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A FLEXIBLE BATHTUB-SHAPED FAILURE TIME MODEL: PROPERTIES AND ASSOCIATED INFERENCE

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1. Introduction

In the statistical literature, a vast proportion is acquired by the lifetime distributions and their applications in real-world phenomena. This theory has paramount importance and hence many new statistical models along with their better adequacy in terms of their goodness of fit are discussed in the literature. However, many real-world problems still exist in which the classical model don't fit the real data adequately. In order to achieve more flexibility in terms of hazard function to study needful properties of the model and for better fitness, many extension have been proposed by researcher by adding a shape parameter in the baseline distribution. In the class of these extended models, Eugene *et al.* (2002) introduced a class of distributions generated from the logit of the beta random variable. Zografos and Balakrishnan (2009) defined a family of univariate distributions generated by Stacy's generalized gamma variables. Alexander *et al.* (2012) introduced generalized beta-generated (GBG) distributions. Amini *et al.* (2014) introduced two new general families of continuous distributions, generated by a distribution F and two positive real parameters α and β which control the skewness and tail weight of the distribution. Tahir *et al.* (2016) introduced a new generator based on the Weibull random variable called the new Weibull-G family. Recently, EL-Morshedy *et al.* (2021) presented a new class of the type I half logistic odd Weibull-G by combining the type I half logistic and odd Weibull families.

The Weibull family is inappropriate when the hazard rate function indicates to be non-monotone particularly bathtub–shaped. As the models with bathtub-shaped failure rates have great practical value, particularly in reliability analysis, this motivates

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researchers to work on the derived models of Weibull distribution which exhibit the tendency of U shaped failure rate functions. Among which the modified Weibull (MW) of Lai *et al.* (2003) has received great attention from many researchers and reliability practitioners. The study in Soliman *et al.* (2012) dealt with the Bayesian estimation using MCMC approach based on progressive censoring data from MW model. Singh and Goel (2018) explored the inferential properties of the MW distribution with Type-II hybrid censoring data. Singh and Goel (2019) did MCMC estimation of the parameters of load-share system model assuming failure time distribution of each component as the MW distribution.

The MW distribution was derived from the cumulative hazard rate function which is a product of Weibull cumulative hazard βx^{ν} and exponentiated function $e^{\lambda x}$. The probability density function (PDF) of the MW distribution is given by

$$f(x) = \beta x^{\nu-1}(\nu + \lambda x)e^{\lambda x}e^{-\beta x^{\nu}e^{\lambda x}}; \quad x > 0, (\nu, \beta) > 0, \lambda \geq 0, \tag{1}$$

where β and ν are the scale and shape parameters respectively while λ is an acceleration parameter.

Though, the MW distribution is frequently used for modeling purposes in survival analysis, we found the real life situations where this distribution didn't provide the comfortable fit (Carrasco *et al.*, 2008; Almalki and Yuan, 2013). In view of this, we introduce the extended form of the MW distribution by incorporating an additional shape parameter $\theta > 0$. We multiply the Weibull cumulative hazard rate function βx^{ν} by $e^{\lambda x^{\theta}}$ instead of $e^{\lambda x}$ and obtain the corresponding PDF of this extended modified Weibull (EMW) distribution as

$$f(x) = \beta x^{\nu-1}(\nu + \lambda\theta\, x^{\theta})e^{\lambda x^{\theta}}e^{-\beta x^{\nu}e^{\lambda x^{\theta}}}; \quad x > 0. \tag{2}$$

The survival and hazard rate functions are given in Equations (3) and (4), respectively,

$$S(x) = 1 - F(x) = e^{-\beta x^{\nu}e^{\lambda x^{\theta}}}, \tag{3}$$

$$h(x) = \beta x^{\nu-1}(\nu + \lambda\theta x^{\theta})e^{\lambda x^{\theta}}. \tag{4}$$

The purpose of introducing an additional shape parameter θ in EMW distribution is to make its density and hazard rate functions more flexible and hence enrich its modeling ability. The significance of this new shape parameter has been proven in the real data analysis. Another important characteristic of the proposed distribution is that it contains various sub-models such as Weibull, modified Weibull by Lai *et al.* (2003), extreme value by Bain (1974) and Chen (2000), Rayleigh and exponential distributions as the special cases. The EMW distribution is not only useful for modeling bathtub-shaped lifetime data, but is also suitable for testing the goodness-of-fit of sub-model of the MW distribution.

The rest of the paper is organized as follows. In Section 2, some statistical properties of EMW distribution including shapes of the density and hazard rate functions, sub-models, moments and distribution of order statistic are presented. Section 3 is devoted to the derivation of maximum likelihood estimates (MLEs) and confidence intervals of the model parameters. In Section 4.1, we derive Bayes estimators of the parameters using Tierney and Kadane (1986) approximation method. Further in Section 4.2, we also obtain Bayes estimators and highest posterior density (HPD) intervals of the parameters using Markov chain Monte Carlo (MCMC) techniques such as the Gibbs and Metropolis-Hastings (M-H) algorithm (Metropolis and Ulam, 1949; Hastings, 1970). In Section 5, we illustrate the real-life applications of the proposed EMW distribution to two real datasets. In Section 6, we have carried out a simulation study to highlight the theoretical developments and to compare the performance of various estimators obtained. Finally, some concluding remarks are addressed in Section 7.

2. Properties of the EMW distribution

2.1. *Shape of the density function*

The possible shapes of the density function of EMW distribution for some selected values of the parameters β, ν, λ and θ are depicted in Figure 1. The PDF of the EMW distribution is quite flexible as it may have different monotonic and non-monotonic shapes, for examples decreasing, increasing-decreasing and decreasing then increasing-decreasing shapes.

2.2. *Shape of the hazard rate function*

The shape of the hazard function of the EMW distribution depends on the shape parameters ν and θ as follows:
Case 1: For $\theta = 1$ and $\nu \geq 1$, Equation (4), shows that

- $h(x) = 0$ for $\nu > 1$ and $h(0) = \beta$ if $\nu = 1$.
- $h^{'}(x) > 0$ for $x > 0$, hence $h(x)$ is monotonically increasing in x, implying a monotone increasing hazard rate function.

Case 2: For $\theta = 1$ and $0 < \nu < 1$, we have

- $h(x) \to \infty$ as $x \to 0$ and $h(x) \to \infty$ as $x \to \infty$.
- $h(x)$ initially decreases and then increases, implies a bathtub shape.

Case 3: For λ=0,

- $h(x)$ is an increasing function of x if $\nu > 1$, it is a decreasing function when $\nu < 1$, and for $\nu = 1$, $h(x) = \beta$ i.e. constant.

Case 4: For $\nu = 0$ and $0 < \theta < 1$, one gets,

- $h(x) \to \infty$ as $x \to 0$ and $h(x) \to \infty$ as $x \to \infty$.

- $h(x)$ initially decreases and then increases, implies a bathtub shape.

Case 5: For $\nu = 0$ and $\theta \geq 0$, we have,

- $h^{'}(x) > 0$ for $x > 0$, hence $h(x)$ is an increasing function of x, implying an increasing hazard rate function.

Thus, in general, the shape of the hazard rate function of EMW distribution can be described as increasing, decreasing, constant and bathtub-shaped depending on different choices of the parameters β, ν, λ and θ as shown in Figure 2(a) and 2(b). The interesting point here is that the bathtub-shape of $h(x)$ can be attained either by setting $[0 < \nu < 1, (\theta, \lambda, \beta) > 0]$ or by setting $[0 < \theta < 1, \nu = 0, (\lambda, \beta) > 0]$.

Although, for two combination of parameters of EMW distribution, we can attain a bathtub-shaped hazard rate, but a continuous distribution becomes more applicable if it's hazard rate function has long useful period. This period plays a important role in various engineering and other related fields. For example, in setting warranty policies of different manufactured products, to decide when to retire a mechanical or other equipment, in preventive maintenance etc,. Thus, here, we have also derived the useful period for the hazard rate function of EMW distribution for some specific values of its parameters.

In Figure 3, we have portrayed the bathtub-shaped hazard rate function of EMW (0.097, 0.785, 0.012, 1.22) distribution to show the long useful period. From this figure, we can easily notice that initially failure rate decreases steeply, after that its curvature starts changing, and becomes approximately constant; and later the curvature starts changing again, and the failure rate function ultimately starts increasing. From these observations, we can define the *useful period* as an interval between the two points where the curvature change most rapidly. Thus, following Bebbington *et al.* (2006), we have observed that the curvature achieves its two local maxima of 13.1711 at t=0.0283 and 0.0002 at t =14.1597. Therefore, the conservative useful period (shown by the solid blue lines in Figure 3) is of length 14.1296. On the contrary, the derivative of the curvature achieves its left most local maxima of 730.0811 at t=0.0094, and its right most local minima of -7.26e-06 at t = 18.8913. So, the useful period (depicted by red dotted lines in Figure 3) is of length 18.8819, which clearly shows that the conservative useful period is the subset of useful period. Additionally, we can say that the hazard rate function of EMW distribution has comfortable bathtub shape as it's conservative useful period is non-empty.

2.3. Sub-models

For the particular values of the different parameters, the PDF in Equation (2), represents a family of distributions as it covers the following lifetime distributions as special cases:

1. $h(x) = \beta x^{\nu-1}(\nu + \lambda x)e^{\lambda x}$; $\theta = 1$, we have the modified Weibull distribution.
2. $h(x) = \beta\theta x^{\theta-1}e^{x^{\theta}}; \nu = 0, \lambda = 1$, we get the Chen distribution.

3. $h(x) = \beta\lambda e^{\lambda x}; \nu = 0, \theta = 1$, it reduces to extreme value distribution.
4. $h(x) = \nu\beta x^{\nu-1}; \lambda = 0$, it turns out to be Weibull distribution.
5. $h(x) = 2\beta x; \lambda = 0, \nu = 2$, we get Rayleigh distribution.
6. $h(x) = \beta; \lambda = 0, \nu = 1$, we get one parameter exponential distribution.

2.4. *Moments*

The k^{th} moment of the EMW distribution is

$$\begin{aligned} \mu_k^{'} &= E[X^k] = \int_0^\infty x^k f(x) dx = k \int_0^\infty x^{k-1} S(x) dx \\ &= k \int_0^\infty x^{k-1} e^{-\beta x^\nu e^{\lambda x^\theta}} dx. \end{aligned} \tag{5}$$

From Equation (5), the closed form expression for $\mu_k^{'}$ is not available and has to be evaluated numerically. However, we can show that $\mu_k^{'}$ is finite. For $x \geq 0$, $S(x) \leq e^{-\beta x^\nu}$, so we have

$$\begin{aligned} \mu_k^{'} < k \int_0^\infty x^{k-1} e^{-\beta x^\nu} &= \frac{k}{\beta^{k/\nu}} \int_0^\infty x^{(k/\nu)-1} e^{-x} dx \\ &= \frac{k}{\beta^{k/\nu}} \Gamma(k/\nu). \end{aligned}$$

Thus, $\mu_k^{'}$ is a finite non-negative value. Denoting $\mu = E(X) = \mu_1^{'}$ as the mean, we have the k^{th} central moment as

$$\mu_k = E(X-\mu)^k = \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} \mu_k^{'} \mu^{k-i},$$

and $\mu_k < \mu_k^{'}$ is also finite.

2.5. *Distribution of order statistics*

Let $X_1, X_2, ..., X_n$ be a random sample drawn from EMW$(\beta, \nu, \lambda, \theta)$ distribution. Since $X_1, X_2, ..., X_n$ are independently and identically distributed (iid) continuous random variables, then $P(X_i = X_j, i \neq j = 1, 2, .., n)$=0, therefore, there exists an unique ordered arrangement of the sample observations according to magnitude. Let $X_{(1:n)} <$

$X_{(2:n)} < ... < X_{(n:n)}$ be the order statistics then, the PDF of $X_{(r:n)}$, $1 \leq r \leq n$, denoted by $f_{r:n}(x)$ is given by

$$f_{r:n}(x) = \frac{1}{B(r, n-r+1)} (1 - S(x))^{r-1} S(x)^{n-r} f(x), \tag{6}$$

where, $f(x)$ and $S(x)$ are the PDF and the survival function of EMW distribution given by Equations (2) and (3), respectively. The PDF $f_{r:n}(x)$ can also be written as

$$f_{r:n}(x) = \frac{1}{B(r, n-r+1)} \sum_{m=0}^{r-1} \binom{r-1}{m} (-1)^m S(x)^{n-r+m} f(x), \quad x > 0. \tag{7}$$

So the i^{th} raw moment of $X_{r:n}$ is given by

$$E(X_{r:n}^i) = \frac{1}{B(r, n-r+1)} \sum_{m=0}^{r-1} \binom{r-1}{m} (-1)^m E[Z^i S(Z)^{n-r+m}], \tag{8}$$

where, $Z \sim$ EMW distribution with PDF given in Equation (2).

3. Maximum likelihood estimation

Among the estimation methods available in the classical statistical inference, the usually preferred method is maximum likelihood estimation due to its better asymptotic properties. Let $\underline{X} = (X_1, X_2, ..., X_n)$ be a random sample from EMW distribution with PDF in Equation (2) and $\gamma = (\beta, \nu, \lambda, \theta)$ be a parameter vector, then the corresponding likelihood function is given by

$$L = L(\underline{x} \mid \beta, \nu, \lambda, \theta) = \beta^n \prod_{i=1}^{n} [x_i^{\nu-1}(\nu + \lambda\theta x_i^{\theta})] e^{\sum_{i=1}^{n} \lambda x_i^{\theta}} e^{-\sum_{i=1}^{n} \beta x_i^{\nu} e^{\lambda x_i^{\theta}}}. \tag{9}$$

The log likelihood function is

$$l = l(\underline{x}|\beta, \nu, \lambda, \theta) = n \log \beta + (\nu - 1) \sum_{i=1}^{n} \log x_i + \sum_{i=1}^{n} \log(\nu + \lambda\theta x_i^{\theta}) + \sum_{i=1}^{n} \lambda x_i^{\theta} - \sum_{i=1}^{n} \beta x_i^{\nu} e^{\lambda x_i^{\theta}}. \tag{10}$$

The MLEs of the parameters β, ν, λ and θ can be obtained by differentiating Equation (10) with respect to β, ν, λ and θ and equating the first-order partial derivatives to zero. Thus, we have

$$\frac{\partial l}{\partial \beta} = \frac{n}{\beta} - \sum_{i=1}^{n} x_i^{\nu} e^{\lambda x_i^{\theta}} = 0, \tag{11}$$

$$\frac{\partial l}{\partial \upsilon} = \sum_{i=1}^{n} \log x_i + \sum_{i=1}^{n} \frac{1}{(\upsilon + \lambda\theta x_i^{\theta})} - \beta \sum_{i=1}^{n} x_i^{\upsilon} \log x_i e^{\lambda x_i^{\theta}} = 0, \tag{12}$$

$$\frac{\partial l}{\partial \lambda} = \sum_{i=1}^{n} \frac{\theta x_i^{\theta}}{(\nu + \lambda\theta x_i^{\theta})} + \sum_{i=1}^{n} x_i^{\theta} - \beta \sum_{i=1}^{n} x_i^{\nu+\theta} e^{\lambda x_i^{\theta}} = 0, \tag{13}$$

$$\frac{\partial l}{\partial \theta} = \sum_{i=1}^{n} \frac{\lambda(x_i^{\theta} + \theta x_i^{\theta} \log x_i)}{(\nu + \lambda\theta x_i^{\theta})} + \lambda \sum_{i=1}^{n} x_i^{\theta} \log x_i - \lambda\beta \sum_{i=1}^{n} x_i^{\nu+\theta} \exp(\lambda x_i^{\theta}) \log x_i = 0. \tag{14}$$

From Equation (11), we get $\beta = n\left(\sum_{i=1}^{n} x_i^{\nu} e^{\lambda x_i^{\theta}}\right)^{-1}$. Now, after putting the expression of β in Equations (12), (13) and (14), one can solve these non-linear equations by any suitable iterative method to get MLEs for ν, λ and θ. Once we obtain the MLEs $(\hat{\nu}, \hat{\lambda}, \hat{\theta})$ of (ν, λ, θ), the MLE of β can be obtained as $\hat{\beta} = n\left(\sum_{i=1}^{n} x_i^{\hat{\nu}} e^{\hat{\lambda} x_i^{\hat{\theta}}}\right)^{-1}$. Alternatively, the maximization of Equation (9) can also be done directly by using well-established routines like maxLik() or mle() available in the statistical package R. Further, under certain regularity conditions, the estimator $\hat{\gamma} = (\hat{\beta}, \hat{\nu}, \hat{\lambda}, \hat{\theta})$ will follow multivariate normal distribution with mean vector γ and variance-covariance matrix $\text{cov}(\hat{\gamma}) = J(\hat{\gamma})^{-1}$, where $J(\hat{\gamma})$ is 4×4 observed information matrix evaluated at $\hat{\gamma}$. The second order partial derivatives of log-likelihood function are given as follows

$$\frac{\partial^2 l}{\partial \beta^2} = -\frac{n}{\beta^2}, \; \frac{\partial^2 l}{\partial \beta \partial \nu} = -\sum_{i=1}^{n} x^{\nu} \log x_i . e^{\lambda x_i^{\theta}}, \; \frac{\partial^2 l}{\partial \beta \partial \lambda} = -\sum_{i=1}^{n} x_i^{(\nu+\theta)} e^{\lambda x_i^{\theta}},$$

$$\frac{\partial^2 l}{\partial \beta \partial \theta} = -\lambda \sum_{i=1}^{n} x_i^{(\nu+\theta)} e^{\lambda x_i^{\theta}} \log x_i, \; \frac{\partial^2 l}{\partial \nu^2} = -\sum_{i=1}^{n} \frac{1}{(\nu + \lambda\theta x_i^{\theta})^2} - \beta \sum_{i=1}^{n} x_i^{\nu} (\log x_i)^2 e^{\lambda x_i^{\theta}},$$

$$\frac{\partial^2 l}{\partial \nu \partial \lambda} = -\sum_{i=1}^{n} \frac{\theta x_i^{\theta}}{(\nu + \lambda\theta x_i^{\theta})^2} - \beta \sum_{i=1}^{n} x_i^{(\nu+\theta)} \log x_i . e^{\lambda x_i^{\theta}},$$

$$\frac{\partial^2 l}{\partial \nu \partial \theta} = -\sum_{i=1}^{n} \frac{\lambda x_i^{\theta}(1 + \theta \log x_i)}{(\nu + \lambda\theta x_i^{\theta})^2} - \beta\lambda \sum_{i=1}^{n} x_i^{(\nu+\theta)} (\log x_i)^2 e^{\lambda x_i^{\theta}},$$

$$\frac{\partial^2 l}{\partial \lambda^2} = -\sum_{i=1}^{n} \frac{\theta^2 x_i^{2\theta}}{(\nu + \lambda\theta x_i^{\theta})^2} - \beta \sum_{i=1}^{n} x_i^{(\nu+2\theta)} e^{\lambda x_i^{\theta}},$$

$$\frac{\partial^2 l}{\partial \lambda \partial \theta} = -\sum_{i=1}^{n} \frac{\nu x_i^{\theta}(\theta \log x_i + 1)}{(\nu + \lambda\theta x_i^{\theta})^2} - \beta \sum_{i=1}^{n} x_i^{(\nu+\theta)} \log x_i e^{\lambda x_i^{\theta}} (1 + \lambda x_i^{\theta}),$$

$$\frac{\partial^2 l}{\partial \theta^2} = \sum_{i=1}^{n} \frac{\lambda x_i^{\theta} \log x_i (2+\theta \log x_i)(\nu + \lambda \theta x_i^{\theta}) - \lambda^2 x_i^{2\theta}(1+\theta \log x_i)^2}{(\nu + \lambda \theta x_i^{\theta})^2}$$
$$+\lambda \sum_{i=1}^{n} x_i^{\theta} (\log x_i)^2 - \lambda \beta \sum_{i=1}^{n} x_i^{(\nu+\theta)} (\log x_i)^2 exp(\lambda x_i^{\theta})(1+\lambda x_i^{\theta}).$$

An asymptotic confidence interval (ACI) with confidence coefficient 1-α for the parameters γ_j; j=1, 2, 3, 4, is obtained by $\hat{\gamma} \pm z_{\alpha/2} \cdot \sqrt{\hat{J}_{jj}}$, where $\hat{J}_{jj}$ denotes the $(j, j)^{\text{th}}$ diagonal elements of $J(\hat{\gamma})^{-1}$ and $z_{\alpha/2}$ is the upper $100\times(\alpha/2)^{\text{th}}$ percentile of the standard normal distribution. The asymptotic normality is also useful for testing goodness-of-fit of the proposed distribution and for comparing this distribution with some of special submodels using the well-known asymptotically equivalent test statistics such as likelihood ratio (L-R) statistic.

4. Bayesian estimation

In this section, we derive Bayes estimators of the parameters of proposed EMW distribution under the squared error loss function (SELF). We assume the following independent Gamma priors for the model parameters β,ν, λ and θ.

$$\omega_1(\beta) \propto \beta^{\eta_1 - 1} e^{-a_1 \beta}; \ \ \beta > 0, (a_1, \eta_1) > 0, \tag{15}$$
$$\omega_2(\nu) \propto \nu^{\eta_2 - 1} e^{-a_2 \nu}; \ \ \nu > 0, (a_2, \eta_2) > 0, \tag{16}$$
$$\omega_3(\lambda) \propto \lambda^{\eta_3 - 1} e^{-a_3 \lambda}; \ \ \lambda > 0, (a_3, \eta_3) > 0, \tag{17}$$
$$\omega_4(\theta) \propto \theta^{\eta_4 - 1} e^{-a_4 \theta}; \ \ \theta > 0, (a_4, \eta_4) > 0. \tag{18}$$

The joint posterior distribution of β, ν, λ and θ given the data can be obtained by multiplying the likelihood in Equation (9) with the priors given in Equations (15)-(18), and is expressed as:

$$\pi(\beta, \nu, \lambda, \theta | \underline{x}) = \frac{h(\beta, \nu, \lambda, \theta | \underline{x})}{\int\int\int\int h(\beta, \nu, \lambda, \theta | \underline{x}) d\beta d\nu d\lambda d\theta}, \tag{19}$$

where $h(\beta, \nu, \lambda, \theta | \underline{x}) = L(\underline{x} | \beta, \nu, \lambda, \theta) \omega_1(\beta) \omega_2(\nu) \omega_3(\lambda) \omega_4(\theta)$.
Thus, Bayes estimator of any function of parameters, say, $W(\beta, \nu, \lambda, \theta)$ under SELF is given by

$$\bar{W}_{BS}(\beta, \nu, \lambda, \theta) = \int\int\int\int W(\beta, \nu, \lambda, \theta) \pi(\beta, \nu, \lambda, \theta | \underline{x}) d\beta d\nu d\lambda d\theta. \tag{20}$$

From Equation (20), it is clear that Bayes estimator for the parameters β, ν, λ and θ cannot be obtained analytically. So we obtain the approximate Bayes estimators using the Tierney and Kadane (1986) method and posterior sample estimators using the Gibbs sampler proposed by Geman and Geman (1984).

4.1. Bayes estimators using Tierney and Kadane's approximation method

Tierney and Kadane (1986) proposed approximation method to evaluate the approximated value of the ratio of multidimensional integrals given in Equation (20).

$$
\begin{aligned}
E[W(\beta,\nu,\lambda,\theta)|\underline{x}] &= \int\int\int\int W(\beta,\nu,\lambda,\theta)\pi(\beta,\nu,\lambda,\theta)d\beta d\nu d\lambda d\theta \\
&= \frac{\int\int\int\int e^{nL^*(\beta,\upsilon,\lambda,\theta)}d\beta d\upsilon d\lambda d\theta}{\int\int\int\int e^{nL(\beta,\upsilon,\lambda,\theta)}d\beta d\upsilon d\lambda d\theta}, \qquad (21)
\end{aligned}
$$

where $L(\beta,\nu,\lambda,\theta)=\frac{1}{n}(l(\underline{x}|\beta,\nu,\lambda,\theta)+\log(\omega(\beta))+\log(\omega(\nu))+\log(\omega(\lambda))+\log(\omega(\theta))$ and $L^*(\beta,\nu,\lambda,\theta)=L(\beta,\nu,\lambda,\theta)+\frac{1}{n}\log W(\beta,\nu,\lambda,\theta)$.
Using Tierney-Kadane (T-K) approximation, the posterior expectation of $W(\beta,\nu,\lambda,\theta)$ can be estimated by

$$
\hat{E}[W(\beta,\nu,\lambda,\theta)]=\left(\frac{|\sum^*|}{|\sum|}\right)^{1/2}e^{n[L^*(\hat{\beta}^*,\hat{\nu}^*,\hat{\lambda}^*,\hat{\theta}^*)-L(\hat{\beta},\hat{\nu},\hat{\lambda},\hat{\theta})]}, \qquad (22)
$$

where $(\hat{\beta}^*,\hat{\nu}^*,\hat{\lambda}^*,\hat{\theta}^*)$ and $(\hat{\beta},\hat{\nu},\hat{\lambda},\hat{\theta})$ maximize $L^*(\beta,\nu,\lambda,\theta)$ and $L(\beta,\nu,\lambda,\theta)$ respectively and $|\sum^*|$ and $|\sum|$ denote the determinants of the inverse Hessian of $L^*(\beta,\nu,\lambda,\theta)$ and $L(\beta,\nu,\lambda,\theta)$ evaluated at $(\hat{\beta}^*,\hat{\nu}^*,\hat{\lambda}^*,\hat{\theta}^*)$ and $(\hat{\beta},\hat{\nu},\hat{\lambda},\hat{\theta})$ respectively. The posterior variance and $100\times(1-\alpha)\%$ confidence interval for $W(\beta,\nu,\lambda,\theta)$ are $V(\hat{W}(\beta,\nu,\lambda,\theta))=E[W^2(\beta,\nu,\lambda,\theta|\underline{x})]-[E(W(\beta,\nu,\lambda,\theta|\underline{x}))]^2$ and $\hat{W}(\beta,\nu,\lambda,\theta)\pm z_{\alpha/2}\sqrt{V(\hat{W}(\beta,\nu,\lambda,\theta))}$ respectively. Using Equation (22), we can deduce approximate Bayes estimators of β, ν, λ, and θ and under SELF. To compute the Bayes estimator of β we take $W(\beta,\nu,\lambda,\theta)=\beta$, then

- $\hat{\beta}_{TK}=\left(\frac{\left|\sum^*(\hat{\beta}^*,\hat{\nu}^*,\hat{\lambda}^*,\hat{\theta}^*)\right|}{\left|\sum(\hat{\beta},\hat{\nu},\hat{\lambda},\hat{\theta})\right|}\right)^{1/2}e^{n[L^*(\hat{\beta}^*,\hat{\nu}^*,\hat{\lambda}^*,\hat{\theta}^*)-L(\hat{\beta},\hat{\nu},\hat{\lambda},\hat{\theta})]}$,
- Posterior variance $V(\hat{\beta})=E(\beta^2|\underline{x})-[E(\beta|\underline{x})]^2$,
- $100\times(1-\alpha)\%$ posterior confidence interval (PCI)is $\hat{\beta}\pm z_{\alpha/2}\sqrt{V(\hat{\beta})}$,

where $(\hat{\beta}^*,\hat{\nu}^*,\hat{\lambda}^*,\hat{\theta}^*)$ and $(\hat{\beta},\hat{\nu},\hat{\lambda},\hat{\theta})$ maximize $L^*(\hat{\beta}^*,\hat{\nu}^*,\hat{\lambda}^*,\hat{\theta}^*)$ and $L(\hat{\beta},\hat{\nu},\hat{\lambda},\hat{\theta})$ respectively and $\sum^*(\hat{\beta}^*,\hat{\nu}^*,\hat{\lambda}^*,\hat{\theta}^*)$ and $\sum(\hat{\beta},\hat{\nu},\hat{\lambda},\hat{\theta})$ are the inverse Hessian of $L^*(\hat{\beta}^*,\hat{\nu}^*,\hat{\lambda}^*,\hat{\theta}^*)$ and $L(\hat{\beta},\hat{\nu},\hat{\lambda},\hat{\theta})$ evaluated at $(\hat{\beta}^*,\hat{\nu}^*,\hat{\lambda}^*,\hat{\theta}^*)$ and $(\hat{\beta},\hat{\nu},\hat{\lambda},\hat{\theta})$, respectively. Here,

$$
L^*(\beta^*,\nu^*,\lambda^*,\theta^*)=L(\beta,\nu,\lambda,\theta)+\frac{1}{n}\log\beta,
$$

$$L(\beta,\nu,\lambda,\theta)=\frac{1}{n}\left(\begin{array}{l} l(\underline{x}|\beta,\nu,\lambda,\theta)+(\eta_1-1)\log\beta-a_1\beta+(\eta_2-1)\log\nu-a_2\nu+ \\ (\eta_3-1)\log\lambda-a_3\lambda+(\eta_4-1)\log\theta-a_4\theta \end{array}\right),$$

$$\Sigma^*(\hat{\beta}^*,\hat{\nu}^*,\hat{\lambda}^*,\hat{\theta}^*)=-\begin{bmatrix} \frac{\partial^2 L^*(\beta^*,\nu^*,\lambda^*,\theta^*)}{\partial\beta^2} & \frac{\partial^2 L^*(\beta^*,\nu^*,\lambda^*,\theta^*)}{\partial\beta\partial\nu} & \frac{\partial^2 L^*(\beta^*,\nu^*,\lambda^*,\theta^*)}{\partial\beta\partial\lambda} & \frac{\partial^2 L^*(\beta^*,\nu^*,\lambda^*,\theta^*)}{\partial\beta\partial\theta} \\ - & \frac{\partial^2 L^*(\beta^*,\nu^*,\lambda^*,\theta^*)}{\partial\nu^2} & \frac{\partial^2 L^*(\beta^*,\nu^*,\lambda^*,\theta^*)}{\partial\nu\partial\lambda} & \frac{\partial^2 L^*(\beta^*,\nu^*,\lambda^*,\theta^*)}{\partial\nu\partial\theta} \\ - & - & \frac{\partial^2 L^*(\beta^*,\nu^*,\lambda^*,\theta^*)}{\partial\lambda^2} & \frac{\partial^2 L^*(\beta^*,\nu^*,\lambda^*,\theta^*)}{\partial\lambda\partial\theta} \\ - & - & - & \frac{\partial^2 L^*(\beta^*,\nu^*,\lambda^*,\theta^*)}{\partial\theta^2} \end{bmatrix}^{-1}_{\left(\hat{\beta}^*,\hat{\nu}^*,\hat{\lambda}^*,\hat{\theta}^*\right)},$$

and

$$\Sigma\left(\hat{\beta},\hat{\nu},\hat{\lambda},\hat{\theta}\right)=-\begin{bmatrix} \frac{\partial^2 L(\hat{\beta},\hat{\nu},\hat{\lambda},\hat{\theta})}{\partial\beta^2} & \frac{\partial^2 L(\hat{\beta},\hat{\nu},\hat{\lambda},\hat{\theta})}{\partial\beta\partial\nu} & \frac{\partial^2 L(\hat{\beta},\hat{\nu},\hat{\lambda},\hat{\theta})}{\partial\beta\partial\lambda} & \frac{\partial^2 L(\hat{\beta},\hat{\nu},\hat{\lambda},\hat{\theta})}{\partial\beta\partial\theta} \\ - & \frac{\partial^2 L(\hat{\beta},\hat{\nu},\hat{\lambda},\hat{\theta})}{\partial\nu^2} & \frac{\partial^2 L(\hat{\beta},\hat{\nu},\hat{\lambda},\hat{\theta})}{\partial\nu\partial\lambda} & \frac{\partial^2 L(\hat{\beta},\hat{\nu},\hat{\lambda},\hat{\theta})}{\partial\nu\partial\theta} \\ - & - & \frac{\partial^2 L(\hat{\beta},\hat{\nu},\hat{\lambda},\hat{\theta})}{\partial\lambda^2} & \frac{\partial^2 L(\hat{\beta},\hat{\nu},\hat{\lambda},\hat{\theta})}{\partial\lambda\partial\theta} \\ - & - & - & \frac{\partial^2 L(\hat{\beta},\hat{\nu},\hat{\lambda},\hat{\theta})}{\partial^2\theta^2} \end{bmatrix}^{-1}_{\left(\hat{\beta},\hat{\nu},\hat{\lambda},\hat{\theta}\right)}.$$

Similarly, $\hat{\nu_{TK}}$, $\hat{\lambda_{TK}}$ and $\hat{\theta_{TK}}$ can be deduced to obtain Bayes estimator of ν, λ and θ under SELF with their posterior variance and confidence intervals.

Here, it is to be noted that the HPD credible intervals for the model parameters $(\beta,\nu,\lambda,\theta)$ are not possible to construct by using T-K procedure. Therefore, MCMC technique such as the Gibbs sampler is to be utilized to compute Bayes estimators as well as HPD credible intervals for the parameters.

4.2. *Bayes estimators using the Gibbs sampler*

In order to obtain Bayes estimators using the Gibbs sampling procedure, we need full conditional distribution of each of the parameter. These full conditional distributions can be easily derived by picking those terms from the joint distribution of the parameters and data in Equation (19), which involve the corresponding parameter, and are given as:

$$\pi_1(\beta|\underline{x},\nu,\lambda,\theta)\propto\beta^{n+\eta_1-1}\exp\left(-\beta\left(a_1+\sum_{i=1}^{n}x_i^{\nu}\exp\left(\lambda\sum_{i=1}^{n}x_i^{\theta}\right)\right)\right),\tag{23}$$

$$\pi_2(\nu|\underline{x},\beta,\lambda,\theta)\propto\nu^{\eta_2-1}\exp\left(-a_2\nu-\beta\sum_{i=1}^{n}x_i^{\nu}\exp\left(\lambda\sum_{i=1}^{n}x_i^{\theta}\right)\right)\times\prod_{i=1}^{n}\left[x_i^{\nu-1}\left(\nu+\lambda\theta x_i^{\theta}\right)\right],\tag{24}$$

$$\pi_3(\lambda|\underline{x},\beta,\nu,\theta) \propto \lambda^{\eta_3-1}\left(\exp\left(-\lambda\left(a_3+\sum_{i=1}^{n}x_i^{\theta}\right)\right)-\beta\sum_{i=1}^{n}x_i^{\nu}\exp\left(\lambda\sum_{i=1}^{n}x_i^{\theta}\right)\right) \times\prod_{i=1}^{n}\left(\nu+\lambda\theta x_i^{\theta}\right), \tag{25}$$

$$\pi_4(\theta|\underline{x},\beta,\nu,\lambda) \propto \theta^{\eta_4-1}\exp\left(-\theta a_4+\lambda\sum_{i=1}^{n}x_i^{\theta}-\beta\sum_{i=1}^{n}x_i^{\nu}\exp\left(\lambda\sum_{i=1}^{n}x_i^{\theta}\right)\right) \times\prod_{i=1}^{n}\left(\nu+\lambda\theta x_i^{\theta}\right). \tag{26}$$

Now, we proceed as follows:

1. Initialize $(\nu_0,\lambda_0,\theta_0)$ as starting values of (ν,λ,θ).
2. Simulate β from $\pi_1(\beta|\underline{x},\nu,\lambda,\theta)$, a well-known Gamma distribution with shape parameter $n+\eta_1$ and scale parameter $a_1+\sum_{i=1}^{n}x_i^{\nu}\exp(\lambda\sum_{i=1}^{n}x_i^{\theta})$.
3. Simulate ν, λ, and θ from the full conditional densities given in Equations (24), (25) and (26) respectively, using well known M-H algorithm.
4. Repeat Steps 2-3, M times and record the sequence of $\gamma=(\beta,\nu,\lambda,\theta)$ after discarding the burn-in-sampler of size, say, N from the sample so that the effect of the initial values is nullified. The simulation study carried out in the next section indicates that the proposed sampling algorithm is quite efficient in terms of mixing and convergence of the generated chains under different starting values.
5. Now, Bayes estimators and posterior variances of β, ν, λ and θ under SELF can be obtained as the means and variances of the generated value of β, ν, λ and θ.
6. Let $\gamma_{(N+1)},\gamma_{(N+2)},...,\gamma_{(M)}$ be the ordered values of $\gamma_{N+1},\gamma_{N+2},...,\gamma_M$. Then, using the method proposed by Chen and Shao (1999), the $(1-\alpha)\times 100\%$ HPD interval for γ is $(\gamma_{N+i^*},\gamma_{N+i^*+[(1-\alpha)(M-N)]})$, where i^* is chosen so that $\gamma_{N+i^*+[(1-\alpha)(M-N)]}-\gamma_{N+i^*}=\min_{N\leq i\leq(M-N)-[(1-\gamma)(M-N)]}(\Theta_{N+i+[(1-\gamma)(M-N)]}-\Theta_{N+i})$.

5. Goodness of Fit

Here, we illustrate the applications of EMW to two well-known datasets. Dataset 1 possesses the increasing hazard rate while dataset 2 exhibits the bathtub-shaped hazard function. Firstly, we demonstrate the goodness of fit of EMW distribution in comparison to the MW distribution. For this, we compute the maximum values of the unrestricted and restricted log-likelihoods to obtain the likelihood ratio (LR) statistic for testing whether the fit using the EMW distribution is statistically superior to a fit using the MW distribution for a given dataset. For this we consider the null hypothesis $H_0 : \theta = 1$ (MW distribution) against $H_1 : \theta \neq 1$ (EMW distribution). Under H_0, the deviance test statistic $D_n = -2(\log H_0 - \log H_1)$ approximately follows χ^2 distribution with k degrees of freedom. Here, k is the difference in the number of the parameters between the two models.

Secondly, we compare the fitting of EMW distribution with other considered models namely, the modified Weibull distribution, the additive Weibull (ADDW), the modified Weibull extension (MWE), the modified Weibull distribution of Sarhan and Zaindin (SZMW), the generalized modified Weibull (GMW) and the Chen distribution. The survival functions of these models are listed in Table 1. The Akaike information criterion (AIC), the Bayesian information criterion (BIC) and the Kolmogorov-Smirnov (K-S) statistic with the corresponding p-value are used to compare the fit of the considered distributions.

Dataset 1: This dataset is taken from Cook and Weisberg (2009). It contains 13 variables on 102 males and 100 females athletes collected at the Australian Institute of sports. Jamalizadeh *et al.* (2011) used the heights for the 100 female athletes and hemoglobin concentration levels for the 202 athletes to illustrate the application of generalized two-piece skew-normal distribution. Almheidat *et al.* (2015) considered the percentage of body fat for the 202 athletes. Here, we consider the weights of all male and female athletes. The value of deviance statistic D_n for the variable weights is observed to be 25.46 with p-value $4.52 \times 10^{-7} (< 0.01)$. Hence, it clearly indicates that the null hypothesis $H_0 : \theta = 1$ (MW distribution) is rejected at 1% level of significance.

Table 2 displays the MLEs with corresponding standard errors (SEs) of the model's parameters along with log-likelihood value, AIC, BIC, K–S statistic and corresponding p-value. Since the values of AIC, BIC and K–S statistic of the EMW model are smallest among those of the six fitted models, therefore our new model can be chosen as the best model. The density and survival plots of the considered models fitted to this dataset is displayed in Figure 4(a) and 4(b). From these plots, it can be seen that the estimated density and the survival functions of EMW model are closely following the patterns of the histogram and the empirical survival function of this dataset respectively. The observation has also been confirmed by plotting the estimated hazard rate function of the model as shown in Figure 4(c).

Dataset 2: This dataset consists of the time to failure of 50 devices by Aarset (1987). It is a widely used dataset in survival analysis due to its bathtub-shaped failure rate property. The value of the deviance statistic D_n for this dataset comes out to be 15.31 with p-value $9.13 \times 10^{-05} (< 0.01)$. Hence, we reject the null hypothesis $H_0 : \theta = 1$ (MW distribution) even at 1% level of significance.

The MLEs with corresponding SEs of the models' parameters along with the log-likelihood value, AIC, BIC and K-S statistic with corresponding p-value are given in Table 3. Again, EMW model turns out to be the best-fitted model among the considered distributions as it has the lowest AIC and BIC values. However, ADDW distribution has a slightly lower value of K–S statistic. When the interpretation is given in terms of the log likelihood, we found the maximum value of the log likelihood for the EMW distribution which reflects that the EMW model fits well the data among the others.

Additionally, the corresponding survival function for the six fitted distributions and the empirical survival function are plotted in Figure 5(a). It can be seen that the EMW distribution is a very competitive model. Further, the plots of the estimated densities and the histogram of this data given in Figure 5(b) reveals that the EMW distribution produces a better fit than the other five models. Finally, the plot of the estimated hazard rate function superimposed on the non-parametric counterpart given in Figure 5(c) confirm that EMW distribution is an adequate model for describing the pattern of the bathtub-shaped hazard rate of this dataset. Since, the present dataset is of bathtub-shaped hazard rate type, so for EMW distribution, we have also obtained the conservative useful period as (0.0882, 50.2587) and useful period as (0.0457, 65.2065).

6. Simulation study

This section is dedicated to examining the performance of the proposed estimators by using the Monte Carlo simulation techniques. By assuming the model parameters $\beta = 0.5$, $\nu = 0.5$, $\lambda = 1.5$ and $\theta = 2$, we simulated the sample of sizes 30, 50 and 100 from the proposed four-parameter EMW distribution. R software is used to perform all the required numerical computations. The MLEs along with their standard errors (SEs) and confidence intervals (CIs) of the parameters are obtained.

In Bayesian estimation, Bayes estimators of the model parameters are appraised by using SELF. Here we first obtain Bayes estimate (BE) assuming Gamma priors for all the parameters and then equate hyper parameters to zero ($a_1 = \eta_1 = a_2 = \eta_2 = a_3 = \eta_3 = a_4 = \eta_4 = 0$) to obtain BEs under non-informative priors. Tierney-Kadane's approximation method has also been used to compute BEs. Since it is not possible to compute HPD intervals in case of T-K method thus we propose the M-H algorithm within the Gibbs sampler to compute BEs along with posterior standard errors (PSEs) and HPD intervals as discussed in Section 4.2. Using the Gibbs sampler we generate 100000 realizations of the Markov chains of β, ν, λ and θ from the respective full conditional posterior distribution. During simulation it was observed that there exist some autocorrelation among the draws of the parameters, therefore equally spaced (every 10^{th}) generated observations are stored. These gaps were taken to reduce the autocorrelation as much as possible. All the statistics related to stationarity of the target distribution is obtained by employing the functions `geweke.diag`, `heidel.diag` and `raftery.diag` of the `coda package` in the `R` software. The resulting MCMC runs, posterior distributions and autocorrelation plots of the model parameters are shown in the Figure 6. All the parameter estimates along with their estimated errors and confidence/HPD intervals are listed in Table 4 and 5. From the computed results, the following trends are observed:

- In comparison to MLEs, BEs perform better in respect of the estimation error.

- BEs with Gamma priors are better in comparison to BEs with non- informative in terms of precision.
- The HPD intervals are comparatively shorter in length than asymptotic confidence intervals.
- The standard errors of T-K estimates are less than those of BEs with the Gibbs sampler.
- The standard errors of MLEs, as well as the posterior errors in Bayes estimation, tend to decrease in each case as we increase the sample size.

7. Concluding remarks

In this study, an extended version of modified Weibull distribution with one additional shape parameter is introduced in order to provide more flexibility to its density and the hazard rate functions. This extension is not only convenient for modeling bathtub-shaped failure rate but it is also suitable to accommodate the decreasing, increasing and constant behavior of the hazard rate function. The proposed model consists of six sub-models that are extensively used in the reliability and the survival analysis such as the modified Weibull, the Chen, extreme value, the Weibull, the Rayleigh, and the exponential distributions. The statistical properties of this model which includes the hazard rate function, moments, and distribution of the order statistics have been explored. The maximum likelihood and Bayesian method of estimation are used to compute estimates of the model parameters. In Bayesian estimation, we assume informative as well as non-informative priors for the parameters and MCMC techniques have been implemented to obtain posterior sample-based estimators and HPD intervals of the parameters. Tierney and Kadane's approximation procedure has also been used to obtain Bayes estimators of the parameters. A simulation study is carried out to assess the performances of the various estimates. Two real datasets analysis is given to illustrate the significance of the model.

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Appendix

A. Figures

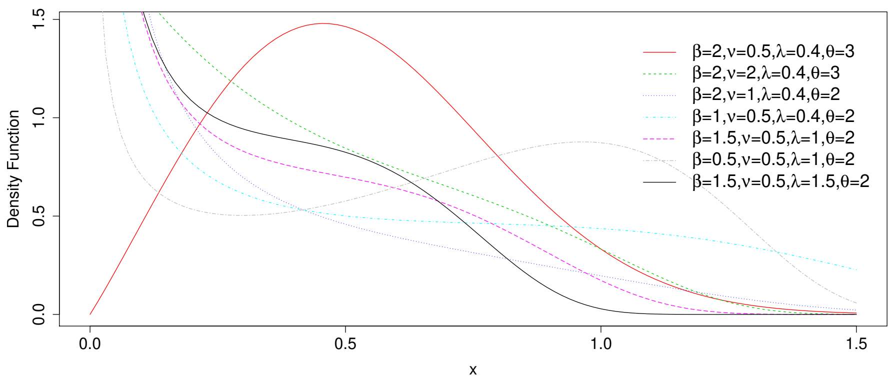
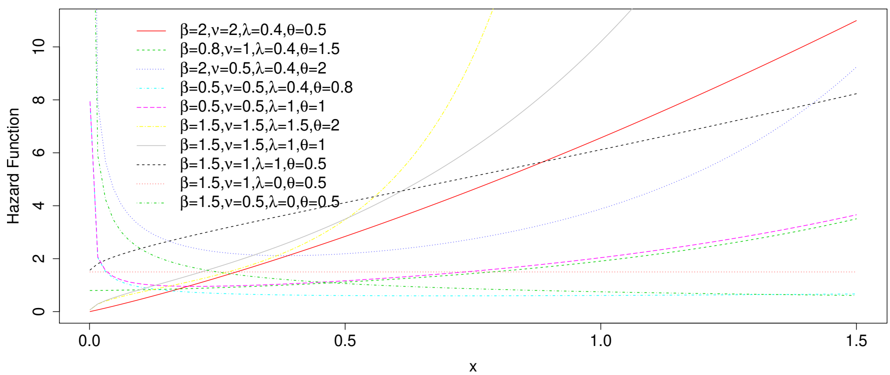
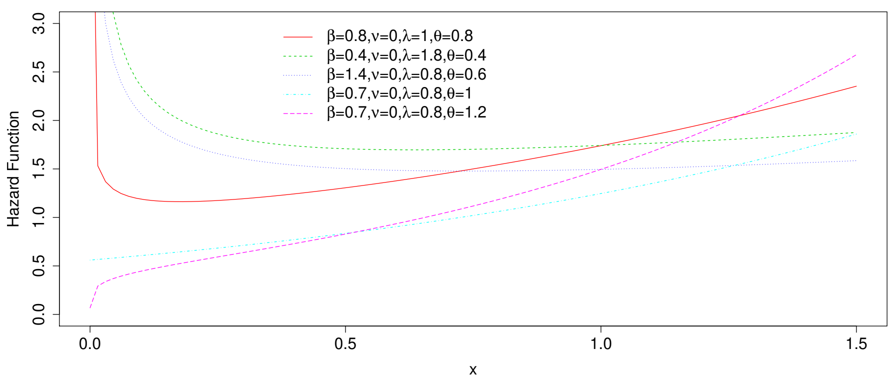


Figure 1 – The plots of density function of the EMW distribution for different choices of the parameters.



(a) Hazard plots when $\nu > 0$.



(b) Hazard plots when $\nu = 0$.

Figure 2 – The plots of hazard rate function of EMW distribution.

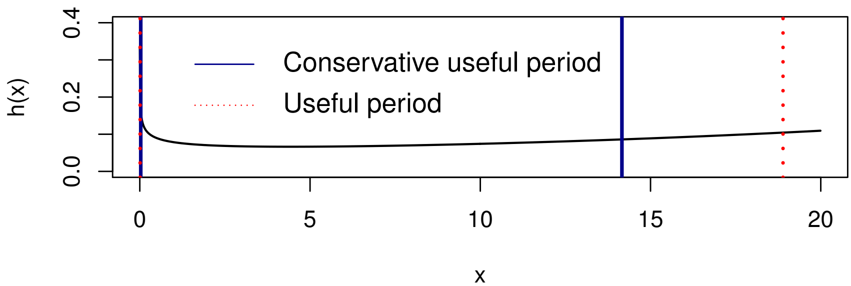
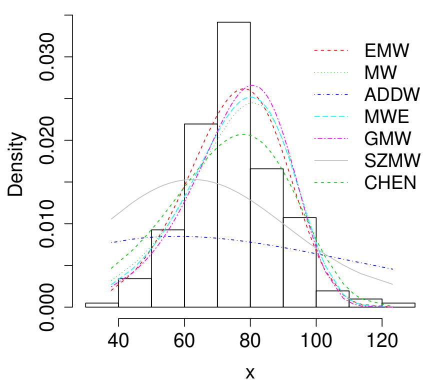
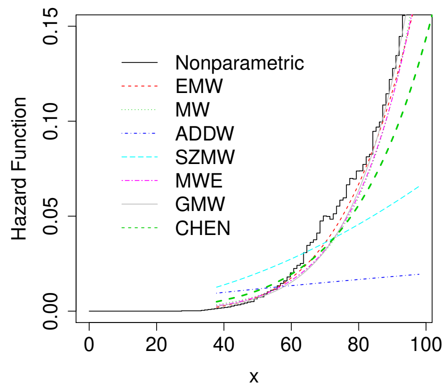


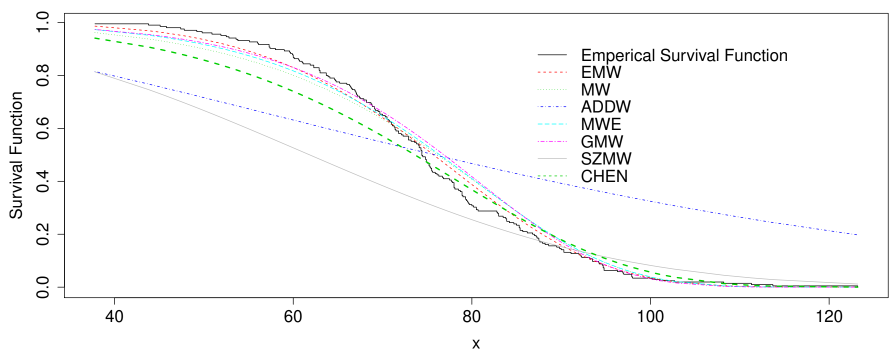
Figure 3 – The useful periods of EMW distribution hazard rate function with $\beta = 0.097, \nu = 0.785, \lambda = 0.012, \theta = 1.22$.



(a) The plots of density function.

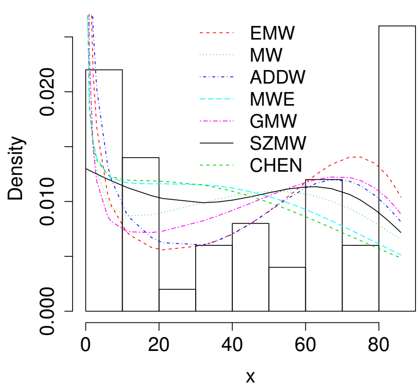


(b) The plots of hazard rate function.

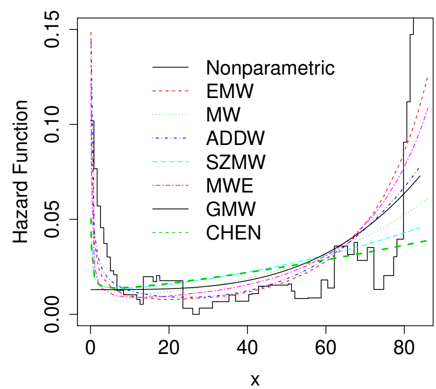


(c) The plots of survival function.

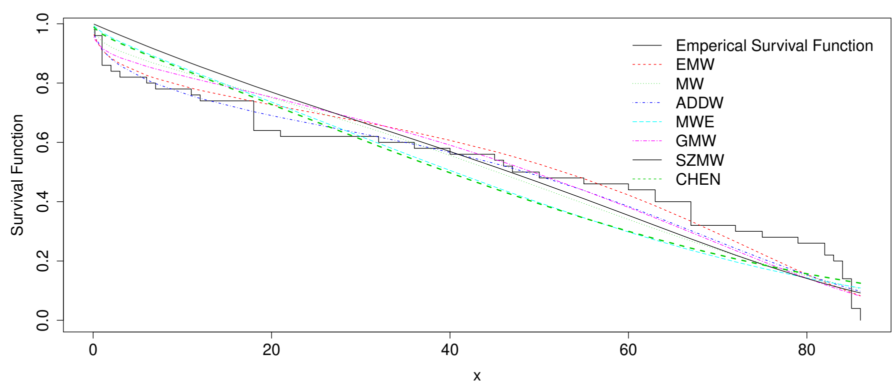
Figure 4 – Model fitting plots for dataset-1.



(a) The plots of density function.



(b) The plots of hazard rate function.



(c) The plots of survival function.

Figure 5 – Model fitting plots for dataset-2.

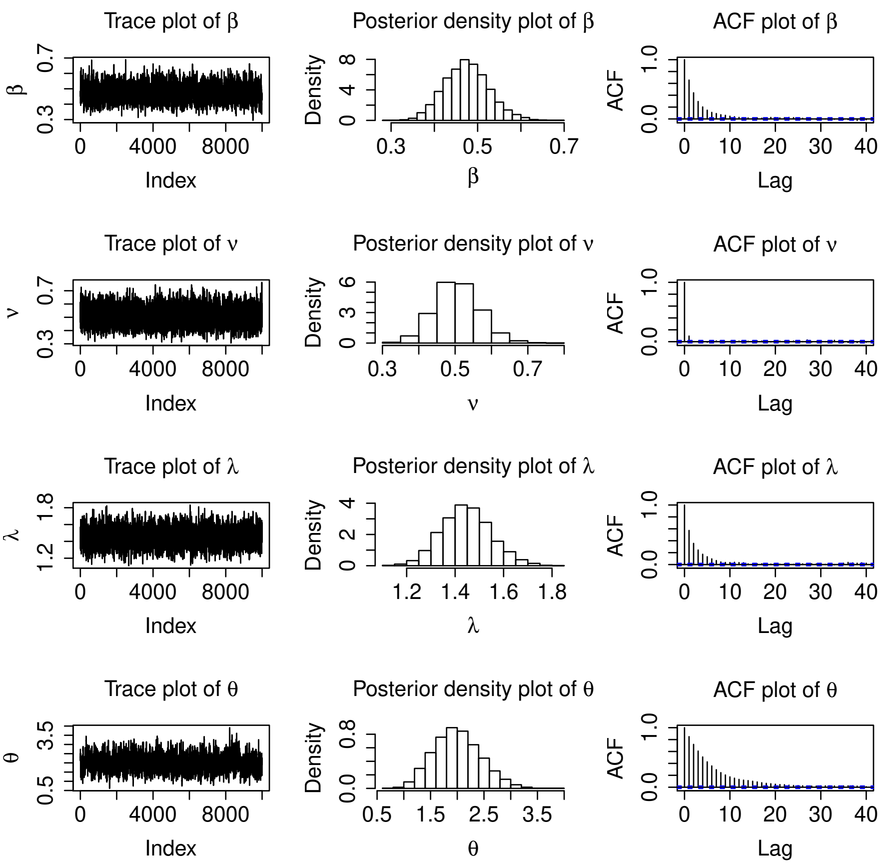


Figure 6 – MCMC diagnostic plots of the parameters.

B. Tables

TABLE 1
Survival functions of considered models.

Model	Parameters	$S(x)$	Reference
MW	$(\beta,\nu,\lambda)>0$	$e^{-\beta x^{\nu} e^{\lambda x}}$	Lai *et al.* (2003)
ADDW	$(\alpha,\theta,\beta,\nu)>0$	$e^{-\alpha x^{\theta}-\beta x^{\nu}}$	Xie and Lai (1996)
SZMW	$(\alpha,\beta,\nu)>0$	$e^{-\alpha x-\beta x^{\nu}}$	Sarhan and Zaindin (2009)
MWE	$(\alpha,\beta,\lambda)>0$	$e^{\lambda\alpha(1-e^{(x/\alpha)^{\beta}})}$	Xie *et al.* (2002)
GMW	$(\alpha,\beta,\lambda,\nu)>0$	$1-(1-e^{-\alpha x^{\nu} e^{\lambda x}})^{\beta}$	Carrasco *et al.* (2008)
Chen	$(\theta,\beta)>0$	$e^{\beta(1-e^{x^{\theta}})}$	Chen (2000)

TABLE 2
Fitting summary for the athlete's data (dataset-1).

Model	MLE (SE)	logL	AIC	BIC	K-S	p-value
EMW	$\hat{\beta} = 2.18 \times 10^{-12}(9.45 \times 10^{-13})$ $\hat{\nu} = 5.648(0.289)$ $\hat{\lambda} = 2.026(0.856)$ $\hat{\theta} = 0.001(2.63 \times 10^{-06})$	-825.45	1658.91	1672.14	0.079	0.161
MW	$\hat{\beta} = 3.01 \times 10^{-05}(1.42 \times 10^{-04})$ $\hat{\nu} = 1.465(1.389)$ $\hat{\lambda} = 0.048(0.017)$	-838.18	1682.36	1692.29	0.098	0.039
ADDW	$\hat{\alpha} = 3.45 \times 10^{-04}(4.33 \times 10^{-05})$ $\hat{\theta} = 1.754(0.033)$ $\hat{\beta} = 5.25 \times 10^{-06}(6.91 \times 10^{-06})$ $\hat{\nu} = 1.661(0.036)$	-979.82	1967.65	1980.88	0.302	2.2×10^{-16}
SZMW	$\hat{\alpha} = 1.58 \times 10^{-03}(4.63 \times 10^{-03})$ $\hat{\beta} = 4.71 \times 10^{-06}(4.19 \times 10^{-07})$ $\hat{\nu} = 2.848(0.013)$	-903.59	1813.19	1836.12	0.358	2.2×10^{-16}
MWE	$\alpha = 0.356(0.264)$ $\hat{\beta} = 0.459(0.052)$ $\hat{\lambda} = 1.53 \times 10^{-05}(3.04 \times 10^{-06})$	-832.85	1671.70	1681.63	0.093	0.061
GMW	$\hat{\alpha} = 6.27 \times 10^{-04}(3.40 \times 10^{-04})$ $\hat{\beta} = 1.334(0.286)$ $\hat{\lambda} = 0.053(0.009)$ $\hat{\nu} = 0.737(0.252)$	-835.26	1678.52	1691.75	0.112	0.012
Chen	$\beta = 0.500(0.013)$ $\hat{\theta} = 1.29 \times 10^{-04}(7.14 \times 10^{-05})$	-841.59	1687.19	1693.80	0.145	0.001

TABLE 3
Fitting summary for the Aarset data (dataset-2).

Model	MLE (SE)	logL	AIC	BIC	K-S	*p*-value
EMW	$\hat{\beta} = 0.093(0.035)$ $\hat{\nu} = 0.400(0.089)$ $\hat{\lambda} = 1.30 \times 10^{-05}(2.46 \times 10^{-06})$ $\hat{\theta} = 2.617(2.45 \times 10^{-04})$	-219.50	447.01	454.66	0.131	0.351
MW	$\hat{\beta} = 0.060(0.026)$ $\hat{\nu} = 0.367(0.115)$ $\hat{\lambda} = 0.023(0.005)$	-227.16	460.32	466.06	0.134	0.329
ADDW	$\hat{\alpha} = 8.45 \times 10^{-09}(1.50 \times 10^{-09})$ $\hat{\theta} = 4.279(0.049)$ $\hat{\beta} = 0.091(0.038)$ $\hat{\nu} = 0.466(0.098)$	-221.35	450.71	458.36	0.127	0.392
SZMW	$\hat{\alpha} = 0.013(0.003)$ $\hat{\beta} = 0.013(0.003)$ $\hat{\nu} = 4.405(0.143)$	-229.41	464.82	470.56	0.151	0.202
MWE	$\alpha = 13.736(5.088)$ $\hat{\beta} = 0.588(0.078)$ $\hat{\lambda} = 0.009(0.003)$	-231.65	469.29	475.03	0.159	0.158
GMW	$\hat{\alpha} = 5.177 \times 10^{-05}(1.44 \times 10^{-05})$ $\hat{\beta} = 0.254(0.048)$ $\hat{\lambda} = 0.063(0.026)$ $\hat{\nu} = 1.033(0.230)$	-223.12	454.24	461.89	0.139	0.283
Chen	$\beta = 0.021(0.009)$ $\hat{\theta} = 0.344(0.021)$	-233.17	470.33	474.16	0.167	0.123

TABLE 4
Various estimates along with their estimated errors.

Sample Size (n)	30	50	100
		$\beta = 0.5, \nu = 0.5, \lambda = 1.5, \theta = 2$	
MLE(SE)	$\hat{\beta} = 0.354(0.199)$ $\hat{\nu} = 0.417(0.182)$ $\hat{\lambda} = 1.785(0.574)$ $\hat{\theta} = 1.917(0.691)$	$\hat{\beta} = 0.486(0.176)$ $\hat{\nu} = 0.425(0.121)$ $\hat{\lambda} = 1.479(0.375)$ $\hat{\theta} = 2.655(0.766)$	$\hat{\beta} = 0.717(0.172)$ $\hat{\nu} = 0.613(0.108)$ $\hat{\lambda} = 1.176(0.250)$ $\hat{\theta} = 2.814(0.655)$
BE(PSE) with Gamma Prior	$\hat{\beta} = 0.509(0.060)$ $\hat{\nu} = 0.518(0.058)$ $\hat{\lambda} = 1.532(0.111)$ $\hat{\theta} = 1.979(0.122)$	$\hat{\beta} = 0.463(0.051)$ $\hat{\nu} = 0.517(0.058)$ $\hat{\lambda} = 0.747(0.072)$ $\hat{\theta} = 1.987(0.156)$	$\hat{\beta} = 0.494(0.047)$ $\hat{\nu} = 0.505(0.055)$ $\hat{\lambda} = 0.742(0.062)$ $\hat{\theta} = 2.069(0.145)$
BE(PSE) with Non-informative Prior	$\hat{\beta} = 0.597(0.076)$ $\hat{\nu} = 0.512(0.066)$ $\hat{\lambda} = 0.749(0.159)$ $\hat{\theta} = 2.296(0.683)$	$\hat{\beta} = 0.608(0.059)$ $\hat{\nu} = 0.528(0.064)$ $\hat{\lambda} = 1.497(0.101)$ $\hat{\theta} = 2.161(0.373)$	$\hat{\beta} = 0.602(0.057)$ $\hat{\nu} = 0.512(0.058)$ $\hat{\lambda} = 1.497(0.093)$ $\hat{\theta} = 2.262(0.292)$
BE using T-K method (PSE) with Gamma Prior	$\hat{\beta} = 0.421(0.021)$ $\hat{\nu} = 0.391(0.013)$ $\hat{\lambda} = 1.316(0.057)$ $\hat{\theta} = 1.967(0.078)$	$\hat{\beta} = 0.375(0.018)$ $\hat{\nu} = 0.430(0.012)$ $\hat{\lambda} = 1.266(0.041)$ $\hat{\theta} = 1.908(0.059)$	$\hat{\beta} = 0.332(0.011)$ $\hat{\nu} = 0.402(0.010)$ $\hat{\lambda} = 1.188(0.038)$ $\hat{\theta} = 1.912(0.057)$

TABLE 5
Various confidence intervals with their corresponding widths.

Sample Size (n)	30	50	100
		$\beta = 0.5, \nu = 0.5, \lambda = 1.5, \theta = 2$	
(ACI) [Width]	$\hat{\beta} \in (0,\ 0.745)[0.745]$ $\hat{\nu} \in (0.059,\ 0.775)[0.716]$ $\hat{\lambda} \in (0.660,\ 2.910)[2.250]$ $\hat{\theta} \in (0.562,\ 3.271)[2.709]$	$\hat{\beta} \in (0.139,\ 0.832)[0.692]$ $\hat{\nu} \in (0.186,\ 0.664)[0.477]$ $\hat{\lambda} \in (0.742,\ 2.215)[1.473]$ $\hat{\theta} \in (1.153,\ 4.157)[3.003]$	$\hat{\beta} \in (0.378,\ 1.055)[0.677]$ $\hat{\nu} \in (0.399,\ 0.826)[0.426]$ $\hat{\lambda} \in (0.684,\ 1.668)[0.983]$ $\hat{\theta} \in (1.529,\ 4.098)[2.568]$
HPD interval with Gamma Prior [Width]	$\hat{\beta} \in (0.390,\ 0.629)[0.238]$ $\hat{\nu} \in (0.411,\ 0.647)[0.235]$ $\hat{\lambda} \in (1.318,\ 1.756)[0.438]$ $\hat{\theta} \in (1.687,\ 2.187)[0.500]$	$\hat{\beta} \in (0.369,\ 0.568)[0.199]$ $\hat{\nu} \in (0.400,\ 0.632)[0.232]$ $\hat{\lambda} \in (0.631,\ 0.903)[0.271]$ $\hat{\theta} \in (1.703,\ 2.289)[0.585]$	$\hat{\beta} \in (0.400,\ 0.586)[0.186]$ $\hat{\nu} \in (0.399,\ 0.611)[0.212]$ $\hat{\lambda} \in (0.637,\ 0.877)[0.239]$ $\hat{\theta} \in (1.800,\ 2.395)[0.594]$
HPD interval with Non informative Prior [Width]	$\hat{\beta} \in (0.446,\ 0.747)[0.300]$ $\hat{\nu} \in (0.389,\ 0.644)[0.254]$ $\hat{\lambda} \in (0.619,\ 0.851)[0.231]$ $\hat{\theta} \in (1.082,\ 3.647)[2.565]$	$\hat{\beta} \in (0.475,\ 0.711)[0.236]$ $\hat{\nu} \in (0.402,\ 0.653)[0.251]$ $\hat{\lambda} \in (1.296,\ 1.691)[0.395]$ $\hat{\theta} \in (1.409,\ 2.868)[1.459]$	$\hat{\beta} \in (0.505,\ 0.760)[0.254]$ $\hat{\nu} \in (0.411,\ 0.631)[0.219]$ $\hat{\lambda} \in (1.317,\ 1.681)[0.364]$ $\hat{\theta} \in (1.713,\ 2.818)[1.105]$
PCI using T-K method with Gamma Prior [Width]	$\hat{\beta} \in (0.379,\ 0.464)[0.085]$ $\hat{\nu} \in (0.364,\ 0.418)[0.053]$ $\hat{\lambda} \in (1.203,\ 1.428)[0.224]$ $\hat{\theta} \in (1.814,\ 2.121)[0.306]$	$\hat{\beta} \in (0.339,\ 0.411)[0.072]$ $\hat{\nu} \in (0.406,\ 0.454)[0.047]$ $\hat{\lambda} \in (1.185,\ 1.348)[0.163]$ $\hat{\theta} \in (1.791,\ 2.025)[0.234]$	$\hat{\beta} \in (0.310,\ 0.354)[0.043]$ $\hat{\nu} \in (0.392,\ 0.423)[0.040]$ $\hat{\lambda} \in (1.113,\ 1.263)[0.149]$ $\hat{\theta} \in (1.800,\ 2.024)[0.223]$

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SUMMARY

In this study, we introduce an extended version of the modified Weibull distribution with an additional shape parameter, in order to provide more flexibility to its density and the hazard rate function. The distribution is capable of modeling the bathtub-shaped, decreasing, increasing and the constant hazard rate function. The proposed model contains sub-models that are widely used in lifetime data analysis such as the modified Weibull, Chen, extreme value, Weibull, Rayleigh, and exponential distributions. We study its statistical properties which include the hazard rate function, moments and distribution of the order statistics. The parameters involved in the model are estimated by using maximum likelihood and the Bayesian method of estimation. In Bayesian estimation, we assume independent Gamma priors for the parameters and MCMC technique such as the Metropolis-Hastings algorithm within Gibbs sampler has been implemented to obtain the sample-based estimators and the highest posterior density intervals of the parameters. Tierney and Kadane (1986) approximation is also used to obtain Bayes estimators of the parameters. In order to highlight the relative importance of various estimates obtained, a simulation study is carried out. The usefulness of the proposed model is illustrated using two real datasets.

Keywords: Extended modified Weibull distribution; Maximum likelihood estimates; Bayesian estimates; Gibbs sampler; Tierney and Kadane's approximation; Highest posterior density intervals.