STATISTICA, anno LXXX, n. 4, 2020

ON THE GENERALIZED ODD TRANSMUTED TWO-SIDED CLASS OF DISTRIBUTIONS

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1. INTRODUCTION

In recent years, extensive efforts have been made to present new models in the area of distribution theory and related statistical applications. These studies are mainly due to the modeling of various data sources and find out the probabilistic structure. In connection with the development of new models, it is worthwhile to note that these new models should have the capability for analyzing a wide range of real observations. Undoubtedly, this is also the most basic concern in the development of new models from the past to the future. Change point models are a kind of statistical models that have the ability to describe complex structure of phenomena with sudden changes in behavior. In the distribution theory, the change point distributions are used in different fields of sciences such as economic, engineering, agriculture and so on. However, the theory of these types of statistical distributions has become less developed over the last one to two decades. Motivated by economic applications, van Dorp and Kotz (2002a) suggested a two-sided distribution that has proved to be of seminal importance in economic theory. The corresponding probability density function (pdf) and cumulative distribution function (cdf) functions are given by

$$f(x;\alpha,\beta)=\begin{cases} \alpha\left(\frac{x}{\beta}\right)^{\alpha-1}, & 0<x\leq\beta, \\ \alpha\left(\frac{1-x}{1-\beta}\right)^{\alpha-1}, & \beta\leq x<1, \end{cases} \tag{1}$$

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and

$$F(x;\alpha,\beta)=\begin{cases}\beta\left(\frac{x}{\beta}\right)^{\alpha}, & 0<x\leq\beta,\\ 1-(1-\beta)\left(\frac{1-x}{1-\beta}\right)^{\alpha}, & \beta\leq x<1,\end{cases} \tag{2}$$

respectively, where $0\leq\beta\leq 1$ and $\alpha>0$. The parameter β is the location parameter called "turning point" and α is the shape parameter that controls the shape of distribution on the left and right of β. This model is known as two-sided power distribution (TSP) in the literature.

An extension of the three-parameter triangular distribution utilized in risk analysis has been introduced by van Dorp and Kotz (2002b). Their model includes the TSP distribution as a special case. van Dorp and Kotz (2003) considered a family of continuous distributions on a bounded interval generated by convolutions of the TSP distributions. In recent years, a number of researchers have studied some generalization of the TSP distribution such as Nadarajah (2005), Oruç and Bairamov (2005), Vicari *et al.* (2008), Herrerías-Velasco *et al.* (2009) and Soltani and Homei (2009).

More recent research associated with TSP distributions has done by Korkmaz and Genç (2017), and Kharazmi and Zargar (2019), to extended the idea of two-sidedness by implementing special cases of general Alzaatreh's method to some other ordinary distributions.

Alzaatreh *et al.* (2013) introduced a technique to drive new family of distributions by using an arbitrary pdf as a baseline generator. To review the Alzaatreh's method, suppose that $G(x)$ is a parent distribution and $m(x)$ is a initial probability distribution function of a random variable X. Then a general model by compounding cdf $G(x)$ and pdf $m(x)$ is given by

$$F(x)=\int_{a}^{W(G(x))} m(t)dt, \tag{3}$$

where $X\in(a,b)$ and $-\infty\leq a<b\leq\infty$ and $W(G(x))$ satisfies the following conditions:

(i) $W(G(x))\in(a,b)$

(ii) $W(G(x))$ is differentiable and monotonically non-decreasing

(iii) $W(G(x))\rightarrow a$ as $x\rightarrow-\infty$ and $W(G(x))\rightarrow b$ as $x\rightarrow\infty$.

A special case of $W(.)$ is the generalized odd ratio function. In recent years some researchers have considered odd ratio and generalized odd ratio functions for introducing more flexible distributions.

Cruz *et al.* (2016) proposed a generalization of the log-logistic distribution, namely the so-called the generalized odd log-logistic family. Also, they introduced the log-odd log-logistic Weibull regression model based on the odd log-logistic Weibull distribution with censored data. Cordeiro *et al.* (2017) considered a class of distributions, i.e. the so-called Burr XII with two extra positive parameters. Alizadeh *et al.* (2018) studied a generator of continuous distributions with one extra parameter called the odd power Cauchy family and they defined a log-odd power Cauchy-Weibull regression model.

As we mentioned formerly, the change point models have been less developed so far, and until now, the change point distributions are not presented based on the general Alzaatreh's technique. The main motivation of the present paper is to apply the generalized odd ratio function of baseline distribution G to the newly distribution introduced by Kharazmi and Zargar (2019). The authors have introduced a new family of distribution by applying the transmutation technique for the two-sided distribution. The pdf and cdf of this model are given by

$$f(x)=\begin{cases} \alpha\left((1+\lambda)\left(\frac{x}{\beta}\right)^{\alpha-1}-2\lambda\left(\frac{x}{\beta}\right)^{2\alpha-1}\right), & 0<x\leq\beta,\\ \alpha\left((1+\lambda)\left(\frac{1-x}{1-\beta}\right)^{\alpha-1}-2\lambda\left(\frac{1-x}{1-\beta}\right)^{2\alpha-1}\right), & \beta\leq x<1,\end{cases} \tag{4}$$

and

$$F(x)=\begin{cases} \beta\left((1+\lambda)\left(\frac{x}{\beta}\right)^{\alpha}-\lambda\left(\frac{x}{\beta}\right)^{2\alpha}\right), & 0<x\leq\beta,\\ 1-(1-\beta)\left((1+\lambda)\left(\frac{1-x}{1-\beta}\right)^{\alpha}-\lambda\left(\frac{1-x}{1-\beta}\right)^{2\alpha}\right), & \beta\leq x<1,\end{cases} \tag{5}$$

respectively, where α is a shape parameter, β is a scale parameter and λ is a transmuted parameter. They called this model transmuted two-sided (TTS) distribution.

The rest of the paper is organized as follows. In Section 2, we first propose a new family of distributions so-called generalized odd transmuted two-sided-G distribution and then the corresponding survival function, quantile function, asymptotics and moments of this distribution are obtained in general setting. Based on the exponential distribution as a parent distribution, we introduce generalized odd transmuted two-sided exponential distribution, in Section 3. In this section, we consider some properties of proposed distribution such as density shape, hazard function, moments and a theoretical discussion about survival regression. We study the performance of the maximum likelihood estimates of the parameters of the generalized odd transmuted two-sided exponential distribution via a simulation study, in Section 4. In Section 5, we fit the proposed distribution to the two real data sets and compare the results with some competitor distributions through the different criteria for model selection. Finally, the paper is concluded in Section 6.

2. Generalized odd transmuted two-sided-G distribution

Suppose that $G(x;\xi)$ is the cdf of a continuous random variable with pdf, parameter vector and inverse function $g(x;\xi)$, ξ and $G^{-1}_{(x;\xi)}(.)$, respectively. Then based on the relation (4) and under the condition that the function $W(G(x;\xi)) = \frac{G^{\eta}_{(x;\xi)}}{1-G^{\eta}_{(x;\xi)}}$ satisfies relations (i), (ii) and (iii), one can see the following definition.

DEFINITION 1. *Let* $\psi(x;\eta,\xi) = \frac{G^{\eta}_{(x;\xi)}}{1-G^{\eta}_{(x;\xi)}}$, *then a random variable* X *is said to be generalized odd transmuted two-sided-G distribution if its pdf is given by*

$$f(x) = \begin{cases} \alpha\psi'(x;\eta,\xi)\Big((1+\lambda)\Big(\frac{\psi(x;\eta,\xi)}{\beta}\Big)^{\alpha-1} - 2\lambda\Big(\frac{\psi(x;\eta,\xi)}{\beta}\Big)^{2\alpha-1}\Big), \\ \qquad\qquad\qquad\qquad\qquad\qquad -\infty < x \leq \Omega_1, \\ \alpha\psi'(x;\eta,\xi)\Big((1+\lambda)\Big(\frac{1-\psi(x;\eta,\xi)}{1-\beta}\Big)^{\alpha-1} - 2\lambda\Big(\frac{1-\psi(x;\eta,\xi)}{1-\beta}\Big)^{2\alpha-1}\Big), \\ \qquad\qquad\qquad\qquad\qquad\qquad \Omega_1 \leq x < \Omega_2, \end{cases} \tag{6}$$

where $\Omega_1 = G^{-1}_{(x;\xi)}\Big(\Big(\frac{\beta}{1+\beta}\Big)^{\frac{1}{\eta}}\Big)$, $\Omega_2 = G^{-1}_{(x;\xi)}\Big(\big(\frac{1}{2}\big)^{\frac{1}{\eta}}\Big)$ *and* $\psi'(x;\eta,\xi) = \frac{\eta g_{(x;\xi)} G^{\eta-1}_{(x;\xi)}}{\big(1-G^{\eta}_{(x;\xi)}\big)^2}$ *is the first derivative of* $\psi(x;\eta,\xi)$.

The corresponding cdf is given by

$$F(x) = \begin{cases} \beta\Big((1+\lambda)\Big(\frac{\psi(x;\eta,\xi)}{\beta}\Big)^{\alpha} - \lambda\Big(\frac{\psi(x;\eta,\xi)}{\beta}\Big)^{2\alpha}\Big), \\ \qquad\qquad\qquad\qquad\qquad\qquad -\infty < x \leq \Omega_1, \\ 1-(1-\beta)\Big((1+\lambda)\Big(\frac{1-\psi(x;\eta,\xi)}{1-\beta}\Big)^{\alpha} - \lambda\Big(\frac{1-\psi(x;\eta,\xi)}{1-\beta}\Big)^{2\alpha}\Big), \\ \qquad\qquad\qquad\qquad\qquad\qquad \Omega_1 \leq x < \Omega_2, \end{cases} \tag{7}$$

We denote the generalized odd transmuted two-sided-G family of distributions by GOTTS-G. Two sub-models of general density function with Eq. (6) are given in Remarks 2 and 3.

REMARK 2. *If* $\lambda = 0$, *we get a sub-model of density with Eq. (6) as*

$$f(x;\alpha,\beta,\eta,\xi) = \begin{cases} \alpha\psi'(x;\eta,\xi)\Big(\frac{\psi(x;\eta,\xi)}{\beta}\Big)^{\alpha-1}, & -\infty < x \leq \Omega_1, \\ \alpha\psi'(x;\eta,\xi)\Big(\frac{1-\psi(x;\eta,\xi)}{1-\beta}\Big)^{\alpha-1}, & \Omega_1 \leq x < \Omega_2, \end{cases} \tag{8}$$

and the corresponding cdf is given as

$$F(x;\alpha,\beta,\eta,\xi) = \begin{cases} \beta\Big(\frac{\psi(x;\eta,\xi)}{\beta}\Big)^{\alpha}, & -\infty < x \leq \Omega_1, \\ 1-(1-\beta)\Big(\frac{1-\psi(x;\eta,\xi)}{1-\beta}\Big)^{\alpha}, & \Omega_1 \leq x < \Omega_2. \end{cases} \tag{9}$$

Notice that this new model is obtained by applying generalized odd quantity for two-sided power distribution and it is called GOTSP-G.

REMARK 3. *If $\lambda = 0$ and $\alpha = \beta = 1$, then we get another sub-model as*

$$f(x;\eta,\xi) = \psi'(x;\eta,\xi), \qquad -\infty < x \leq \Omega_2, \tag{10}$$

and its cdf is given by

$$F(x;\eta,\xi) = \psi(x;\eta,\xi), \qquad -\infty \leq x < \Omega_2. \tag{11}$$

Following this section, we get some fundamental properties of proposed model such as survival function, quantile function and rth moment. It is seen that all of these measures have closed expression.

2.1. *Survival function*

The survival function (SF) is a key concept in reliability analysis and measuring the aging process. The SF of the general model in Eq. (6) is given as

$$\bar{F}(x) = \begin{cases} 1-\beta\Big((1+\lambda)\big(\frac{\psi(x;\eta,\xi)}{\beta}\big)^{\alpha} - \lambda\big(\frac{\psi(x;\eta,\xi)}{\beta}\big)^{2\alpha}\Big), & -\infty < x \leq \Omega_1, \\ (1-\beta)\Big((1+\lambda)\big(\frac{1-\psi(x;\eta,\xi)}{1-\beta}\big)^{\alpha} - \lambda\big(\frac{1-\psi(x;\eta,\xi)}{1-\beta}\big)^{2\alpha}\Big), & \Omega_1 \leq x < \Omega_2. \end{cases} \tag{12}$$

2.2. *Quantile function*

To generate samples from the GOTTS-G distribution, one can use the inverse transformation method. The quantile of order q of the GOTTS-G distribution is given by

$$x_q = F^{-1}(q;\alpha,\beta,\lambda,\eta,\xi) = \begin{cases} G^{-1}_{(x;\xi)}\Big[\big(\frac{A_1}{1+A_1}\big)^{\frac{1}{\eta}}\Big], & 0 < q \leq \beta, \\ G^{-1}_{(x;\xi)}\Big[\big(\frac{A_2}{1+A_2}\big)^{\frac{1}{\eta}}\Big], & \beta \leq q < 1, \end{cases}$$

where

$$A_1 = \beta\left(\frac{1+\lambda-\sqrt{(1+\lambda)^2 - \frac{4\lambda q}{\beta}}}{2\lambda}\right)^{\frac{1}{\alpha}}$$

and

$$A_2 = 1-(1-\beta)\left(\frac{1+\lambda-\sqrt{(1+\lambda)^2 - \frac{4\lambda(1-q)}{1-\beta}}}{2\lambda}\right)^{\frac{1}{\alpha}}.$$

Let U be a random variable generated from a uniform distribution on$(0,1)$, then the data generator for GOTTS-G distribution is given as

$$x_q = F^{-1}(u;\alpha,\beta,\lambda,\eta,\xi) = \begin{cases} G^{-1}_{(x;\xi)}\left[\left(\frac{A_1}{1+A_1}\right)^{\frac{1}{\eta}}\right], & 0 < U \leq \beta, \\ G^{-1}_{(x;\xi)}\left[\left(\frac{A_2}{1+A_2}\right)^{\frac{1}{\eta}}\right], & \beta \leq U < 1. \end{cases}$$

2.3. Moments

Some of the most important features and characteristics of a distribution can be studied through its moments. The rth moment of the GOTTS-G distribution is

$$\begin{aligned} E(X^r) &= \alpha \int_0^{\beta} \left(\psi^{-1}(y)\right)^r \left((1+\lambda)\left(\frac{y}{\beta}\right)^{\alpha-1} - 2\lambda\left(\frac{y}{\beta}\right)^{2\alpha-1}\right) dy \\ &+\alpha \int_{\beta}^{1} \left(\psi^{-1}(y)\right)^r \left((1+\lambda)\left(\frac{1-y}{1-\beta}\right)^{\alpha-1} - 2\lambda\left(\frac{1-y}{1-\beta}\right)^{2\alpha-1}\right) dy \\ &= E[\left(\psi^{-1}(Z)\right)^r], \end{aligned}$$

where $\psi^{-1}(y) = G^{-1}\left[\left(\frac{y}{1+y}\right)^{\frac{1}{\eta}}\right]$ and the random variable Z has the density function as Eq. (4).

2.4. Asymptotics

In order to evaluate the effect of parameters on tails of GOTTS-G family, we obtain the asymptotics of pdf, cdf and the hazard rate function of the proposed distribution. The asymptotics of the first part of Eqs. (6) and (7) as $x \to \gamma$, where $\gamma = \inf\{x|\ G(x) > 0\}$, are given by

- $F(x;\alpha,\beta,\lambda,\eta,\xi) \sim \frac{1+\lambda}{\beta^{\alpha-1}} G(x;\xi)^{\alpha\eta}$,
- $f(x;\alpha,\beta,\lambda,\eta,\xi) \sim \frac{(1+\lambda)\alpha\eta}{\beta^{\alpha-1}} g(x;\xi) G(x;\xi)^{\alpha\eta-1}$,
- $r(x;\alpha,\beta,\lambda,\eta,\xi) \sim \frac{(1+\lambda)\alpha\eta}{\beta^{\alpha-1}} g(x;\xi) G(x;\xi)^{\alpha\eta-1}$,

where $r(x;\alpha,\beta,\lambda,\eta,\xi) = \frac{f(x;\alpha,\beta,\lambda,\eta,\xi)}{1-F(x;\alpha,\beta,\lambda,\eta,\xi)}$ is the corresponding hazard rate function of GOTTS-G family. Due to the finite support of the second part of Eqs. (6) and (7), there is no need to consider its asymptotic behavior.

3. GENERALIZED ODD TRANSMUTED TWO-SIDED EXPONENTIAL DISTRIBUTION

In this section, we first introduce a special case of GOTT-G distribution, and then we provide some main statistical and reliability properties of this specialized sub-model.

Suppose that the parent distribution G has an exponential distribution with pdf, cdf and inverse cdf functions $g(x;\theta) = \frac{1}{\theta}e^{-\frac{x}{\theta}}$, $G(x;\theta) = 1 - e^{-\frac{x}{\theta}}, x > 0, \theta > 0$ and $G^{-1}(x;\theta) = -\theta \log(1-x)$, respectively. By substituting $g(x;\theta)$ and $G(x;\theta)$ in equations $\psi(x;\eta,\xi)$ and $\psi'(x;\eta,\xi)$, the pdf of GOTTS-E distribution is given as

$$f(x) = \begin{cases} \alpha\psi'(x;\eta,\theta)\Big((1+\lambda)\Big(\frac{\psi(x;\eta,\theta)}{\beta}\Big)^{\alpha-1} - 2\lambda\Big(\frac{\psi(x;\eta,\theta)}{\beta}\Big)^{2\alpha-1}\Big), & 0 < x \leq \Omega_3, \\ \alpha\psi'(x;\eta,\theta)\Big((1+\lambda)\Big(\frac{1-\psi(x;\eta,\theta)}{1-\beta}\Big)^{\alpha-1} - 2\lambda\Big(\frac{1-\psi(x;\eta,\theta)}{1-\beta}\Big)^{2\alpha-1}\Big), & \Omega_3 \leq x < \Omega_4, \end{cases} \tag{13}$$

and its cdf is given by

$$F(x) = \begin{cases} \beta\Big((1+\lambda)\Big(\frac{\psi(x;\eta,\theta)}{\beta}\Big)^{\alpha} - \lambda\Big(\frac{\psi(x;\eta,\theta)}{\beta}\Big)^{2\alpha}\Big), & -\infty < x \leq \Omega_3, \\ 1-(1-\beta)\Big((1+\lambda)\Big(\frac{1-\psi(x;\eta,\theta)}{1-\beta}\Big)^{\alpha} - \lambda\Big(\frac{1-\psi(x;\eta,\theta)}{1-\beta}\Big)^{2\alpha}\Big), & \Omega_3 \leq x < \Omega_4, \end{cases} \tag{14}$$

where $\Omega_3 = -\theta \ln\Big(1 - \Big(\frac{\beta}{1+\beta}\Big)^{\frac{1}{\eta}}\Big)$ and $\Omega_4 = -\theta \ln\Big(1 - \Big(\frac{1}{2}\Big)^{\frac{1}{\eta}}\Big)$.

We call this distribution the generalized odd transmuted two-sided exponential distribution and is denoted by GOTTS-E.

3.1. *Density shape of GOTTS-E distribution*

Here, we consider a discussion about the shape of the proposed density function. Some shapes of GOTTS-E distribution for the selected values of parameters are given in Figures 1 and 2.

In the end points of the support, the behaviour of the pdf of GOTTS-E distribution is given as follows.

$$\lim_{x\to 0} f(x;\alpha,\beta,\lambda,\eta,\theta) = \begin{cases} \infty, & \alpha\eta < 1 \\ \frac{1+\lambda}{\theta\beta^{\alpha-1}}, & \alpha\eta = 1 \\ 0, & \alpha\eta > 1 \end{cases}$$

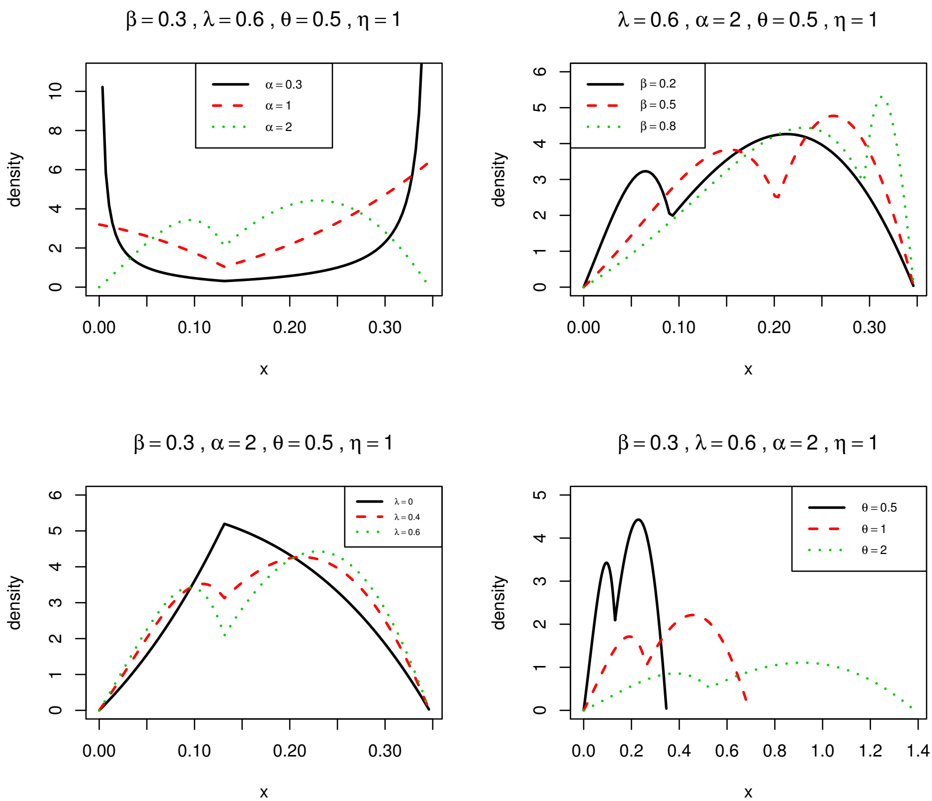

Figure 1 – The graphs of the densities of the GOTTS-E distribution with $0 \leq \lambda \leq 1$.

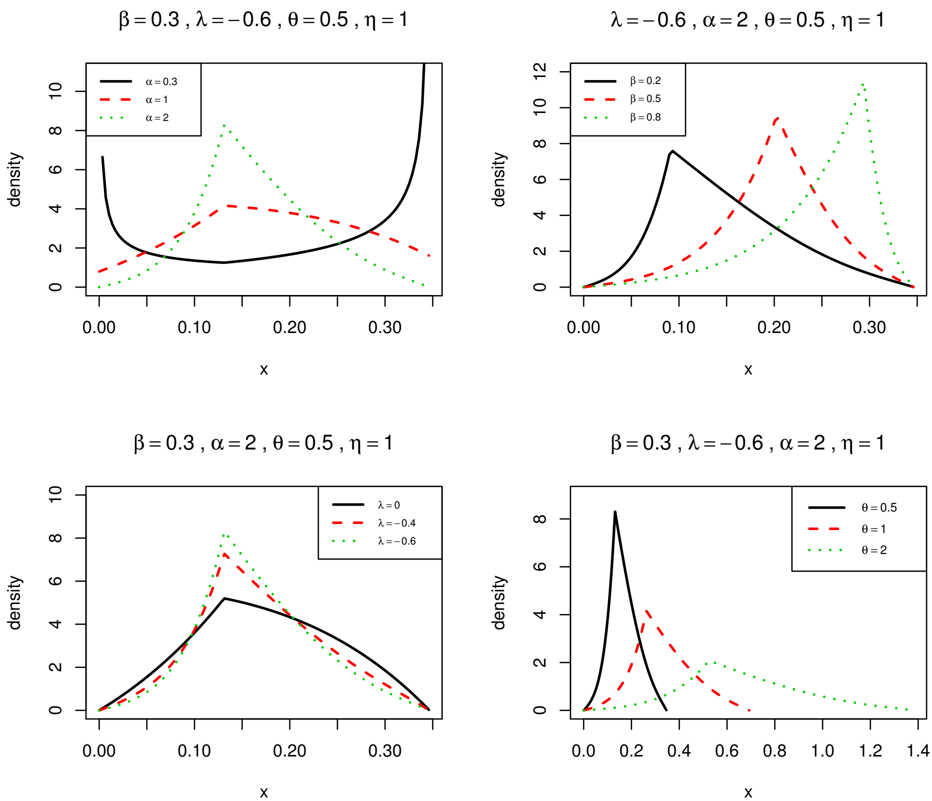

Figure 2 – The graphs of the densities of the GOTTS-E distribution with $-1 \leq \lambda \leq 0$.

$$\lim_{x\to\Omega_4} f(x;\alpha,\beta,\lambda,\eta,\theta)=\begin{cases}\infty, & \alpha<1\\ \frac{4\eta(1+\lambda)}{\theta}\left(1-\left(\frac{1}{2}\right)^{\frac{1}{\eta}}\right)\left(\frac{1}{2}\right)^{1-\frac{1}{\eta}}, & \alpha=1\\ 0, & \alpha>1\end{cases}$$

The right and left hand limits of f' at $x=\Omega_3$ are given by

$$\lim_{x\to\Omega_3^-} f'(x;\alpha,\beta,\lambda,\eta,\theta)=\frac{\omega_1\eta\alpha}{\theta^2(\frac{\beta}{1+\beta})^{2+\frac{2}{\eta}}}\left\{(1-\lambda)\frac{\omega_1\eta\beta-1}{1+\beta}+(1-3\lambda)\omega_1\eta\alpha\right\},$$

where $\omega_1=1-\left(\frac{\beta}{1+\beta}\right)^{\frac{1}{\eta}}$ and

$$\begin{aligned}\lim_{x\to\Omega_3^+} f'(x) \;=\; & \frac{4\eta(1+\beta)^4\left(\frac{\beta}{1+\beta}\right)^{1-\frac{1}{\eta}}\left(-1+\left(\frac{\beta}{1+\beta}\right)^{\frac{1}{\eta}}\right)}{(1-\beta)\theta^2\left(\frac{\beta}{1+\beta}\right)^{\frac{1}{\eta}}}\\ & \times\left[\lambda\left\{\omega_2\left(\frac{\beta}{1+\beta}\right)^2-\alpha\left(\frac{3}{2\alpha}+1+\omega_2\right)\frac{\beta}{1+\beta}-\frac{1}{2}(\omega_2+2)\right\}\right.\\ & \times\left(\frac{1-\beta}{\beta^2-2\beta-1}\right)^{2\alpha-1}-\frac{1}{2}(1+\lambda)\\ & \times\left\{\omega_2\left(\frac{\beta}{1+\beta}\right)^2+\frac{\alpha}{2}\left(\frac{3}{\alpha}+1+\omega_2\right)\frac{\beta}{1+\beta}-\frac{1}{2}(\omega_2+2)\right\}\\ & \left.\times\left(\frac{1-\beta}{\beta^2-2\beta-1}\right)^{\alpha-1}\right],\end{aligned}$$

where $\omega_2=\eta\left(\frac{\beta}{1+\beta}\right)^{\frac{1}{\eta}}-\eta-1$.

These limits are not equal. So, $f'(\Omega_3)$ does not exist and the GOTTS-E distribution has a corner at $x=\Omega_3$. Figures 1 and 2 indicate that for $\alpha>1$ and $\lambda>0$, the GOTTS-E distribution is a bimodal distribution and for $\alpha\geq 1$ and $\lambda\leq 0$, the GOTTS-E distribution is a unimodal distribution.

3.2. *Hazard function of the GOTTS-E distribution*

The hazard rate is a key concept in analysis of the aging process of different phenomena with probabilistic structure. Knowing the shape and behavior of the hazard rate in reliability theory, risk analysis, and so on, is very important. The hazard rate function of the general distribution GOTTS-G is given as

$$r(x) \;=\; \frac{f(x;\alpha,\beta,\lambda,\eta,\xi)}{1-F(x;\alpha,\beta,\lambda,\eta,\xi)}. \tag{15}$$

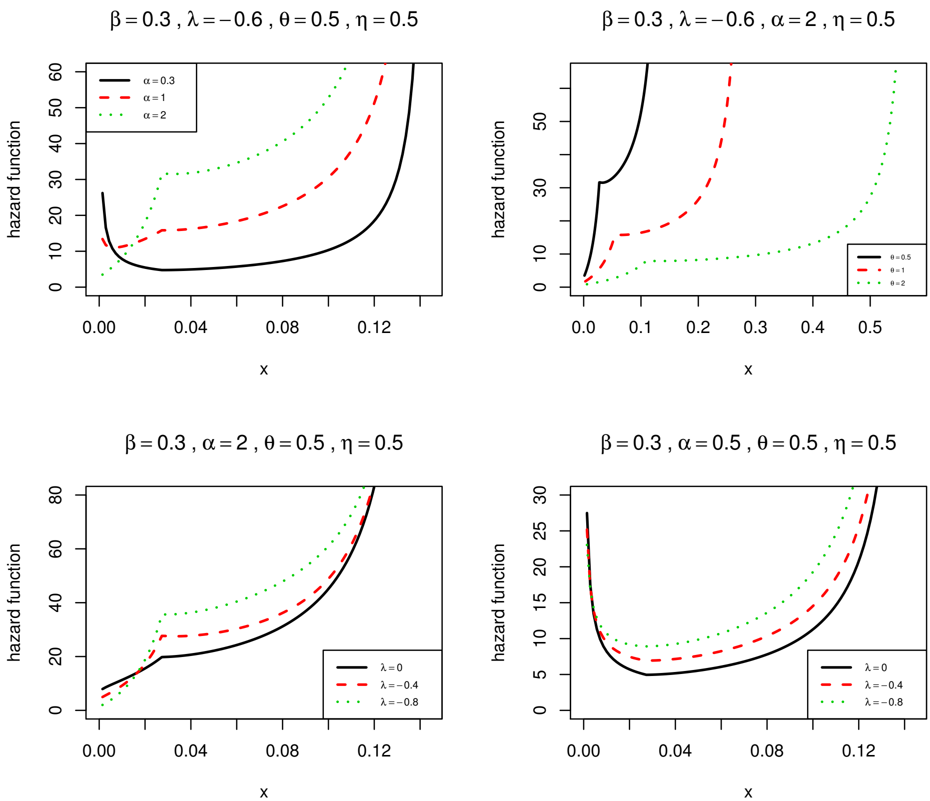

Figure 3 – The graphs of the hazard function of the GOTTS-E distribution with $0 \leq \lambda \leq 1$.

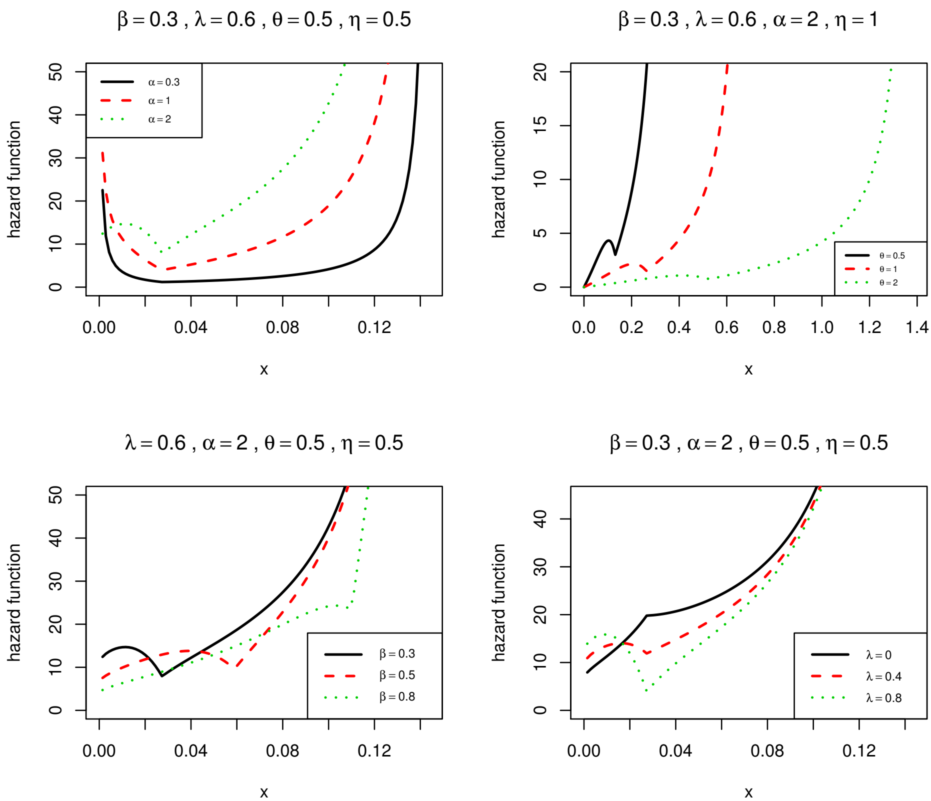

Figure 4 – The graphs of the hazard function of the GOTTS-E distribution with $-1 \leq \lambda \leq 0$.

In special case, when the parent distribution is exponential, one can calculate the hazard rate function of the GOTTS-E by substituting Eqs. (13) and (14) in Eq. (15). Because of complicated form of hazard function, we couldn't explore this function analytically. We only consider the end points of the support. The behaviour of hazard function in the end points is given as follows.

$$\lim_{x\to 0} r(x) = \begin{cases} \infty, & \alpha\eta < 1 \\ \frac{1+\lambda}{\theta\beta^{\alpha-1}}, & \alpha\eta = 1 \\ 0, & \alpha\eta > 1 \end{cases}$$

$$\lim_{x\to\Omega_4} r(x) = \infty, \qquad \forall\, \alpha > 0.$$

Some shapes of hazard function of GOTTS-E for the selected values of parameters are given in Figures 3 and 4. Figures 3 and 4 show that the hazard rate function of GOTTS-E distribution can be IFR (increasing failure rate), DFR (decreasing failure rate), BUT (bathtub shaped: first increasing and then decreasing), and UBT (upside-down bathtub shaped: first increasing and then decreasing). These classes of hazard rate function have been found very useful in reliability theory.

3.3. *Moments of the GOTTS-E distribution*

In this subsection, moments and related measures including mean, median, variance, skewness and kurtosis are presented. We provide these partial measures for the some selected value of parameters in Table 1. Based on the results of Table 1, it is easy to see the following.

- The mean and median values increase for increasing α, θ or η.
- On the other hand, for $\lambda > 0$, the skewness increases if $\beta \leq 0.5$ and decreases if $\beta > 0.5$. For $\lambda < 0$ and every $\beta \geq .5$, the skewness decreases for increasing β.
- For $\lambda > 0$, the kurtosis increases for $\beta \leq 0.5$. For $\lambda < 0$ and $\beta \geq .5$, the kurtosis increases for increasing β.
- For $\lambda > 0$ and $\beta \leq .5$ and every α, θ and η, the skewness of distribution is negative. For $\lambda > 0$ and each α, β, θ and η, the kurtosis of distribution is negative.

TABLE 1
The mean, median, variance, skewness and kurtosis of the GOTTS-E distribution for some parameter values.

β	λ	α	θ	η	Mean	Median	Variance	Skewness	Kurtosis
0.25	0.6	0.5	0.5	0.5	-	-	-	-	-
0.25	0.6	1	0.5	1	0.207	0.238	0.012	-0.533	-1.098
0.25	0.6	2	0.5	2	0.392	0.412	0.015	-0.446	-0.718
0.25	0.6	0.5	2	0.5	0.346	0.440	0.050	-0.545	-1.361
0.25	0.6	1	2	1	0.831	0.955	0.188	-0.532	-1.099
0.25	0.6	2	2	2	1.566	1.649	0.246	-0.446	-0.719
0.5	0.6	0.5	0.5	0.5	-	-	-	-	-
0.5	0.6	1	0.5	1	0.190	0.203	0.013	-0.169	-1.462
0.5	0.6	2	0.5	2	0.414	0.431	0.014	-0.466	-0.638
0.5	0.6	0.5	2	0.5	-	-	-	-	-
0.5	0.6	1	2	1	0.761	0.811	0.204	-0.169	-1.461
0.5	0.6	2	2	2	1.657	1.722	0.221	-0.467	-0.631
0.75	0.6	0.5	0.5	0.5	0.046	0.014	0.003	0.867	-1.189
0.75	0.6	1	0.5	1	0.174	0.164	0.012	0.119	-1.308
0.75	0.6	2	0.5	2	0.435	0.448	0.013	-0.568	-0.314
0.75	0.6	0.5	2	0.5	0.184	0.056	0.049	0.836	-1.053
0.75	0.6	1	2	1	0.696	0.655	0.187	0.117	-1.309
0.75	0.6	2	2	2	1.741	1.792	0.202	-0.569	-0.314
0.25	-0.6	0.5	0.5	0.5	0.068	0.068	0.002	0.086	-1.519
0.25	-0.6	1	0.5	1	0.178	0.176	0.007	0.034	-0.921
0.25	-0.6	2	0.5	2	0.359	0.351	0.008	0.169	-0.163
0.25	-0.6	0.5	2	.5	-	-	-	-	-
0.25	-0.6	1	2	1	0.714	0.702	0.118	0.034	-0.921
0.25	-0.6	2	2	2	1.437	1.403	0.133	0.169	-0.163
0.5	-0.6	0.5	0.5	0.5	0.063	0.059	0.002	0.237	-1.507
0.5	-0.6	1	0.5	1	0.196	0.203	0.007	-0.301	-0.667
0.5	-0.6	2	0.5	2	0.423	0.431	0.006	-0.709	1.126
0.5	-0.6	0.5	2	0.5	-	-	-	-	-
0.5	-0.6	1	2	1	0.784	0.811	0.109	-0.301	-0.667
0.5	-0.6	2	2	2	1.693	1.722	0.100	-0.703	1.077
0.75	-0.6	0.5	0.5	0.5	0.058	0.050	0.002	0.385	-1.391
0.75	-0.6	1	0.5	1	0.212	0.228	0.007	-0.582	-0.550
0.75	-0.6	2	0.5	2	0.476	0.497	0.006	-1.380	2.257
0.75	-0.6	0.5	2	0.5	0.233	0.201	0.037	0.334	-1.299
0.75	-0.6	1	2	1	0.849	0.914	0.114	-0.581	-0.549
0.75	-0.6	2	2	2	1.906	1.990	0.101	-1.379	2.258

3.4. Survival regression model of GOTTS-E distribution

In this section, we develop the survival regression theory for the GOTTS-E distribution. The survival regression model a well-known model in survival analysis and applied statistics. Recently, several regression models have been introduced in the literature by considering the class of location models. A number of researchers have introduced new families of distributions by using regression models such as Hashimoto *et al.* (2012), Ramires *et al.* (2013), Cordeiro *et al.* (2015) and Cordeiro *et al.* (2017). Based on the odd log-logistic Weibull distribution, Cruz *et al.* (2016) introduced the log-odd log-logistic Weibull regression model with censored data. Cordeiro *et al.* (2018) introduced a family of regression models based on a class of distributions called the Burr XII system of densities with two extra positive parameters. Alizadeh *et al.* (2018) also proposed a log-odd power Cauchy-Weibull regression model.

Let X be a random variable with pdf GOTTS-E in Eq. (13) for $x > 0$. By applying the log transformation $Y = \ln(X)$, the pdf of variable Y is given by

$$f(y) = \begin{cases} \alpha e^{y} \psi'(e^{y};\eta,\theta)\Big((1+\lambda)\Big(\frac{\psi(e^{y};\eta,\theta)}{\beta}\Big)^{\alpha-1} - 2\lambda\Big(\frac{\psi(e^{y};\eta,\theta)}{\beta}\Big)^{2\alpha-1}\Big), \\ \qquad\qquad -\infty < y \leq \ln(\Omega_3), \\ \alpha e^{y} \psi'(e^{y};\eta,\theta)\Big((1+\lambda)\Big(\frac{1-\psi(e^{y};\eta,\theta)}{1-\beta}\Big)^{\alpha-1} - 2\lambda\Big(\frac{1-\psi(e^{y};\eta,\theta)}{1-\beta}\Big)^{2\alpha-1}\Big), \\ \qquad\qquad \ln(\Omega_3) \leq y < \ln(\Omega_4). \end{cases}$$

By replacing $\mu = \ln(\theta)$, the pdf of variable Y is given by

$$f(y) = \begin{cases} \frac{\alpha\eta e^{y-\mu} e^{-e^{y-\mu}}\left(1-e^{-e^{y-\mu}}\right)^{\eta-1}}{\left(1-\left(1-e^{-e^{y-\mu}}\right)^{\eta}\right)^{2}} \\ \times\Big((1+\lambda)\Big(\frac{D_1}{\beta}\Big)^{\alpha-1} - 2\lambda\Big(\frac{D_1}{\beta}\Big)^{2\alpha-1}\Big), \\ \qquad -\infty < y \leq \mu + \ln\Big(-\ln\Big(1-(\frac{\beta}{1+\beta})^{\frac{1}{\eta}}\Big)\Big), \\ \frac{\alpha\eta e^{y-\mu} e^{-e^{y-\mu}}\left(1-e^{-e^{y-\mu}}\right)^{\eta-1}}{\left(1-\left(1-e^{-e^{y-\mu}}\right)^{\eta}\right)^{2}} \\ \times\Big((1+\lambda)\Big(\frac{1-D_1}{1-\beta}\Big)^{\alpha-1} - 2\lambda\Big(\frac{1-D_1}{1-\beta}\Big)^{2\alpha-1}\Big), \\ \mu + \ln\Big(-\ln\Big(1-(\frac{\beta}{1+\beta})^{\frac{1}{\eta}}\Big)\Big) \leq y < \mu + \ln\Big(-\ln\Big(1-(\frac{1}{2})^{\frac{1}{\eta}}\Big)\Big), \end{cases} \tag{16}$$

where $D_1 = \frac{\left(1-e^{-e^{y-\mu}}\right)^{\eta}}{1-\left(1-e^{-e^{y-\mu}}\right)^{\eta}}$ and $\mu \in R$ is the location parameter.

According to this transformation, one can define a new survival regression model as follows.

$$Y = \mu(\mathbf{v}) + Z, \tag{17}$$

where variable vector $\mathbf{v}$ is a auxiliary variable and Z is a error variable in the regression model such that its distribution does not depend on $\mathbf{v}$. By letting $Z = Y - \mu$ in Eq. (16), the pdf of Z is given by

$$f_Z(z) = \begin{cases} \frac{\alpha\eta e^z e^{-e^Z}\left(1-e^{-e^Z}\right)^{\eta-1}}{\left(1-\left(1-e^{-e^Z}\right)^{\eta}\right)^2}\left((1+\lambda)\left(\frac{D_2}{\beta}\right)^{\alpha-1} - 2\lambda\left(\frac{D_2}{\beta}\right)^{2\alpha-1}\right), \\ \qquad \Omega_1, \\ \frac{\alpha\eta e^z e^{-e^Z}\left(1-e^{-e^Z}\right)^{\eta-1}}{\left(1-\left(1-e^{-e^Z}\right)^{\eta}\right)^2}\left((1+\lambda)\left(\frac{1-D_2}{1-\beta}\right)^{\alpha-1} - 2\lambda\left(\frac{1-D_2}{1-\beta}\right)^{2\alpha-1}\right), \\ \qquad \Omega_2, \end{cases} \tag{18}$$

where

$\Omega_1 = -\infty < z \leq \ln\left(-\ln\left(1-(\frac{\beta}{1+\beta})^{\frac{1}{\eta}}\right)\right)$,

$\Omega_2 = \ln\left(-\ln\left(1-(\frac{\beta}{1+\beta})^{\frac{1}{\eta}}\right)\right) \leq z < \ln\left(-\ln\left(1-(\frac{1}{2})^{\frac{1}{\eta}}\right)\right)$, $D_2 = \frac{\left(1-e^{-e^Z}\right)^{\eta}}{1-\left(1-e^{-e^Z}\right)^{\eta}}$ and Z has a distribution that does not depend on $\mathbf{v}$.

Based on the linear regression model in Eq. (17), we have $y_i = \mu_i + z_i$, $i = 1, \ldots, n$ and we can define $\mu_i = \mathbf{v_i}^T\tau$, where the parameter vector $\tau = \left(\tau_1, \ldots, \tau_p\right)^T$ is a vector associated with the auxiliary variable vector $\mathbf{v_i}^T = \left(v_{i1}, \ldots, v_{ip}\right)$, $i = 1, \ldots, n$. The variable z_i is a noise variable with distribution in Eq. (18) in the regression model. The matrix form of the location parameter vector $\mu = (\mu_1, \ldots, \mu_n)^T$ is given by $\mu = \mathbf{V}\tau$, where $\mathbf{V} = (\mathbf{v_1}, \ldots, \mathbf{v_n})^T$ is a known matrix that contains the auxiliary variables.

4. SIMULATION

We consider the performance of the MLE's of the parameters with respect to sample size n for the GOTTS-E distribution. The evaluation of performance is based on a simulation study by using the Monte Carlo method. Let $\hat{\alpha}, \hat{\beta}, \hat{\lambda}, \hat{\theta}$ and $\hat{\eta}$ be the MLE's of the parameters $\alpha, \beta, \lambda, \theta$ and η, respectively. We calculate the mean square error (MSE) and bias of the MLE's of the parameters $\alpha, \beta, \lambda, \theta$ and η based on the simulation results of 4000 independence replications. results are summarised in Table 2 for different values of $n, \alpha, \beta, \lambda, \theta$ and η. From Table 2 the results verify that MSE of the MLE's of the parameters decrease with respect to sample size n for all parameters. So, it is concluded that the MLE's of $\alpha, \beta, \lambda, \theta$ and η are consistent estimators.

TABLE 2
MSE and bias (values in parentheses) of the MLE's of the parameters β,λ,α, θ and η.

n	$\beta=0.3$	$\lambda=0.25$	$\alpha=0.5$	$\theta=0.5$	$\eta=0.5$
30	0.021 (0.032)	0.711 (-0.629)	0.057 (-0.209)	0 (0)	0 (0)
50	0.012 (0.019)	0.750 (-0.651)	0.049 (-0.188)	0 (0)	0 (0)
100	0.006 (0.012)	0.716 (-0.593)	0.040 (-0.156)	0 (0)	0 (0)
200	0.003 (0.006)	0.444 (-0.369)	0.024 (-0.095)	0 (0)	0 (0)
n	$\beta=0.3$	$\lambda=0.25$	$\alpha=0.5$	$\theta=1.5$	$\eta=1.5$
30	0.023 (0.010)	0.851 (-0.706)	0.063 (-0.205)	0 (0)	0 (-0.001)
50	0.013 (0.008)	0.889 (-0.748)	0.056 (-0.195)	0 (0)	0 (0)
100	0.006 (0.004)	0.914 (-0.759)	0.051 (-0.187)	0 (0)	0 (0)
200	0.003 (0.003)	0.551 (-0.460)	0.029 (-0.116)	0 (0)	0 (0)
n	$\beta=0.3$	$\lambda=0.75$	$\alpha=2$	$\theta=0.5$	$\eta=0.5$
30	0.027 (0.030)	0.210 (0.039)	0.614 (-0.009)	0.115 (0.014)	0.021 (0.045)
50	0.016 (0.014)	0.170 (0.031)	0.332 (-0.001)	0.049 (0.007)	0.011 (0.026)
100	0.007 (0.003)	0.079 (0.024)	0.141 (-0.002)	0.017 (-0.006)	0.005 (0.017)
200	0.002 (-0.002)	0.024 (0.026)	0.049 (0.001)	0.007 (-0.009)	0.002 (0.011)
n	$\beta=0.3$	$\lambda=0.75$	$\alpha=2$	$\theta=1.5$	$\eta=1.5$
30	0.027 (0.030)	0.180 (0.050)	0.854 (-0.002)	0.222 (-0.044)	0.429 (0.240)
50	0.018 (0.019)	0.149 (0.037)	0.337 (-0.017)	0.141 (-0.036)	0.203 (0.148)
100	0.008 (0.003)	0.085 (0.025)	0.153 (-0.008)	0.062 (-0.024)	0.087 (0.079)
200	0.002 (-0.001)	0.023 (0.019)	0.053 (-0.016)	0.026 (-0.026)	0.040 (0.052)
n	$\beta=0.3$	$\lambda=-0.25$	$\alpha=0.5$	$\theta=0.5$	$\eta=0.5$
30	0.047 (0.060)	0.232 (-0.368)	0.047 (-0.199)	0 (-0.002)	0 (0)
50	0.029 (0.046)	0.237 (-0.366)	0.035 (-0.165)	0 (0)	0 (0)
100	0.015 (0.029)	0.244 (-0.367)	0.027 (-0.136)	0 (0)	0 (0)
200	0.008 (0.015)	0.251 (-0.369)	0.024 (-0.120)	0 (0)	0 (0)
n	$\beta=0.3$	$\lambda=-0.75$	$\alpha=2$	$\theta=0.5$	$\eta=0.5$
30	0.017 (-0.012)	0.169 (0.150)	16.078 (0.895)	335.934 (0.496)	0.065 (0.101)
50	0.010 (-0.008)	0.149 (0.159)	3.884 (0.538)	85.413 (0.312)	0.038 (0.061)
100	0.005 (-0.005)	0.113 (0.150)	0.592 (0.322)	0.402 (0.060)	0.016 (0.030)
200	0.003 (-0.002)	0.092 (0.135)	0.380 (0.259)	0.071 (0.030)	0.007 (0.013)
n	$\beta=0.3$	$\lambda=-0.75$	$\alpha=2$	$\theta=1.5$	$\eta=1.5$
30	0.017 (-0.003)	0.165 (0.148)	20.869 (0.927)	2.546 (0.192)	0.954 (0.377)
50	0.010 (-0.007)	0.149 (0.154)	2.288 (0.480)	1.885 (0.128)	0.588 (0.255)
100	0.005 (-0.004)	0.115 (0.140)	0.813 (0.333)	1.150 (0.080)	0.265 (0.130)
200	0.003 (-0.002)	0.084 (0.117)	0.405 (0.239)	0.434 (0.037)	0.107 (0.058)

5. Application of the GOTTS-E distribution

To investigate the advantage of the proposed distribution, we consider two real data sets. The first data set provided by Bjerkedal (1960). This data set devoted to the failure times of 84 aircraft windshield. The windshield on a large aircraft is a complex piece of equipment, comprised basically of several layers of material, including a very strong outer skin with a heated layer just beneath it, all laminated under high temperature and pressure. Failures of these items are not structural failures. Instead, they typically involve damage of the nonstructural outer ply or failure of the heating system. The second data set represents the strengths of 1.5 cm glass fibers, measured at the National Physical Laboratory, England. Unfortunately, the units of measurement are not given in the paper. It is obtained from Smith and Naylor (1987) and also analyzed by Barreto-Souza *et al.* (2010). These data are given below.

First data set: Failure times of 84 Aircraft Windshield

0.040, 1.866, 2.385, 3.443, 0.301, 1.876, 2.481, 3.467, 0.309, 1.899, 2.610, 3.478, 0.557, 1.911, 2.625, 3.578, 0.943, 1.912, 2.632, 3.595, 1.070, 1.914, 2.646, 3.699, 1.124, 1.981, 2.661, 3.779,1.248, 2.010, 2.688, 3.924, 1.281, 2.038, 2.82,3, 4.035, 1.281, 2.085, 2.890, 4.121, 1.303, 2.089, 2.902, 4.167, 1.432, 2.097, 2.934, 4.240, 1.480, 2.135, 2.962, 4.255, 1.505, 2.154, 2.964, 4.278, 1.506, 2.190, 3.000, 4.305, 1.568, 2.194, 3.103, 4.376, 1.615, 2.223, 3.114, 4.449, 1.619, 2.224, 3.117, 4.485, 1.652, 2.229, 3.166, 4.570, 1.652, 2.300, 3.344, 4.602, 1.757, 2.324, 3.376, 4.663.

Second data set: Strengths of 1.5 cm Glass Fibers

0.55 0.93 1.25 1.36 1.49 1.52 1.58 1.61 1.64 1.68 1.73 1.81 2.00 0.74 1.04 1.27 1.39 1.49 1.53 1.59 1.61 1.66 1.68 1.76 1.82 2.01 0.77 1.11 1.28 1.42 1.50 1.54 1.60 1.62 1.66 1.69 1.76 1.84 2.24 0.81 1.13 1.29 1.48 1.50 1.55 1.61 1.62 1.66 1.70 1.77 1.84 0.84 1.24 1.30 1.48 1.51 1.55 1.61 1.63 1.67 1.70 1.78 1.89.

5.1. *Bootstrap inference for the parameters of the GOTTS-E distribution*

In this section, we obtain point and 95% confidence interval (CI) estimation of parameters of the GOTTS-E distribution by parametric and non-parametric bootstrap methods for the two data sets. We provide results of bootstrap estimation based on 10000 bootstrap replicates in Tables 3 and 4. It is interesting to look at the joint distribution of the bootstrapped values in a scatter plot in order to understand the potential structural correlation between parameters (see Figures 5, 6, 7 and 8).

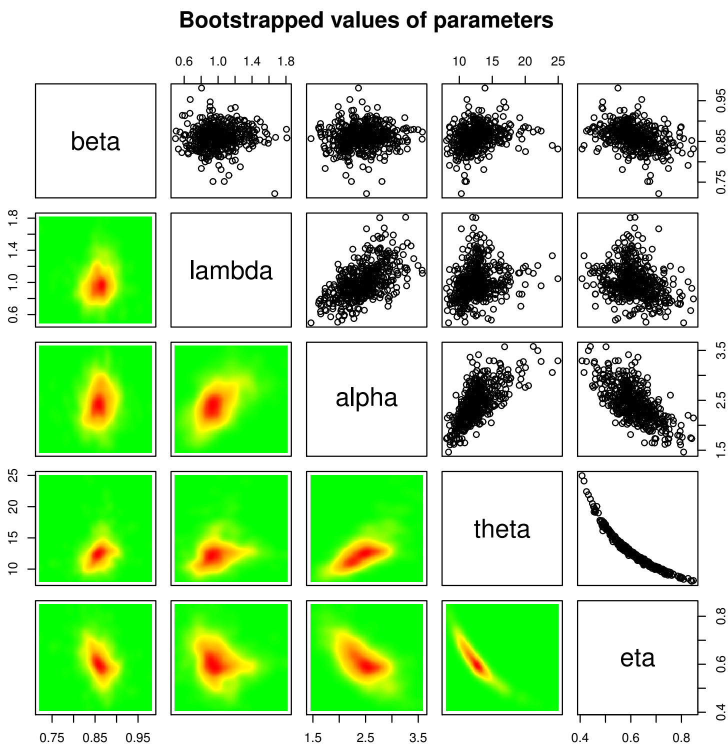

Figure 5 – Parametric bootstrapped values of parameters of GOTTS-E distribution for the first data set.

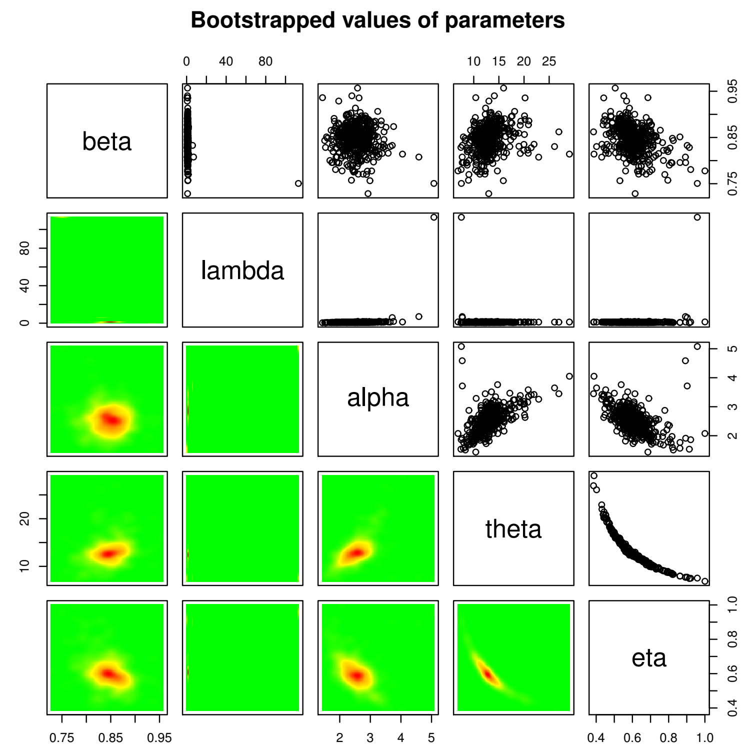

Figure 6 – Non-parametric bootstrapped values of parameters of GOTTS-E distribution for the first data set.

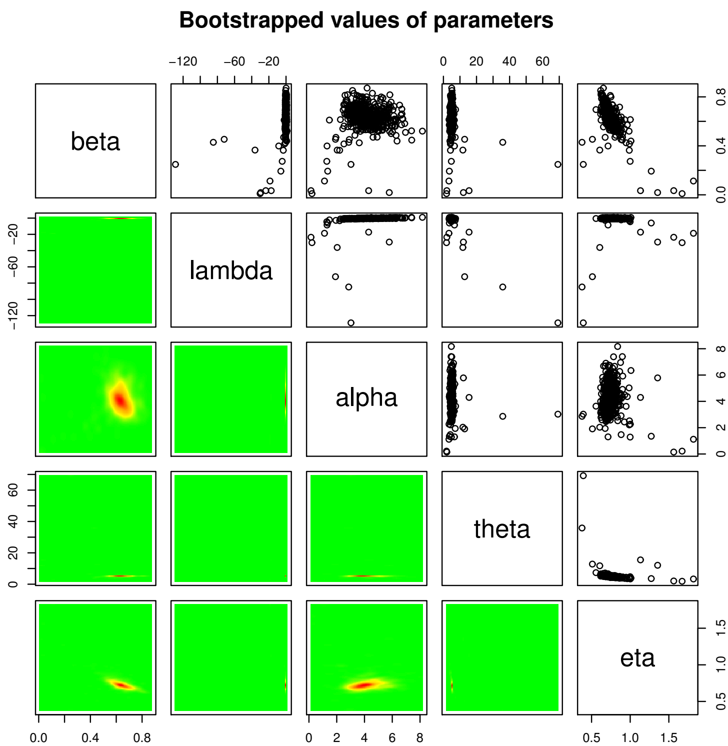

Figure 7 – Parametric bootstrapped values of parameters of GOTTS-E distribution for the second data set.

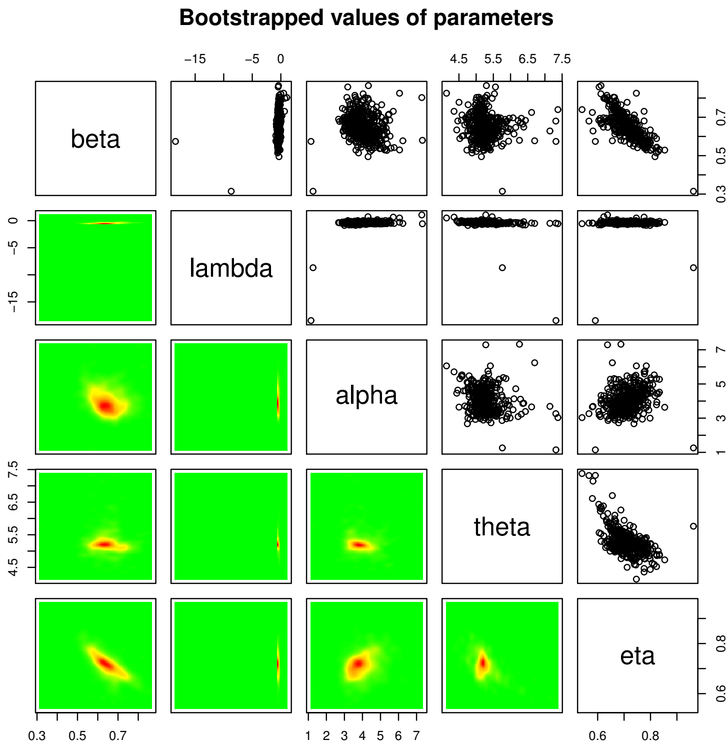

Figure 8 – Non-parametric bootstrapped values of parameters of GOTTS-E distribution for the second data set.

TABLE 3
Bootstrap point and interval estimation of the parameters α, β, λ, θ and η for the first data set.

Parametric bootstrap		Non-parametric bootstrap		
Parameters	Point estimate	CI	Point estimation	CI
α	2.429	(1.732, 3.244)	2.531	(1.770, 3.421)
β	0.860	(0.806, 0.913)	0.848	(0.778, 0.898)
λ	0.975	(0.624,1.475)	1.016	(0.630, 1.622)
θ	12.457	(8.945, 19.119)	12.799	(8.552, 20.193)
η	0.606	(0.463, 0.776)	0.596	(0.452, 0.818)

TABLE 4
Bootstrap point and interval estimation of the parameters α, β, λ, θ and η for the second data set.

	Parametric bootstrap		Non-parametric bootstrap	
Parameters	Point estimate	CI	Point estimation	CI
α	4.109	(2.444, 6.324)	3.870	(0.538, 0.789)
β	0.639	(0.426, 0.797)	0.645	(0.538, 0.789)
λ	-0.400	(-3.094, 0.556)	-0.439	(-0.621, -0.036)
θ	5.131	(3.999, 6.096)	5.203	(4.734, 6.190)
η	0.727	(0.628, 0.957)	0.713	(0.618, 0.803)

5.2. *MLE inference and comparing with other models*

We fit the proposed distribution to the two real data sets by MLE method and compare the results with the gamma, Weibull, two-sided generalized exponential (TSGE), transmuted two-sided generalized exponential (TTSGE) and generalized exponential (GE) distributions with respective densities

$$f_{\text{gamma}}(x) = \frac{1}{\Gamma(\alpha)} \lambda^{\alpha} x^{\alpha-1} e^{-\lambda x}, \qquad x > 0,$$

$$f_{\text{Weibull}}(x) = \frac{\beta}{\lambda^{\beta}} x^{\beta-1} e^{-(\frac{x}{\lambda})^{\beta}}, \qquad x > 0,$$

$$f_{\text{TSGE}}(x) = \begin{cases} \alpha \frac{1}{\theta} e^{-\frac{x}{\theta}} (\frac{1-e^{\frac{-x}{\theta}}}{\beta})^{\alpha-1}, & 0 < x \leq -\theta \log(1-\beta), \\ \alpha \frac{1}{\theta} e^{-\frac{x}{\theta}} (\frac{e^{\frac{-x}{\theta}}}{1-\beta})^{\alpha-1}, & -\theta \log(1-\beta) \leq x < \infty, \end{cases}$$

$$f_{\mathrm{TTSGE}}(x)=\begin{cases}\alpha\frac{1}{\theta}e^{-\frac{x}{\theta}}\left((1+\lambda)\left(\frac{1-e^{-\frac{x}{\theta}}}{\beta}\right)^{\alpha-1}-2\lambda\left(\frac{1-e^{-\frac{x}{\theta}}}{\beta}\right)^{2\alpha-1}\right), \\ \qquad 0<x\leq-\theta\log(1-\beta), \\ \alpha\frac{1}{\theta}e^{-\frac{x}{\theta}}\left((1+\lambda)\left(\frac{e^{-\frac{x}{\theta}}}{1-\beta}\right)^{\alpha-1}-2\lambda\left(\frac{e^{-\frac{x}{\theta}}}{1-\beta}\right)^{2\alpha-1}\right), \\ \qquad -\theta\log(1-\beta)\leq x<\infty,\end{cases}$$

$$f_{\mathrm{GE}}(x)=\alpha\lambda e^{-\lambda x}(1-e^{-\lambda x})^{\alpha-1}, \qquad x>0.$$

Here, we first provide the numerical results for the first data set. For each model, Table 5 includes the MLE's of parameters, corresponding log-likelihood, Akaike information criterion (AIC) and Bayesian information criterion (BIC) for the first data set. We fit the GOTTS-E (GOTTSE) distribution to the this data set and compare it with the mentioned distributions. The selection criterion is that the lowest AIC and BIC statistic corresponding to the best fitted model. The GOTTS-E distribution provides the best fit for the data set as it has lower AIC and BIC statistic than the other competitor models. The histogram of data set, fitted pdf of GOTTS-E distribution and fitted pdfs of other competitor distributions for the real data set are plotted in Figure 9. The plots of empirical and fitted cdfs functions, P-P plots and Q-Q plots for the GOTTS-E and other fitted distributions are displayed in Figure 9. These plots also support the results in Table 5.

TABLE 5
The MLE's of parameters for the first data set.

Model	Estimates	Log-likelihood	AIC	BIC
GOTTS-E	$(\hat{\alpha},\hat{\beta},\hat{\lambda},\hat{\theta},\hat{\eta})$=(2.362, 0.855, 0.893,12.807, 0.596)	-122.286	254.571	266.726
TTSGE	$(\hat{\alpha},\hat{\beta},\hat{\lambda},\hat{\theta})$=(2.330,0.953, -0.790, 1.471)	-127.144	262.287	272.011
Gamma	$(\hat{\alpha},\hat{\lambda})$=(3.492, 1.365)	-136.937	277.874	282.735
Weibull	$(\hat{\beta},\hat{\lambda})$=(2.374,2.863)	-130.053	264.107	268.968
TSGE	$(\hat{\alpha},\hat{\beta},\hat{\theta})$=(3.211, 0.922, 1.691)	-130.979	267.958	275.250
GE	$(\hat{\alpha},\hat{\lambda})$=(3.562, 0.758)	-139.841	283.681	288.543

Analogously, we provide the numerical results for second data set. Table 6 includes the MLE's of parameters, Kolmogorov-Smirnov (K-S) distance between the empirical distribution and the fitted model, its corresponding p-value, log-likelihood and Akaike information criterion (AIC) for candidate models for fitting. We fit the GOTTS-E distribution to the real data set and compare it with the distributions which mentioned formerly. The GOTTS-E distribution provides the best fit for the second data set as it has lower AIC and K-S statistic than the other competitor models. The histogram of the current data set, fitted pdf of the GOTTS-E distribution and fitted pdfs of other competitor distributions are plotted in Figure 10. Also, the plots of empirical and fitted cdfs functions, P-P plots and Q-Q plots for the GOTTS-E and other fitted distributions are displayed in Figure 10. These plots also support the results in Table 6.

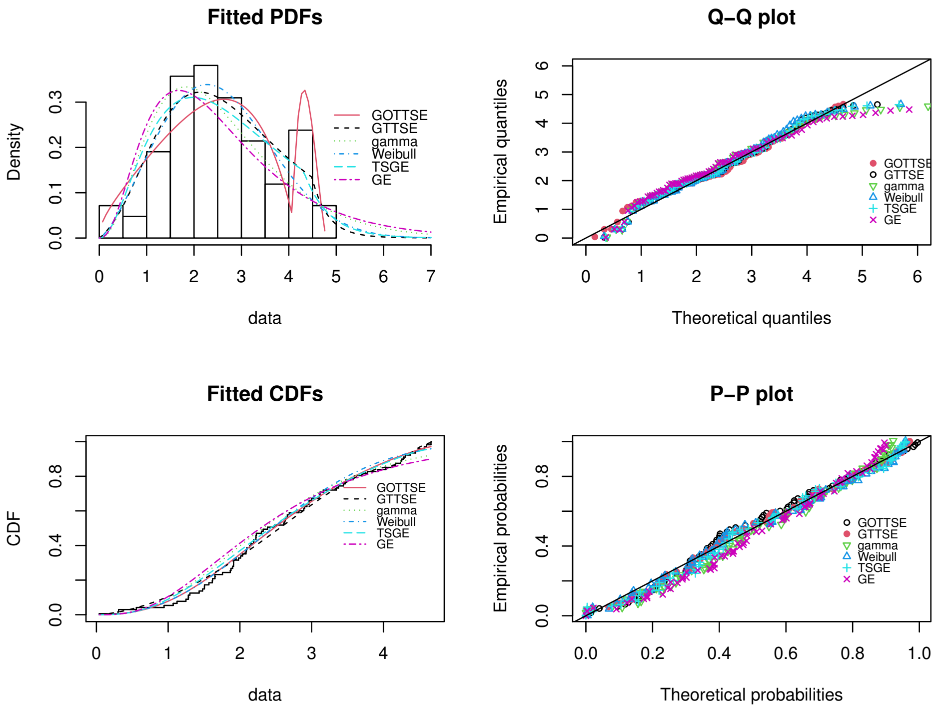

Figure 9 – Histogram, fitted pdfs, empirical and fitted cdfs, Q-Q plots and P-P plots of the GOTT-E distribution and other fitted distributions for the first data set.

TABLE 6
The MLE's of parameters for the second data set.

Model	Estimates	Log-likelihood	AIC	K-S statistic	*p*-value
GOTTS-E	$(\hat{\alpha},\hat{\beta},\hat{\lambda},\hat{\theta},\hat{\eta})$=(3.660,0.642,-0.493, 5.141,0.724)	-9.937	29.873	0.087	0.720
TTSG-E	$(\hat{\alpha},\hat{\beta},\hat{\lambda},\hat{\theta})$=(4.424, 0.737,-0.953, 1.256)	-12.064	32.129	0.141	0.160
Gamma	$(\hat{\alpha},\hat{\lambda})$=(17.441, 11.575)	-23.951	51.903	0.216	0.005
Weibull	$(\hat{\beta},\hat{\lambda})$=(5.781,1.628)	-15.207	34.414	0.152	0.108
TSGE	$(\hat{\alpha},\hat{\beta},\hat{\theta})$=(8.214, 0.729, 1.287)	-12.311	30.622	0.143	0.152
GE	$(\hat{\alpha},\hat{\lambda})$=(31.351, 2.612)	-31.383	66.767	0.229	0.003

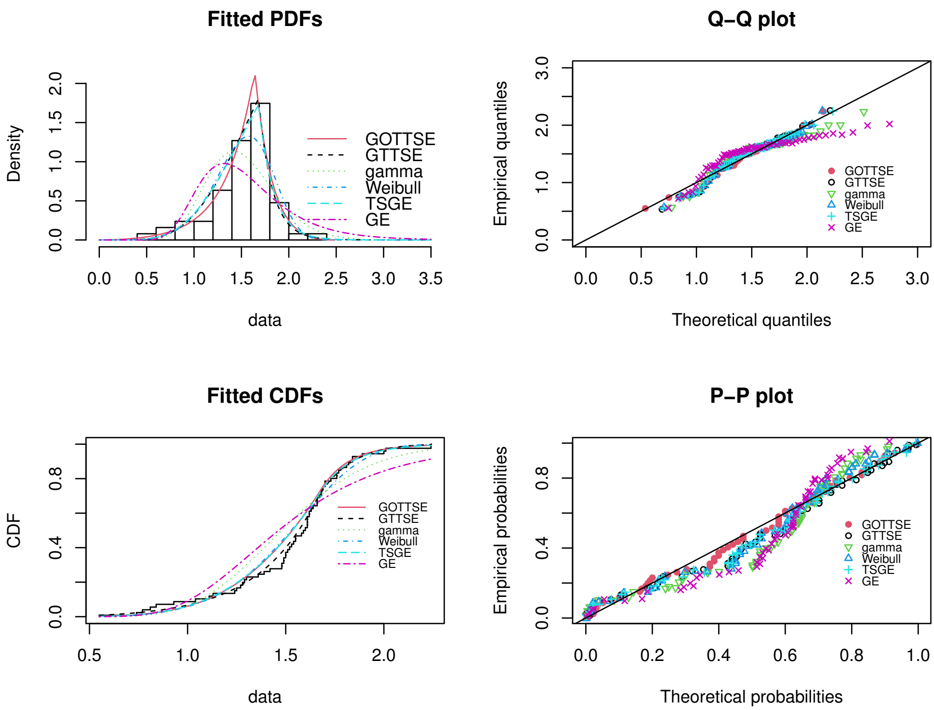

Figure 10 – Histogram, fitted pdfs, empirical and fitted cdfs, Q-Q plots and P-P plots of the GOTT-E distribution and other fitted distributions for the second data set.

6. Conclusion

In this article, taking into account the odd ratio function, the fundamental concept in survival analysis, a new two-sided family of lifetime distributions is introduced and its main properties are derived. A special example of this family is introduced by considering the exponential model as the baseline distribution. We also showed that the proposed distribution has various hazard rate shapes such as increasing, decreasing and bathtub shapes. Numerical results of the maximum likelihood and bootstrap procedures for two real data sets are presented in separate tables. From a practical point of view, we showed that the proposed distribution is more flexible than some common statistical distributions. In particular, we showed that the proposed model has the ability to fit into bimodal data structures.

Acknowledgements

The authors thank two reviewers and the Associate Editor for their helpful comments.

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SUMMARY

In this paper, a general class of two-sided lifetime distributions is introduced via odd ratio function, the well-known concept in survival analysis and reliability engineering. Some statistical and reliability properties including survival function, quantiles, moments function, asymptotic and maximum likelihood estimation are provided in a general setting. A special case of this new family is taken up by considering the exponential model as the parent distribution. Some characteristics of this specialized model and also a discussion associated with survival regression are provided. A simulation study is presented to investigate the bias and mean square error of the maximum likelihood estimators. Moreover, two examples of real data sets are studied; point and interval estimations of all parameters are obtained by maximum likelihood and bootstrap (parametric and non-parametric) procedures. Finally, the superiority of the proposed model over some common statistical distributions is shown through the different criteria for model selection including log-likelihood values, Akaike information criterion and Kolmogorov-Smirnov test statistic values.

Keywords: Hazard rate function; Survival function; Maximum likelihood estimation; Odd ratio function; Regression.