)\in N_x\times N_x$, $\forall(y_1,y_2)\in\mathcal{S}_{\mathbb{R}}^2$, for $b_1>0$, $b_2>0$

$$|f(\theta,y_1,x_1)-f(\theta,y_2,x_2)|\leq C_{\theta,x}\left(||x_1-x_2||^{b_1}+|y_1-y_2|^{b_2}\right).$$

(A4) (i) The Kernel H is a positive bounded function with:

- $\int_{\mathbb{R}}|t|^{b_2}H(t)dt<\infty \quad and \quad \int_{\mathbb{R}}tH(t)dt=0.$
- For all $(t_1,t_2)\in\mathbb{R}^2, \quad |H(t_1)-H(t_2)|\leq C|t_1-t_2|.$

(ii) $H^{(1)}(t)$ and $H^{(2)}(t)$ are bounded with $\int(H^{(1)}(t))^2dt<\infty$

(A5) For $j=0,1,2,$ and any $k\geq 1$,
$\mathbb{E}\left[(h_H^{-1}H^{(j)}(h_H^{-1}(t-Y_i)))^k|\mathcal{G}_{i-1}\right]=\mathbb{E}\left[(h_H^{-1}H^{(j)}(h_H^{-1}(t-Y_i)))^k|<\theta,X_i>\right]$

(A6) $p(\theta,x)$ is continuous in a neighborhood of x.

(A7) (i) $\exists\epsilon_0$, such that $f(\theta,\cdot,x)$ is strictly increasing on $(M_\theta(x)-\epsilon_0,M_\theta(x))$ and strictly decreasing on $(M_\theta(x),M_\theta(x)+\epsilon_0)$, with respect to x.

(ii) $f(\theta,y,x)$ is twice continuously differentiable around $M_\theta(x)$ with $f^{(1)}(\theta,M_\theta(x),x)=0$, and $f^{(2)}(\theta,M_\theta(x),x)\neq 0$, where $f^{(q)}(\theta,y,x)$, $(q=1,2)$ is the qth derivative of $f(\theta,y,x)$ with respect to $y\in\mathcal{S}_{\mathbb{R}}$.

3. Main results

In this part of the paper, the convergence of the conditional density function and asymptotic normality are established. To this end, we consider the same decomposition used in Chaouch and Khardani (2015).

Proposition 1. *Under assumptions (A1)-(A6) and if* $\exists\xi>0, n^\xi h_H^2 \underset{n\to\infty}{\to}\infty$, *and if*

$$\frac{\log n}{n\,h_H^2\,\phi_\theta(h_K)}\to 0, \quad as \quad n\to\infty. \tag{1}$$

then we have,

$$\sup_{y\in\mathcal{S}_{\mathbb{R}}}|\widehat{f}_n(\theta,y,x)-f(\theta,y,x)|=\mathcal{O}_{a.s.}(h_K^{b_1}+h_H^{b_2})+\mathcal{O}_{a.s.}\left(\sqrt{\frac{\log n}{n\,h_H^2\,\phi_\theta(h_K)}}\right).$$

PROPOSITION 2. *Under assumptions of Proposition 1, in addition if*

$$\sqrt{n h_H \phi_\theta(h_K)}(h_K^{b_1} + h_H^{b_2}) \to 0, \quad as \quad n \to \infty. \tag{2}$$

then we have,

$$\sqrt{n h_H \phi_\theta(h_K)}(\widehat{f}_n(\theta,y,x) - f(\theta,y,x)) \xrightarrow{D} \mathcal{N}(0, \sigma^2(\theta,y,x)), \quad as \quad n \to \infty,$$

where " $\xrightarrow{D}$ *" means the convergence in distribution, and*

$$\sigma^2(\theta,y,x) = \frac{M_2}{M_1^2} \frac{f(\theta,y,x)}{p(\theta,x) f_1(\theta,x)} \int_{\mathbb{R}} H^2(u)du$$

with $M_j = K^j(1) - \int_0^1 (K^j)'(u)\tau_0(u)du, \quad for \quad j = 1,2.$

The proof of propositions Proposition 1 and Proposition 2 is based on the following decomposition. Let

$$\widehat{f}_n(\theta,y,x) - f(\theta,y,x) = \frac{Q_n(\theta,y,x) + R_n(\theta,y,x)}{\widehat{F}_D(\theta,x)} + B_n(\theta,y,x)$$

where

$$Q_n(\theta,y,x) = \left(\widehat{f}_N(\theta,y,x) - \bar{f}_N(\theta,y,x)\right) - f(\theta,y,x)\left(\widehat{F}_D(\theta,x) - \bar{F}_D(\theta,x)\right), \tag{3}$$

and

$$\begin{aligned} R_n(\theta,y,x) &= -B_n(\theta,y,x)\left(\widehat{F}_D(\theta,x) - \bar{F}_D(\theta,x)\right), B_n(\theta,y,x) \\ &= \frac{\bar{f}_N(\theta,y,x)}{\bar{F}_D(\theta,x)} - f(\theta,y,x), \end{aligned}$$

with,

$$\bar{f}_N(\theta,y,x) = \frac{1}{n\, h_H\, \mathbb{E}(K_1(\theta,x))} \sum_{i=1}^n \mathbb{E}\left[\delta_i K_i(\theta,x) H_i(y) |, \mathcal{F}_{i-1}\right]$$

$$\bar{F}_D(\theta,x) = \frac{1}{n\, \mathbb{E}(K_1(\theta,x))} \sum_{i=1}^n \mathbb{E}\left[\delta_i K_i(\theta,x) | \mathcal{F}_{i-1}\right].$$

The asymptotic rates of conditional mode estimation in the single index model with MAR response is given in the following theorem.

THEOREM 3. *Under conditions of Proposition 1, and if (A7) holds true, then we have*

$$|\widehat{M}_\theta(x) - M_\theta(x)| = \mathcal{O}_{a.s.}(h_K^{b_1} + h_H^{b_2})^{1/2} + \mathcal{O}_{a.s.}\left(\frac{\log n}{n\, h_H^2 \phi(h_K)}\right)^{1/4}.$$

PROOF. Under (A7) and by using a Taylor expansion of order two of function $f(\theta, M_\theta(x), x)$ in the neighborhood of $M_\theta(x)$ with $f^{(1)}(\theta, M_\theta(x), x) = 0$, it follows that

$$\widehat{f}(\theta, \widehat{M}_\theta(x), x) - f(\theta, M_\theta(x), x) = \frac{1}{2} f^{(2)}(\theta, M_\theta^*(x), x)(\widehat{M}_\theta(x) - M_\theta(x))^2 \tag{4}$$

where $M_\theta^*(x)$ is between $M_\theta(x)$ and $\widehat{M}_\theta(x)$. Noting that by Theorem 6.6 in Ferraty and Vieu (2006), we have

$$|\widehat{f}(\theta, \widehat{M}_\theta(x), x) - f(\theta, M_\theta(x), x)| \leq 2 \sup_{\theta \in \Theta_{\mathscr{H}}} \sup_{y \in \mathscr{S}_{\mathbb{R}}} |\widehat{f}(\theta, y, x) - f(\theta, y, x)|.$$

Same as above, combining Eq. (4) with Ferraty and Vieu (2006), we obtain

$$|\widehat{M}_\theta(x) - M_\theta(x)|^2 f^{(2)}(\theta, M_\theta^*(x), x) = \mathcal{O}\left(\sup_{\theta \in \Theta_{\mathscr{H}}} \sup_{y \in \mathscr{S}_{\mathbb{R}}} |\widehat{f}(\theta, y, x) - f(\theta, y, x)| \right).$$

By (A7)(ii), we get

$$|\widehat{M}_\theta(x) - M_\theta(x)|^2 = \mathcal{O}_{a.s.}\left(\sup_{\theta \in \Theta_{\mathscr{H}}} \sup_{y \in \mathscr{S}_{\mathbb{R}}} |\widehat{f}(\theta, y, x) - f(\theta, y, x)| \right).$$

□

4. CONSTRUCTING CONFIDENCE BANDS

Noting that, both the asymptotic variance $\sigma^2(\theta, y, x)$ and $\rho^2(\theta, M_\theta(x), x)$ contain some unknown quantities $M_\theta(x), p(\theta, x), f(\theta, y, x)$, $f^{(2)}(\theta, y, x)$ and M_j for $j = 1, 2$, and unknown functions $\phi_\theta(h_K)$ and $f_1(\theta, x)$ that we have to estimate.
Therefore, same as above, $f(\theta, y, x)$ and $f^{(2)}(\theta, y, x)$ can be estimated respectively by $\widehat{f}_n(\theta, y, x)$ and $\widehat{f}_n^{(2)}(\theta, y, x)$. Moreover by assumptions (A2)-(i) and (A2)-(iv), we can estimate $\tau_0(\theta, x)$ by

$$\tau_n(u) = \frac{F_{\theta,x,n}(uh)}{F_{\theta,x,n}(h)}$$

where

$$F_{\theta,x,n}(h) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}_{(<x - X_i, \theta> \leq u)}$$

can be used to estimate $\phi_\theta(h_K)$. Finally, the estimator of $p(\theta, x)$ is denoted by

$$P_n(\theta, x) = \frac{\sum_{i=1}^{n} \delta_i K\left(\frac{<\theta, x - X_i>}{h_K}\right)}{\sum_{i=1}^{n} K\left(\frac{<\theta, x - X_i>}{h_K}\right)}.$$

Then, as above, the following corollaries are obtained by using estimate term of unknown term, which allows to obtain a confidence band.

THEOREM 4. *Under conditions of Proposition 2 and if (A7) holds true, and if in addition $nh_H^3\phi_\theta(h_K) \to 0$, then we have*

$$\sqrt{nh_H^3\phi_\theta(h_K)}(\widehat{M}_\theta(x) - M_\theta(x)) \xrightarrow{D} \mathcal{N}(0, \rho^2(\theta, M_\theta(x), x)), \quad as \quad n \to \infty,$$

where

$$\rho^2(\theta, M_\theta(x), x) = \frac{M_2}{M_1^2} \frac{f(\theta, M_\theta(x), x)}{p(\theta, x) f_1(\theta, x)(f^{(2)}(\theta, M_\theta(x), x))^2} \int (H^{(1)}(u))^2 du.$$

PROOF. By the first order Taylor expansion for $\widehat{f}^{(1)}(\theta, y, x)$ at point $M_\theta(x)$ and the fact that $\widehat{f}^{(1)}(\theta, \widehat{M}_\theta(x), x) = 0$, it follows that

$$\sqrt{nh_H^3\phi(h_K)}|\widehat{M}_\theta(x) - M_\theta(x)| = \frac{-\sqrt{nh_H^3\phi(h_K)}\widehat{f}^{(1)}(\theta, M_\theta(x), x)}{\widehat{f}^{(2)}(\theta, M_\theta^*(x), x)}$$

where $M_\theta^*(x)$ is between $\widehat{M}_\theta(x)$ and $M_\theta(x)$. Similar to the proof of Proposition 2, it follows that

$$-\sqrt{nh_H^3\phi(h_K)}\widehat{f}^{(1)}(\theta, M_\theta(x), x) \xrightarrow{D} \mathcal{N}(0, \rho_0^2(\theta, M_\theta(x), x))$$

where

$$\rho_0^2(\theta, M_\theta(x), x) = \frac{M_2}{M_1^2} \frac{f(\theta, M_\theta(x), x)}{p(\theta, x) f_1(\theta, x)} \int_{\mathbb{R}} (H^{(1)}(u))^2 du.$$

Thus, as above, similar to Ferraty and Vieu (2006), we can obtain $\widehat{f}^{(2)}(\theta, y, x) \xrightarrow{P} f^{(2)}(\theta, y, x)$ as $n \to \infty$, which implies that $\widehat{M}_\theta(x) \to M_\theta(x)$.

Therefore, we get

$$\widehat{f}^{(2)}(\theta, M_\theta^*(x), x) \longrightarrow f^{(2)}(\theta, M_\theta(x), x) \neq 0 \quad as \quad n \to \infty.$$

□

COROLLARY 5. *Under the condition of Proposition 2, we have*

$$\sqrt{\frac{n\, h_H\, F_{\theta,x,n}(h_K)}{\widehat{\sigma}^2(\theta, y, x)}}(\widehat{f}_n(\theta, y, x) - f(\theta, y, x)) \xrightarrow{D} \mathcal{N}(0, 1), \quad as \quad n \to \infty. \tag{5}$$

where,

$$\widehat{\sigma}^2(\theta, y, x) = \frac{M_{2n}}{M_{1n}^2} \frac{\widehat{f}_n(\theta, y, x)}{P_n(\theta, x)} \int_{\mathbb{R}} H^2(u) du.$$

PROOF. First, observe that

$$\begin{aligned}\Omega(\theta,y,x) &= \frac{M_{1,n}}{M_1}\frac{\sqrt{M_2}}{\sqrt{M_{2,n}}}\\ &\quad \sqrt{\frac{n\,h_H\,F_{\theta,x,n}(h_K)P_n(\theta,x)f(\theta,y,x)}{p(\theta,x)\widehat{f}_n(\theta,y,x)f_1(\theta,x)nh_H\phi_\theta(h_K)}}\\ &\times \sqrt{\frac{nh_H\phi(h_K)}{\sigma^2(\theta,y,x)}}(\widehat{f}_n(\theta,y,x)-f(\theta,y,x)).\end{aligned}$$

where $\Omega(\theta,y,x)=\sqrt{\frac{n\,h_H\,F_{\theta,x,n}(h_K)}{\widehat{\sigma}^2(\theta,y,x)}}(\widehat{f}_n(\theta,y,x)-f(\theta,y,x))$

By Proposition 2, we have as $n\to\infty$

$$\sqrt{\frac{n\,h_H\,\phi(h_K)}{\sigma^2(\theta,y,x)}}(\widehat{f}_n(\theta,y,x)-f(\theta,y,x))\xrightarrow{D}\mathcal{N}(0,1).$$

In order to prove Eq. (5), we need to show that

$$\frac{M_{1,n}}{M_1}\frac{\sqrt{M_2}}{\sqrt{M_{2,n}}}\sqrt{\frac{n\,h_H\,F_{\theta,x,n}(h_K)P_n(\theta,x)f(\theta,y,x)}{p(\theta,x)\widehat{f}_n(\theta,y,x)f_1(\theta,x)nh_H\phi_\theta(h_K)}}(\widehat{f}_n(\theta,y,x)-f(\theta,y,x))\xrightarrow{P}1,$$

using results given by Laib and Louani (2010), we have

$$M_{1,n}\to M_1,\quad M_{2,n}\to M_2,\quad \frac{F_{\theta,x,n}}{\phi_\theta(h_K)f_1(\theta,x)}\to 1,\quad as\quad n\to\infty.$$

On the other hand, by Proposition 2 in Laib and Louani (2010), it follows that

$$P_n(\theta,x)\underset{n\to\infty}{\to}\mathbb{E}(\delta|<\theta,X>=<\theta,x>)=P(\delta=1|<\theta,X>=<\theta,x>)=p(\theta,x).$$

In addition, by Proposition 1, we have $\widehat{f}_n(\theta,y,x)\to f(\theta,y,x),\quad as\quad n\to\infty$. Then, the proof is completed. □

COROLLARY 6. *Under the condition of Theorem 4, we have*

$$\sqrt{\frac{n\,h_H^3\,F_{\theta,x,n}(h_K)}{\widehat{\rho}^2(\theta,M_\theta(x),x)}}(\widehat{M}_\theta(x)-M_\theta(x))\xrightarrow{D}\mathcal{N}(0,1),\quad as\quad n\to\infty,$$

where

$$\widehat{\rho}^2(\theta,M_\theta(x),x)=\frac{M_{2n}}{M_{1n}^2}\frac{\widehat{f}_n(\theta,\widehat{M}_\theta(x),x)}{P_n(\theta,x)(f_n^{(2)}(\theta,\widehat{M}_\theta(x),x))^2}\int_{\mathbb{R}}(H^{(1)}(u))^2du.$$

REMARK 7. *By corollaries Corollary 5 and Corollary 6, the asymptotic* $(1-\alpha)$ *confidence interval of the conditional density* $f(\theta, y, x)$ *and conditional mode* $M_\theta(x)$ *are presented by*

$$\widehat{f}_n(\theta, y, x) \pm \mu_{\alpha/2}\sqrt{\frac{\widehat{\sigma}^2(\theta, y, x)}{n\, h_H F_{\theta,x,n}(h_K)}},$$

and

$$\widehat{M}_\theta(x) \pm \mu_{\alpha/2}\sqrt{\frac{\widehat{\rho}^2(\theta, M_\theta(x), x)}{n\, h_H^3 F_{\theta,x,n}(h_K)}},$$

where $\widehat{\sigma}^2(\theta, y, x)$, $\widehat{\rho}^2(\theta, M_\theta(x), x)$ *are defined in corollaries, Corollary 5 and Corollary 6 respectively and* $\mu_{\alpha/2}$ *is the upper* $\alpha/2$ *quantile of the normal distribution* $\mathcal{N}(0,1)$.

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APPENDIX

A. PROOFS

First, we introduce some Lemmas and their proofs which are necessary to prove the main results.

LEMMA 8. *Assume that assumptions (A1) and (A2)(i)(ii)(iv) hold true. For any real numbers* $1 \le j \le 2+\delta$ *with* $\delta > 0$, *as* $n \to \infty$, *we have*

(i) $\frac{1}{\phi_\theta(h_K)}\mathbb{E}[K_i^j(\theta, x)|\mathscr{F}_{i-1}] = M_j f_{i,1}(\theta, x) + O_{a.s}\left(\frac{g_{i,\theta,x}(h_K)}{\phi_\theta(h_K)}\right)$.

(ii) $\frac{1}{\phi_\theta(h_K)}\mathbb{E}[K_1^j(\theta, x)] = M_j f_1(\theta, x) + o(1)$.

(iii) $\frac{1}{\phi_\theta^k(h_K)}(\mathbb{E}(K_1^j(\theta, x)))^k = M_1^k f_1^k(\theta, x) + o(1)$.

where M_j *is defined in Proposition 2.*

PROOF. See the proof of Lemma 1 in Laib and Louani (2010). □

LEMMA 9. *Under assumptions (A1)-(A2) and (A6), we have*

$$\widehat{F}_D(\theta, x) - \bar{F}_D(\theta, x) = \mathcal{O}_{a.s.}\left(\sqrt{\frac{\log n}{n\,\phi_\theta(h_K)}}\right) \tag{6}$$

and

$$\lim_{n\to\infty} \widehat{F}_D(\theta,x) = \lim_{n\to\infty} \bar{F}_D(\theta,x) = p(\theta,x) \quad a.s. \tag{7}$$

PROOF. First, we have

$$\widehat{\bar{\Upsilon}}(\theta,x) = \widehat{F}_D(\theta,x) - \bar{F}_D(\theta,x) = \frac{1}{n\,\mathbb{E}(K_1(\theta,x))}\sum_{i=1}^{n} W_{n,i}(\theta,x),$$

where $W_{n,i}(\theta,x) = \delta_i K_i(\theta,x) - \mathbb{E}[\delta_i K_i(\theta,x)|\mathscr{F}_{i-1}]$, is a triangular array of martingale differences sequence with respect to the σ-fields $\mathscr{F}_{i-1}$. By Jensen inequality and (A6), it follows that

$$\begin{aligned}
\mathbb{E}[W_{n,i}^2(\theta,x)|\mathscr{F}_{i-1}] &\leq 2\mathbb{E}[\delta_i K_i^2(\theta,x)|\mathscr{F}_{i-1}] \\
&= 2\mathbb{E}(\mathbb{E}[\delta_i K_i^2(\theta,x)|\mathscr{G}_{i-1}]|\mathscr{F}_{i-1}) \\
&= 2(p(\theta,x)+o(1))\mathbb{E}[K_i^2(\theta,x)|\mathscr{F}_{i-1}].
\end{aligned}$$

Hence, by (A2) and Lemma 8, for n large enough, it follows that

$$\begin{aligned}
\mathbb{E}[W_{n,i}^2(\theta,x)|\mathscr{F}_{i-1}] &\leq 2(p(\theta,x)+o(1))[M_2\phi_\theta(h_K)f_{i,1}(\theta,x)+O_{a.s}(g_{i,\theta,x}(h_K))] \\
&\leq 2(p(\theta,x)+o(1))\phi_\theta(h_K)[M_2 b_i(\theta,x)+1] = d_i^2.
\end{aligned}$$

Thus, by Lemma 8 and Lemma 1 in Laib and Louani (2010) and (A2)-(v), we have

$$\begin{aligned}
\mathbb{P}\left(|\widehat{\bar{\Upsilon}}(\theta,x)| > \varepsilon\right) &= \mathbb{P}\left(|\sum_{i=1}^{n} W_{n,i}(\theta,x)| > \varepsilon\, n\,\mathbb{E}(K_1(\theta,x))\right) \\
&\leq 2\exp\left(-\frac{(\varepsilon\, n\,\mathbb{E}(K_1(\theta,x)))^2}{2(D_n + C\varepsilon n\mathbb{E}(K_1(\theta,x)))}\right) \\
&= 2\exp\left(-n\varepsilon^2\frac{(\mathbb{E}(K_1(\theta,x)))^2}{2(D_n/n + C\varepsilon\mathbb{E}(K_1(\theta,x)))}\right) \\
&= 2\exp\left(-n\varepsilon^2\frac{(\mathbb{E}(K_1(\theta,x)))^2}{2D_n/n}\left[\frac{1}{1+\frac{C\varepsilon\mathbb{E}(K_1(\theta,x))}{D_n/n}}\right]\right),
\end{aligned}$$

then, choosing

$$\varepsilon = \varepsilon_n = \left(\frac{4(p(\theta,x)+o(1))[M_2 D(\theta,x)+1]\log n}{M_1^2 f_1^2(\theta,x) n\phi_\theta(h_K)}\right)^{1/2}\varepsilon_0,$$

with $\varepsilon_0 > 0$, for n large enough, we get

$$\mathbb{P}(|\widehat{F}_D(\theta,x) - \bar{F}_D(\theta,x)| > \varepsilon) \leq 2\exp\left(-C\varepsilon_0^2\log n\right) \leq \frac{2}{n^{C\varepsilon_0^2}}.$$

Finally, by taking ε_0 large enough, if follows that by Eq. (1), and by Borel-Cantelli Lemma,

$$\widehat{F}_D(\theta,x)-\bar{F}_D(\theta,x)=\mathcal{O}_{a.s.}\left(\sqrt{\frac{\log n}{n\,\phi_\theta(h_K)}}\right)=o_{a.s.}(1).$$

To establish Eq. (7), we have the decomposition as follows $\widehat{F}_D(\theta,x)=K_n(\theta,x)+\bar{F}_D(\theta,x)$, where $K_n(\theta,x)=\widehat{F}_D(\theta,x)-\bar{F}_D(\theta,x)$.
Thus, by Eq. (6), we have $K_n(\theta,x)\xrightarrow{\mathbb{P}}0 \quad as \quad n\to\infty$, so we will prove that

$$\bar{F}_D(\theta,x)\xrightarrow{\mathbb{P}}p(\theta,x), \quad as \quad n\to\infty. \tag{8}$$

Thus, by the properties of conditional expectation and the mechanism of MAR, combining the assumptions (A2)(ii)(iii) and the continuous property of $p(\theta,x)$ with Lemma 8, we have

$$\begin{aligned}
\bar{F}_D(\theta,x) &= \frac{1}{n\mathbb{E}(K_1(\theta,x))}\sum_{i=n}^{n}\mathbb{E}[\mathbb{E}[(\delta_i K_i(\theta,x))|\mathcal{F}_{i-1}]|\mathcal{G}_{i-1}]\\
&= \frac{1}{n\mathbb{E}(K_1(\theta,x))}\sum_{i=n}^{n}\mathbb{E}[p(\theta,x)+o(1)K_i(\theta,x)|\mathcal{F}_{i-1}]\\
&= (p(\theta,x)+o(1))\frac{1}{n\mathbb{E}(K_1(\theta,x))}\sum_{i=n}^{n}\mathbb{E}[K_i(\theta,x)|\mathcal{F}_{i-1}]\\
&= \frac{(p(\theta,x)+o(1))}{n\mathbb{E}(K_1(\theta,x))}\sum_{i=n}^{n}(\phi_\theta(h_K)M_1 f_{i1}(\theta,x)+O_{a.s}(g_{i,\theta,x}))\\
&= (p(\theta,x)+o(1))\frac{\phi_\theta(h_K)}{\mathbb{E}(K_1(\theta,x))}\left(\frac{1}{n}\sum_{i=n}^{n}M_1 f_{i1}(\theta,x)\right)\\
&\quad +(p(\theta,x)+o(1))\frac{\phi_\theta(h_K)}{\mathbb{E}(K_1(\theta,x))}\left(\frac{1}{n}\sum_{i=n}^{n}O_{a.s}\left(\frac{g_{i,\theta,x}(h_K)}{\phi_\theta(h_K)}\right)\right)\\
&= (p(\theta,x)+o(1))\frac{1}{M_1 f_1(\theta,x)+o(1)}(M_1(f_1(\theta,x)+o(1))+O_{a.s}(1))\\
&\to p(\theta,x) \quad a.s., \quad as \quad n\to\infty.
\end{aligned}$$

□

LEMMA 10. *Under assumptions (A1)-(A5), we have*

$$\sup_{y\in\mathcal{S}_\mathbb{R}}|B_n(\theta,y,x)|=\mathcal{O}_{a.s.}(h_K^{b_1}+h_H^{b_2}), \tag{9}$$

and

$$\sup_{y\in\mathcal{S}_\mathbb{R}}|R_n(\theta,y,x)|=\mathcal{O}_{a.s.}\left((h_K^{b_1}+h_H^{b_2})\left(\frac{\log n}{n\phi_\theta(h_K)}\right)^{1/2}\right).$$

PROOF. First we have

$$B_n(\theta,y,x)=\frac{\bar{f}_n(\theta,y,x)-\bar{F}_D(\theta,x)f(\theta,y,x)}{\bar{F}_D(\theta,x)}=\frac{\bar{B}_n(\theta,y,x)}{\bar{F}_D(\theta,x)}.$$

Thus, by Eq. (8), we need to show that

$$\bar{B}_n(\theta,y,x)=\mathcal{O}_{a.s.}(h_K^{b_1}+h_H^{b_2}). \tag{10}$$

Next, making use of the condition (A5), we have

$$\begin{aligned}
&\bar{B}_n(\theta,y,x)\\
&= \frac{(nh_H)^{-1}}{\mathbb{E}(K_1(\theta,x))}\sum_{i=1}^{n}\mathbb{E}\left[(H_i(y)-h_Hf(\theta,y,x))\delta_iK_i(\theta,x)|\mathscr{F}_{i-1}\right]\\
&= \frac{(nh_H)^{-1}}{\mathbb{E}(K_1(\theta,x))}\sum_{i=1}^{n}\mathbb{E}\left[\mathbb{E}\left((H_i(y)-h_Hf(\theta,y,x))\delta_iK_i(\theta,x)|\mathscr{G}_{i-1}\right)|\mathscr{F}_{i-1}\right]\\
&= \frac{(nh_H)^{-1}}{\mathbb{E}(K_1(\theta,x))}\sum_{i=1}^{n}\mathbb{E}\left[\mathbb{E}((H_i(y)-h_Hf(\theta,y,x))\delta_iK_i(\theta,x)|<\theta,X_i>)|\mathscr{F}_{i-1}\right].
\end{aligned}$$

Because of conditions (A3) and (A4), we get:

$$\mathbb{E}(H_i(\theta,y,x)-h_Hf(\theta,y,x)|<\theta,X_i>)\leq C_{\theta,x}h_H\int_{\mathbb{R}}H(v)(h_K^{b_1}+|v|^{b_2}h_H^{b_2})dv. \tag{11}$$

Hence, we obtain

$$\begin{aligned}
\bar{B}_n(\theta,y,x) &\leq \frac{1}{nh_H\mathbb{E}(K_1(\theta,x))}\\
&\qquad \sum_{i=1}^{n}\mathbb{E}\left(\delta_iK_i(\theta,x)\{h_H\int_{\mathbb{R}}H(v)(h_K^{b_1}+|v|^{b_2}h_H^{b_2})dv\}|\mathscr{F}_{i-1}\right)\\
&\leq C_{\theta,x}\{h_H^{b_2}\int_{\mathbb{R}}|v|^{b_2}H(v)dv+h_K^{b_1}\}\times\bar{F}_D(\theta,x)\\
&= \mathcal{O}(h_K^{b_1}+h_H^{b_2}).
\end{aligned}$$

Hence, Eq. (9) follows from Eq. (8) and Eq. (10).

Finally, from Eq. (6) and Eq. (9), we easily obtain the second part of Lemma 10. □

LEMMA 11. *If the assumptions (A1)-(A2) and (A4)-(A5) are satisfied, then we have*

$$\sup_{y\in\mathscr{S}_{\mathbb{R}}}|\widehat{f}_N(\theta,y,x)-\bar{f}_N(\theta,y,x)|=\mathcal{O}_{a.s}\left(\sqrt{\frac{\log n}{n\,h_H^2\,\phi_\theta(h_K)}}\right).$$

PROOF. By the compactness property of $\mathscr{S}_{\mathbb{R}} \subset \mathbb{R}$, we can write that $\mathscr{S}_{\mathbb{R}} \subset \bigcup_{k=1}^{\tau_n} \mathscr{S}_k$, where $\mathscr{S}_k = (t_k - l_n, t_k + l_n)$ with l_n and τ_n can be selected such as $l_n = O\left(\tau_n^{-1}\right)$. Taking $t_y = \arg \min_{t \in \{t_1, \ldots, t_{\tau_n}\}} |y - t|$, we have the following decomposition:

$$
\begin{aligned}
\sup_{y \in \mathscr{S}_{\mathbb{R}}} |\widehat{f}_N(\theta, y, x) - \bar{f}_N(\theta, y, x)| &\leq \sup_{y \in \mathscr{S}_{\mathbb{R}}} |\widehat{f}_N(\theta, y, x) - \widehat{f}_N(\theta, t_y, x)| \\
&\quad + \sup_{y \in \mathscr{S}_{\mathbb{R}}} |\widehat{f}_N(\theta, t_y, x) - \bar{f}_N(\theta, t_y, x)| \\
&\quad + \sup_{y \in \mathscr{S}_{\mathbb{R}}} |\bar{f}_N(\theta, t_y, x) - \bar{f}_N(\theta, y, x)| \\
&= T_1 + T_2 + T_3.
\end{aligned}
$$

As the first and the third terms can be treated in the same manner, we deal only first term. By (A4)(ii) which implies in particular that H is Hölder continuous with order one, we can write

$$
\begin{aligned}
T_1 &\leq \frac{1}{n\, h_H\, \mathbb{E}(K_1(\theta, x))} \sup_{y \in \mathscr{S}_{\mathbb{R}}} \sum_{i=1}^{n} \delta_i K_i(\theta, x) \left[H_i(y) - H_i(t_y) \right], \\
&\leq \frac{C}{n\, h_H\, \mathbb{E}(K_1(\theta, x))} \sup_{y \in \mathscr{S}_{\mathbb{R}}} \frac{|y - t_y|}{h_H} \sum_{i=1}^{n} \delta_i K_i(\theta, x), \\
&\leq \frac{l_n}{n\, h_H^2\, \mathbb{E}(K_1(\theta, x))} \sum_{i=1}^{n} \delta_i K_i(\theta, x).
\end{aligned}
$$

Employing Eq. (7) and $\lim_{n \to \infty} n^{\xi} h_H^2 = \infty$, it follows that

$$
T_1 \longrightarrow 0 \quad a.s., \quad as \quad n \to \infty.
$$

Similar to the argument as above, for T_3, we have as $n \longrightarrow \infty$

$$
T_3 \leq \frac{l_n}{n\, h_H^2\, \mathbb{E}(K_1(\theta, x))} \sum_{i=1}^{n} \mathbb{E}\left(\delta_i K_i(\theta, x) | \mathscr{F}_{i-1} \right) \longrightarrow 0.
$$

Finally, let us treat term T_2. Observing that

$$
\widehat{f}(\theta, t_k, x) - \bar{f}(\theta, t_k, x) = \frac{1}{n\, \mathbb{E}(K_1(\theta, x))} \sum_{i=1}^{n} L_{n,i}(\theta, t_k, x).
$$

where $L_{n,i}(\theta, t_k, x) = \delta_i\, h_H^{-1} K_i(\theta, x) H_i(t_k) - \mathbb{E}\left[\delta_i\, h_H^{-1} K_i(\theta, x) H_i(t_k) | \mathscr{F}_{i-1} \right]$ forms a triangular array of stationary martingale differences with respect to the σ-field $\mathscr{F}_{i-1}$. Using the same arguments to the proof of Lemma 5 in Laib and Louani (2011) and that of Lemma 6.3 in Chaouch and Khardani (2015), allowing us to write, under (A1)-(A2), (A5) and Lemma 2, that

$$
\begin{aligned}
|\mathbb{E}(L^p_{i,n}(\theta,t_k,x)|\mathscr{F}_{i-1})| &= (2\max(1,a_1^2))^p h_H^{-p}\Big[c_2^2\phi_\theta(h_K)f_{i,1}(\theta,x) \\
&\quad +o_{a.s}(g_{i,\theta,x}(h_K))+o_{a.s}(g_{i,\theta,x}(h_K)\phi_\theta(h_K))\Big] \\
&= p!C^{p-2}h_H^{-p}[M\phi_\theta(h_K)f_{i,1}(\theta,x)+o_{a.s}(g_{i,\theta,x}(h_K))] \\
&\leq p!C^{p-2}h_H^{-p}\phi_\theta(h_K)[Mb_i(\theta,x)+1].
\end{aligned}
$$

where $C=2\max(1,a_1^2)$ and $M=(C_2C)^2$. The Kernel K and the function τ_0 are bounded by positive constants a_1 and C_2 respectively.
Taking $D_n=\sum_{i=1}^n d_i^2$, whith $d_i^2=\phi_\theta(h_K)h_H^{-p}[Mb_i(\theta,x)+1]$. By assumption (A2)(ii) and (A2)(v) we have $n^{-1}D_n=\phi_\theta(h_K)h_H^{-p}[MD(\theta,x)+o(1)]$ as $n\to\infty$.
Then, by using Lemma 1 in Laib and Louani (2011), where $D_n=\mathcal{O}(n\,h_H^{-p}\phi(h_K))$ a.s, $S_n=\sum_{i=1}^n L_{n,i}(\theta,t_k,x)$, and for $\varepsilon_0>0$ and C_1 a positive constant, we get

$$
\mathbb{P}\left(\sup_{t\in\mathscr{S}}|\widehat{f}_N(\theta,t_k,x)-\bar{f}_N(\theta,t_k,x)|>\varepsilon_0\sqrt{\frac{\log n}{n\,h_H^2\,\phi_\theta(h_K)}}\right)
$$

$$
\begin{aligned}
&\leq \mathbb{P}\left(\max_{k\in 1\ldots\tau_n}|\widehat{f}_N(\theta,t_k,x)-\bar{f}_N(\theta,t_k,x)|>\varepsilon_0\sqrt{\frac{\log n}{n\,h_H^2\,\phi_\theta(h_K)}}\right) \\
&\leq \tau_n\max_{k\in 1\ldots\tau_n}\mathbb{P}\left(|\sum_{i=1}^n L_{n,i,\theta}(\theta,t_k,x)|>n\varepsilon_o\mathbb{E}(K_1(\theta,x))\sqrt{\frac{\log n}{n\,h_H^2\,\phi_\theta(h_K)}}\right) \\
&\leq 2\tau_n\exp\left(-\frac{(n\varepsilon_0\mathbb{E}(K_1(\theta,x)))^2\frac{\log n}{nh_H^2\phi_\theta(h_K)}}{2D_n+2Cn\varepsilon_0\mathbb{E}(K_1(\theta,x))\sqrt{\frac{\log n}{nh_H^2\phi_\theta(h_K)}}}\right) \\
&\leq 2\tau_n\exp\{-C_1\varepsilon_0^2\log n\}\leq\frac{2}{n^{C_1\varepsilon_0^2-2\varsigma}}.
\end{aligned}
$$

Finally, by taking ε_0 large enough and by using Borel-Cantelli Lemma, the result can be easily deduced. □

LEMMA 12. *Under the assumptions (A1)-(A6) and the condition Eq. (1), we have*

$$
\sqrt{nh_H\phi_\theta(h_k)}Q_n(\theta,y,x)\xrightarrow{D}\mathcal{N}(0,\sigma_0^2(\theta,y,x)),
$$

where

$$\sigma_0^2 = \frac{M_2}{M_1^2}\frac{p(\theta,x)}{f_1(\theta,x)} f(\theta,y,x)\int_{\mathbb{R}} H^2(u)du.$$

PROOF. Let's denote

$$\zeta_{ni} = \left(\frac{\phi_\theta(h_K)}{nh_H}\right)^{1/2} \delta_i\,(H_i(y) - h_H f(\theta,y,x))\frac{K_i(\theta,x)}{\mathbb{E}(K_1(\theta,x))},$$

and define

$$\xi_{ni} = \zeta_{ni} - \mathbb{E}[\zeta_{ni}|\mathscr{F}_{i-1}].$$

It is easy to see that

$$(nh_H\phi_\theta(h_k))^{1/2}Q_n(\theta,y,x) = \sum_{i=1}^{n}\xi_{ni}.$$

Thus, the $\xi_{ni}, 1 \le i \le n$ forms a triangular array of stationary martingale differences with respect to the σ-field $\mathscr{F}_{i-1}$. By apply the central limit theorem for discrete-time arrays of real-valued martingales (Hall and Heyde (1980)), the asymptotic normality of $Q_n(\theta,y,x)$ can be obtained if we establish the following statements:

(a) $\sum_{i=1}^{n}\mathbb{E}[\xi_{ni}^2|\mathscr{F}_{i-1}] \xrightarrow{\mathbb{P}} \sigma_0^2(\theta,y,x)$.

(b) $n\mathbb{E}\left[\xi_{ni}^2 \mathbf{1}_{[|\xi_{ni}|>\varepsilon]}\right] = o(1) \quad for \quad \forall \varepsilon > 0$.

Proof of part (a) observe that

$$\left|\sum_{i=1}^{n}\mathbb{E}\left[\zeta_{ni}^2|\mathscr{F}_{i-1}\right] - \sum_{i=1}^{n}\mathbb{E}\left[\xi_{ni}^2|\mathscr{F}_{i-1}\right]\right| \le \sum_{i=1}^{n}\left(\mathbb{E}[\zeta_{ni}|\mathscr{F}_{i-1}]\right)^2.$$

Then, similar to the proof of Eq. (10) and using Lemma 8, we have

$$\begin{aligned}
|\mathbb{E}[\zeta_{ni}|\mathscr{F}_{i-1}]| &= \frac{(\phi_\theta(h_K)/nh_H)^{1/2}}{\mathbb{E}(K_1(\theta,x))}\left|\mathbb{E}\left[\delta_i K_i(\theta,x)(H_i(y) - h_H f(\theta,y,x))|\mathscr{F}_{i-1}\right]\right| \\
&= \frac{1}{\mathbb{E}(K_1(\theta,x))}\left(\frac{\phi_\theta(h_K)}{nh_H}\right)^{1/2}\Big|\mathbb{E}\Big[\delta_i K_i(\theta,x) \\
&\qquad \mathbb{E}[(H_i(y) - h_H f(\theta,y,x))| < \theta, X_i >]|\mathscr{F}_{i-1}\Big]\Big| \\
&\le C\,(h_K^{b_1} + h_H^{b_2})\left(\frac{\phi_\theta(h_K)h_H}{n}\right)^{1/2}(p(\theta,x) + o(1)) \\
&\qquad \left(\frac{f_{i,1}(\theta,x)}{f_1(\theta,x)} + \mathscr{O}_{a.s.}\left(\frac{g_{i,\theta,x}(h_K)}{\phi(h_K)}\right)\right).
\end{aligned}$$

Thus, by (A2)(ii)-(iii), we get

$$\sum_{i=1}^{n}\Big(\mathbb{E}[\zeta_{ni}|\mathscr{F}_{i-1}]\Big)^2=\mathcal{O}_{a.s.}\Big(h_H\phi_\theta(h_K)(h_K^{b_1}+h_H^{b_2})^2\Big).$$

Hence, the statement of (a) follows if we show that

$$\sum_{i=1}^{n}\mathbb{E}\big[\zeta_{ni}^2|\mathscr{F}_{i-1}\big]\xrightarrow{\mathbb{P}}\sigma_0^2(\theta,y,x),\quad as\quad n\to\infty.$$

By assumption (A5) we have

$$\begin{aligned}\sum_{i=1}^{n}\mathbb{E}\big(\zeta_{ni}^2|\mathscr{F}_{i-1}\big) &= \frac{\phi_\theta(h_K)}{nh_H(\mathbb{E}(K_1(\theta,x)))^2}\sum_{i=1}^{n}\mathbb{E}\Big\{\delta_i K_i^2(\theta,x)\Big(H_i(y)\\ &\qquad -h_H f(\theta,y,x)\Big)^2|\mathscr{F}_{i-1}\Big\}\\ &= \frac{\phi_\theta(h_K)}{nh_H(\mathbb{E}(K_1(\theta,x)))^2}\sum_{i=1}^{n}\mathbb{E}\Big\{\delta_i K_i^2(\theta,x)\\ &\qquad \mathbb{E}\big((H_i(y)-h_H f(\theta,y,x))^2\,|<\theta,X_i>\big)|\mathscr{F}_{i-1}\Big\}.\end{aligned}$$

Thus, by the properties of conditional expectation and (A5) for $j=0$ and $k=2$, we obtain that

$$\sum_{i=1}^{n}\mathbb{E}\big[\zeta_{ni}^2|\mathscr{F}_{i-1}\big]=V_{1,n}(\theta,y,x)+V_{2,n}(\theta,y,x),$$

where

$$\begin{aligned}V_{1,n}(\theta,y,x) &= \frac{\phi_\theta(h_K)}{n\,h_H\,(\mathbb{E}(K_1(\theta,x)))^2}\sum_{i=1}^{n}\mathbb{E}\Big[\delta_i K_i^2(\theta,x)\Big(\mathbb{E}(H_i^2(y)|<\theta,X_i>)\\ &\qquad -(\mathbb{E}(H_i(y)|<\theta,X_i>))^2\Big)|\mathscr{F}_{i-1}\Big],\end{aligned}$$

and

$$\begin{aligned}V_{2,n}(\theta,y,x) &= \frac{\phi_\theta(h_K)}{n\,h_H\,(\mathbb{E}(K_1(\theta,x)))^2}\\ &\qquad \sum_{i=1}^{n}\mathbb{E}\Big[\delta_i K_i^2(\theta,x)\mathbb{E}[(H_i(y)-h_H f(\theta,y,x))|<\theta,X_i>]^2|\mathscr{F}_{i-1}\Big].\end{aligned}$$

By inequality Eq. (11), and assumption (A3) and Lemma 8, it follows that, as $n\to\infty$

$$V_{2,n}(\theta,y,x)=\mathcal{O}_{a.s}((h_K^{b_1}+h_H^{b_2})^2)h_H(p(\theta,x)+o(1))\left(\frac{M_2}{M_1^2}\frac{1}{f_1(\theta,x)}+\mathcal{O}_{a.s}(1)\right)\to 0.$$

For $V_{1,n}$, notice that by changing variables, and by assumptions (A3)-(A4), we have

$$\begin{aligned}
\mathbb{E}(H_i^2(y)|<\theta,X_i>) &= \int_{\mathbb{R}} H^2\left(\frac{y-v}{h_H}\right) f(\theta,v,x)dv \\
&\leq h_H \int_{\mathbb{R}} H^2(u)\big[f(\theta,y-uh_H,x)-f(\theta,y,x)\big]du \\
&\quad + h_H f(\theta,y,x)\int_{\mathbb{R}} H^2(u)du \\
&\leq h_H^{1+b_2}\int_{\mathbb{R}} |u|^{b_2} H^2(u)du + h_H f(\theta,y,x)\int_{\mathbb{R}} H^2(u)du \\
&= h_H\Big(o(1)+f(\theta,y,x)\Big(\int_{\mathbb{R}} H^2(u)du\Big)\Big),
\end{aligned}$$

which implies that,

$$\frac{1}{h_H}\mathbb{E}(H_i^2(y)|<\theta,X_i>) \to f(\theta,y,x)\int_{\mathbb{R}} H^2(u)du, \quad as \quad n\to\infty. \tag{12}$$

Similarly, as $n \to \infty$, we have

$$\begin{aligned}
\mathbb{E}(H_i(t)|<\theta,X_i>) &= \frac{1}{h_H}\int H\left(\frac{t-v}{h_H}\right) f(\theta,v,x)dv \\
&= \int H(u)f(\theta,t-uh_H,x)du \to f(\theta,t,x)\int H(u)du. \quad (13)
\end{aligned}$$

Then, by Eq. (12)-Eq. (13) and Lemma 8, we arrive at

$$\begin{aligned}
V_{1,n}(\theta,y,x) &= \frac{\phi_\theta(h_K)}{n\,(\mathbb{E}(K_1(\theta,x)))^2}(p(\theta,x)+o(1))f(\theta,y,x) \\
&\qquad \int_{\mathbb{R}} H^2(u)du \sum_{i=1}^{n}\mathbb{E}[K_i^2(\theta,x)|\mathscr{F}_{i-1}] \\
&\to \frac{M_2}{M_1^2}\frac{p(\theta,x)f(\theta,y,x)}{f_1(\theta,x)}\int_{\mathbb{R}} H^2(u)du, \quad as \quad n\to\infty.
\end{aligned}$$

Proof of part (b). The definition of ξ_{ni}, implies that $n\mathbb{E}\Big[\xi_{ni}^2\mathbf{1}_{[|\xi_{ni}>\varepsilon|]}\Big] \leq 4n\mathbb{E}\Big[\zeta_{ni}^2\mathbf{1}_{[|\zeta_{ni}>\varepsilon/2|]}\Big]$, where $\mathbf{1}_A$ is an indicator function of a set A. Let $a>1$ and $b>1$ such that $1/a+1/b=1$. By Hölder and Markov inequalities, one can write, for all $\varepsilon>0$,

$$\mathbb{E}\Big[\zeta_{ni}^2\mathbf{1}_{[|\zeta_{ni}>\varepsilon/2|]}\Big] \leq \frac{\mathbb{E}|\zeta_{ni}|^{2a}}{(\varepsilon/2)^{2a/b}}.$$

Taking C_0 a positive constant and $2a = 2+\delta$ (with δ as in (A6)), we obtain

$$\begin{aligned}
4n\mathbb{E}\Big[\zeta_{ni}^2 \mathbf{1}_{[|\zeta_{ni}>\varepsilon/2|]}\Big] &\leq C_0\Gamma(\theta,x)\mathbb{E}\Big([|H_i(y)-h_H f(\theta,y,x)|\delta_i K_i(\theta,x)]^{2+\delta}\Big)\\
&\leq C_0\Gamma(\theta,x)\mathbb{E}\Big((K_i(\theta,x))^{2+\delta}\, p(\theta,X_i)\\
&\qquad \mathbb{E}\Big[|H_i(y)-h_H f(\theta,y,x)|^{2+\delta}\,|<\theta,X_i>\Big]\Big),
\end{aligned}$$

where $\Gamma(\theta,x) = \left(\frac{\phi_\theta(h_K)}{nh_H}\right)^{(2+\delta)/2} \frac{n}{(\mathbb{E}(K_1(\theta,x)))^{2+\delta}}$.

By changing variables, we get

$$\begin{aligned}
&\mathbb{E}\Big[|H_i(y)-h_H f(\theta,y,x)|^{2+\delta}\,|<\theta,X_i>\Big]\\
&= \int_{\mathbb{R}} \left(H\left(\frac{y-v}{h_H}\right) - h_H f(\theta,y,x)\right)^{2+\delta} f(\theta,v,x)dv\\
&\leq C\int_{\mathbb{R}} H^{2+\delta}\left(\frac{y-v}{h_H}\right) f(\theta,v,x)dv + h_H^{2+\delta} f^{2+\delta}(\theta,y,x)\\
&= Ch_H\int_{\mathbb{R}} H^{2+\delta}(u) f(\theta,y-uh_H,x)du + h_H^{2+\delta} f^{2+\delta}(\theta,y,x)\\
&= h_H\int_{\mathbb{R}} H^{2+\delta}(u) f(\theta,y-uh_H,x)du + h_H^{2+\delta} f^{2+\delta}(\theta,y,x),
\end{aligned}$$

$$\begin{aligned}
4n\mathbb{E}\Big[\zeta_{ni}^2 \mathbf{1}_{[|\zeta_{ni}>\varepsilon/2|]}\Big] &\leq C_0\left(\frac{\phi_\theta(h_K)}{n}\right)^{(2+\delta)/2} \frac{n(p(\theta,x)+o(1))}{h_H^{\delta/2}(\mathbb{E}(K_1(\theta,x)))^{2+\delta}}\\
&\qquad \mathbb{E}\Big(K_i^{2+\delta}(\theta,x)\Big[\int_{\mathbb{R}} H^{2+\delta}(u) f(\theta,y-uh_H,x)du\\
&\qquad +h_H^{1+\delta} f^{2+\delta}(\theta,y,x)\Big]\Big)\\
&\leq C_0\left(\frac{\phi_\theta(h_K)}{n}\right)^{(2+\delta)/2} \frac{n\mathbb{E}(K_i^{2+\delta}(\theta,x))(p(\theta,x)+o(1))}{h_H^{\delta/2}(\mathbb{E}(K_1(\theta,x)))^{2+\delta}}.
\end{aligned}$$

Thus, by Lemma 8, it follows that

$$\begin{aligned}
4n\mathbb{E}\Big[\zeta_{ni}^2 \mathbf{1}_{[|\zeta_{ni}>\varepsilon/2|]}\Big] &\leq C_0\frac{(p(\theta,x)+o(1))}{(nh_H\phi(h_K))^{\delta/2}} \frac{M_{2+\delta} f_1(\theta,x)+o(1)}{M_1^{2+\delta} f_1^{2+\delta}(\theta,x)+o(1)}\\
&= \mathcal{O}((nh_H\phi(h_K))^{-\delta/2}).
\end{aligned}$$

□

PROOF (PROPOSITION 1). By the decomposition Eq. (3), Proposition 1 follows directly from lemmas, Lemma 9-Lemma 11. □

PROOF (PROPOSITION 2). By Eq. (2) it follows that
$\sqrt{n\, h_H\, \phi_\theta(h_K)}R_n(\theta,y,x) \underset{n\to\infty}{\to} 0$ and $\sqrt{n\, h_H\, \phi_\theta(h_K)}B_n(\theta,y,x) \underset{n\to\infty}{\to} 0$.
In addition by the decomposition Eq. (3) and the second part of Lemma 9 and Lemma 12. Then, the proof of Proposition 2 is completed. □

REFERENCES

A. AIT SAÏDI, F. FERRATY, R. KASSA, P. VIEU (2008). *Cross-validated estimations in the single-functional index model*. Statistics, 42, no. 6, pp. 475–494.

S. ATTAOUI (2014). *On the nonparametric conditional density and mode estimates in the single functional index model with strongly mixing data*. Sankhya A, 76, no. 2, pp. 356–378.

S. ATTAOUI, A. LAKSACI, E. OULD-SAÏD (2011). *A note on the conditional density estimate in the single functional index model*. Statistics & Probability Letters, 81, no. 1, pp. 45–53.

A. A. BOUCHENTOUF, T. DJEBBOURI, A.RABHI, K. SABRI (2014). *Strong uniform consistency rates of some characteristics of the conditional distribution estimator in the functional single-index model*. Applicationes Mathematicae, 41, no. 4, pp. 301–322.

H. CARDOT, C. CRAMBES, P. SARDA (2004). *Estimation spline de quantiles conditionnels pour variables explicatives fonctionnelles.* Comptes Rendus. Mathématique. Académie des Sciences, Paris, 339, no. 2, pp. 141–144.

M. CHAOUCH, S. KHARDANI (2015). *Randomly censored quantile regression estimation using functional stationary ergodic data*. Journal of Nonparametric Statistics, 27, no. 1, pp. 65–87.

P. CHAUDHURI, K. DOKSUM, A. SAMAROV (1997). *On average derivative quantile regression*. The Annals of Statistics, 25, no. 2, pp. 715–744.

P. E. CHENG (1994). *Nonparametric estimation of mean functionals with data missing at random*. Journal of the American Statistical Association, 89, no. 425, pp. 81–87.

S. EFROMOVICH (2011a). *Nonparametric regression with responses missing at random*. Journal of Statistical Planning and Inference, 141, no. 12, pp. 3744–3752.

S. EFROMOVICH (2011b). *Nonparametric regression with predictors missing at random*. Journal of the American Statistical Association, 106, pp. 306–319.

M. EZZAHRIOUI, E. OULD SAÏD (2010). *Some asymptotic results of a non-parametric conditional mode estimator for functional time-series data*. Statistica Neerlandica, 64, no. 2, pp. 171–201.

F. FERRATY, A. LAKSACI, P. VIEU (2005). *Functional time series prediction via conditional mode estimation*. Comptes Rendus. Mathématique. Académie des Sciences, Paris, 340, no. 5, pp. 389–392.

F. FERRATY, A. LAKSACI, P. VIEU (2006). *Estimating some characteristics of the conditional distribution in nonparametric functional models*. Statistical Inference for Stochastic Processes, 9, no. 1, pp. 47–76.

F. FERRATY, A. PEUCH, P. VIEU (2003). *Modèle à indice fonctionnel simple. (single functional index model)*. Comptes Rendus. Mathématique. Académie des Sciences, Paris, 336, no. 12, pp. 1025–1028.

F. FERRATY, M. SUED, P. VIEU (2013). *Mean estimation with data missing at random for functional covariables*. Statistics, 47, no. 4, pp. 688–706.

F. FERRATY, P. VIEU (2003). *Curves discrimination: a nonparametric functional approach*. Computational Statistics and Data Analysis, 44, no. 1-2, pp. 161–173.

F. FERRATY, P. VIEU (2006). *Nonparametric functional data analysis. Theory and practice*. Springer Series in Statistics, New York.

P. HALL, C. C. HEYDE (1980). *Martingale Limit Theory and its Application*. Probability and Mathematical Statistics: A Series of Monographs and Textbooks, Academic Press, New York.

W. HÄRDLE, J. S. MARRON (1985). *Optimal bandwidth selection in nonparametric regression function estimation*. The Annals of Statistics, 13, pp. 1465–1481.

M. HRISTACHE, A. JUDITSKY, V. SPOKOINY (2001). *Direct estimation of the index coefficient in a single-index model*. The Annals of Statistics, 29, no. 3, pp. 595–623.

S. KHARDANI, M. LEMDANI, E. OULD SAÏD (2010). *Some asymptotic properties for a smooth kernel estimator of the conditional mode under random censorship*. Journal of the Korean Statistical Society, 39, no. 4, pp. 455–469.

S. KHARDANI, M. LEMDANI, E. OULD SAÏD (2011). *Uniform rate of strong consistency for a smooth kernel estimator of the conditional mode for censored time series*. Journal of Statistical Planning and Inference, 141, no. 11, pp. 3426–3436.

S. KHARDANI, M. LEMDANI, E. OULD SAÏD (2012). *On the strong uniform consistency of the mode estimator for censored time series*. Metrika, 75, no. 2, pp. 229–241.

N. LAIB, D. LOUANI (2010). *Nonparametric kernel regression estimation for functional stationary ergodic data: Asymptotic properties*. Journal of Multivariate Analysis, 101, no. 10, pp. 2266–2281.

N. LAIB, D. LOUANI (2011). *Rates of strong consistencies of the regression function estimator for functional stationary ergodic data*. Journal of Statistical Planning and Inference, 141, no. 1, pp. 359–372.

M. LEMDANI, E. OULD SAÏD, N. POULIN (2009). *symptotic properties of a conditional quantile estimator with randomly truncated data*. Journal of Multivariate Analysis, 100, no. 3, pp. 546–559.

H. LIANG, J. DE UÑA ÁLVAREZ (2010). *Asymptotic normality for estimator of conditional mode under left-truncated and dependent observations*. Metrika, 72, no. 1, pp. 1–19.

H. LIANG, S. WANG, R. J. CARROLL (2007). *Partially linear models with missing response variables and error-prone covariates*. Biometrika, 94, no. 1, pp. 185–198.

N. LING, L. LIANG, P. VIEU (2015). *Nonparametric regression estimation for functional stationary ergodic data with missing at random*. Journal of Statistical Planning and Inference, 162, pp. 75–87.

N. LING, Y. LIU, P. VIEU (2016). *Conditional mode estimation for functional stationary ergodic data with responses missing at random*. Statistics, 50, no. 5, pp. 991–1013.

R. J. A. LITTLE, D. B. RUBIN (2002). *Statistical analysis with missing data*. Chichester: John Wiley Series in Probability and Statistics, New York.

T. NITTNER (2003). *Missing at random (mar) in nonparametric regression a simulation experiment*. Statistical Methods and Applications, 12, no. 2, pp. 195–210.

E. OULD SAÏD, Z. CAI (2005). *Strong uniform consistency of nonparametric estimation of the censored conditional mode function*. Journal of Nonparametric Statistics, 17, no. 7, pp. 797–806.

E. OULD SAÏD, Y. DJABRANE (2011). *Asymptotic normality of a kernel conditional quantile estimator under strong mixing hypothesis and left-truncation*. Communications in Statistics. Theory and Methods, 40, no. 14, pp. 2605–2627.

A. TATACHAK, E. OULD SAÏD (2011). *A nonparametric conditional mode estimate under rlt model and strong mixing condition*. International Journal of Statistics and Economics, 6, pp. 76–92.

A. A. TSIATIS (2006). *Semiparametric theory and missing data.* Springer Series in Statistics, New York, NY: Springer.

SUMMARY

The main objective of this work is to estimate, semi-parametrically, the mode of a conditional density when the response is a real valued random variable subject to censored phenomenon and the predictor takes values in a semi-metric space. We assume that the explanatory and the response variables are linked through a single-index structure. First, we introduce a type of kernel estimator of the conditional density function when the data are supposed to be selected from an underlying stationary and ergodic process with missing at random (MAR). Under some general conditions, both the uniform almost-complete consistencies with convergence rates of the model are established. Further, the asymptotic normality of the considered model is given. As an application, the asymptotic $(1-\alpha)$ confidence interval of the conditional density function and the conditional mode are also presented for $0 < \alpha < 1$.

Keywords: Ergodic processes; Functional data analysis; Functional single-index process; Missing at random; Small ball probability.